COMPARATIVE PROBABILITY ORDERINGS

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1. Introduction

The term comparative or qualitative probability is used to denote an ordering relation among events which actually means a comparison of probability. This term has been used from three different points of view: the first one is the introduction of the concept as a rational and natural approach to numerical probability theory, as by de Finetti, Savage and Koopman. The second one is the idea of an autonomous theory of probability, possibly weaker than the classical numerical theory of probability (e.g. Keynes, Fishburn, Fine). The third one proposes comparative probability as a first natural step of the (numerical) probability elicitation process. This point of view is is explicit in (Good, 1950 [25]).

De Finetti (1931 [14]), from the first point of view, proposed four axioms for a comparative probability (CP) ordering which have been considered the basis for any axiomatic theory of comparative probability. His purpose was to show that "starting out from a purely qualitative system of axioms, one arrives at a quantitative measure of probability, and then at the theorem of total probability which permits the construction of the whole calculus of probability" (de Finetti, 1937 [15]). In fact, with the additional assumption that there is an uniform partition on an arbitrary number of events, he proved the existence of a unique probability measure representing the given comparisons. This point of view has led toward the search for general conditions for the existence and unicity of a probability measure representing the given ordering.

From the second point of view, there are some attempts to select qualitative axioms of probability and to develop alternative concepts of probability which take into account the limited precision of our beliefs (Fine, 1973 [19]; Kaplan and Fine, 1977 [30]; Walley and Fine, 1979 [63]). On the other hand, the four axioms have been weakened and linked with numerical tools different from probability, in order to incorporate various behavioral attitudes. In his excellent review on comparative probability, Fishburn (1986 [21]) also surveys those attempts and exhibits examples of behavioral "inconsistency" with de Finetti's axioms. In most of the examples, the inconsistency seems to depend on the requirement of completeness.

Using comparisons as the first step of an elicitation process, completeness is obviously abandoned. From this point of view it is quite natural to consider also comparative prevision or preference (Buehler, 1976 [2]; Giron and Rios, 1980 [24]; Walley, 1991 [61]; Regoli, 1994b [46]), that is, comparisons amongst random quantities.

In the first and third approaches, the main issue is the consistency of comparative assessments with a numerical probability measure. A more general tool for representing a comparative prevision (probability) is de Finetti's concept of coherent prevision (probability) (de Finetti, 1937 [15], 1970 [16]; Regazzini, 1988 [45]). This concept does not need any structure on the set of random quantities (events) where the comparisons are made; therefore, such a set can be extended and updated. If all the random quantities are bounded, a coherent prevision P can be considered as the expected value with respect to a suitable finitely additive probability measure (for short finitely additive probability). In fact if P is a coherent prevision on a set of bounded random quantities, $\mathcal{F}$, then there is a finitely additive probability P', defined on the algebra $\mathcal{A}$ spanned by $\mathcal{F}$, such that P equals the extension of P' as a nonnegative linear functional to the linear space of all bounded $\mathcal{A}$-measurable random quantities. De Finetti's coherent conditional prevision (probability) can be seen as the restriction of a complete conditional finitely additive expectation (probability), e.g. (Regazzini, 1987 [44]), where *complete* means that it is permissible to condition on every possible event, including those with zero probability (Dubins, 1975 [18]). Links between complete conditional finitely additive probability and probability taking values on a nonstandard model of the real field can be found in (Krauss, 1968 [36]).

All the necessary and sufficient conditions for the representability of a comparative probability ordering have been proved by embedding the events in a linear space of random variables, which is equipped with the linear order induced by the given comparisons. Separating hyperplane theorems or alternative theorems for systems of linear inequalities give the representation criteria; therefore the same criteria and proofs can be used for events and random quantities. Therefore, those conditions are necessary and sufficient conditions for the representability of a comparative prevision, and we review them in this form in the next section.

2. Comparative probability and comparative prevision

Let $\mathcal{F}$ be an arbitrary family of events (possibly represented as a family of subsets of a set Ω). A *comparative probability* on $\mathcal{F}$ is a partial binary relation $\mathcal{R}$ on $\mathcal{F}$. For every $(B, A) \in \mathcal{R}$ we denote by $B \preceq A$ the assertion that "*B is not more probable than A*". The assertion that "*B is equally likely as A*" summarizes the two assertions that $B \preceq A$ and $A \preceq B$, and it will be denoted by $B \sim A$. Observe that *not all* the pairs of events are necessarily compared.

An additional *strict comparative probability*, $\mathcal{R}'$, can be elicitated by assertions such as "*B is strictly less probable than A*", denoted by $B \prec A$. Let $\mathcal{R}^*$ be the asymmetric relation formally deduced from $\mathcal{R}$, namely $\mathcal{R}^* = \mathcal{R} \setminus \mathcal{R}^T$, where $\mathcal{R}^T = \{(A, B) \colon (B, A) \in \mathcal{R}\}$ and let us denote it by $B \prec^* A$. If the relations $\mathcal{R}$ and $\mathcal{R}'$ represent the opinion of an expert, then it is natural to have $\mathcal{R}' \subset \mathcal{R}^*$: in fact, in case of partial or gradual information, it is possible that, at a first stage of judgment, the expert has not decided yet if $B \prec A$ or $B \sim A$ and he summarizes his opinion by $B \preceq A$.

A comparative probability is said to be *consistent on* $\mathcal{F}$ if there exists a finitely additive probability P on the algebra generated by $\mathcal{F}$ such that

$$B \preceq A \Rightarrow P(B) \leq P(A).$$

In such a case we say that P *almost represents* (or is *compatible with*, or *almost agrees with*) $\mathcal{R}$. Moreover, if also

$$B \prec A \Rightarrow P(B) < P(A),$$

we say that P *represents* (or is *strictly compatible with*, or *agrees with*) $\mathcal{R}$. Note that in case of ambiguity, we would write: P *represents* $(\mathcal{R}, \mathcal{R}')$.

Let $\mathcal{F}$ be a family of real bounded random quantities and let $\mathcal{A}_{\mathcal{F}}$ be the spanned algebra of events, that is the minimal algebra containing the events $(X \in S)$ for all $X \in \mathcal{F}$ and for all real sets S. Let $\mathcal{K}_{\mathcal{F}}$ be the linear space of $\mathcal{A}_{\mathcal{F}}$-measurable random quantities. An event and its indicator function will be denoted by the same letter.

A *comparative prevision* on $\mathcal{F}$ is a finite or infinite list, $\mathcal{R}$, of comparisons among random quantities, equivalently it is a partial binary relation on $\mathcal{F}$. For every $(Y, X) \in \mathcal{R}$ we denote by $Y \preceq X$ the assertion that "*Y is not bigger than X on average*", or "*I expect Y not to be bigger than X*". The restriction of $\mathcal{R}$ to the set of the indicators of events is a comparative probability. If $\mathcal{R}' \subset \mathcal{R}$ is a given subset of strict comparisons, then, for every $(Y, X) \in \mathcal{R}'$ we denote by $Y \prec X$ the assertion that "*Y is smaller than X on average*".

A comparative prevision, $\mathcal{R}$, is said to be *consistent on $\mathcal{F}$* if there exists a finitely additive probability P on $\mathcal{A}_{\mathcal{F}}$ such that

$$Y \preceq X \Rightarrow P(Y) \leq P(X)$$

where $P(X)$ is the expected value of X with respect to P. In such a case we say that P *almost represents* (or is *compatible with*) $\mathcal{R}$. Moreover, if also

$$Y \prec X \Rightarrow P(Y) < P(X),$$

we say that P *represents* (or is *strictly compatible with*) $\mathcal{R}$ and that $\mathcal{R}$ is *strongly consistent on $\mathcal{F}$*.

In particular, when only indicators are considered, a consistent comparative prevision is a consistent comparative probability.

2.1. **Representation and almost representation.**

Every comparative prevision $\mathcal{R}$ on $\mathcal{F}$ has an extension defined by $\hat{\mathcal{R}} = \mathcal{R} \cup \{(\mathbf{0}, X) \colon X \in \mathcal{F}, \mathbf{0} \leq X\}$; if $\mathcal{R}$ is a comparative probability then its extension can be written as $\hat{\mathcal{R}} = \mathcal{R} \cup \{(\emptyset, A) \colon A \in \mathcal{F}\}$. The obvious meaning is that for every event $A \in \mathcal{F}$ and every nonnegative random quantity $X \in \mathcal{F}$, the relations $\emptyset \preceq A$ and $\mathbf{0} \preceq X$ are in $\hat{\mathcal{R}}$.

Fact 1. *With this notation, if $\mathcal{F}$ is an arbitrary set of bounded random quantities the following conditions are equivalent:*

a) *The comparative prevision $\mathcal{R}$ is consistent.*
b) *For every finite set F, every set of comparisons $\{(Y_j, X_j) \in \mathcal{R}\}$, and all positive real numbers y_j, $j \in F$,*

$$\sup \sum_{j \in F} y_j (X_j - Y_j) \geq 0. \tag{1}$$

c) *For every finite set F, every set of comparisons $\{(Y_j, X_j) \in \hat{\mathcal{R}}\}$, and all positive real numbers y_j, $j \in F$,*

$$1 + \sum_{j \in F} y_j (X_j - Y_j) \neq 0. \tag{2}$$

To obtain a representation which takes into account the strict elicited comparisons, $\mathcal{R}'$, analogous conditions work only for simple sets of random quantities, for example, a finite set of events.

Fact 2. *When $\mathcal{F}$ is a* finite *set of* finite-valued *random quantities, the following conditions are equivalent:*

a') *The comparative prevision $\mathcal{R}$ is strongly consistent.*

b') *For every finite set F, every set of comparisons $\mathcal{S}_F = \{(Y_j, X_j) \in \mathcal{R}, j \in F\}$, $\mathcal{S}_F \cap \mathcal{R}' \neq \emptyset$, and all positive real numbers y_j, $j \in F$,*

$$\sup \sum_{j \in F} y_j (X_j - Y_j) > 0. \tag{3}$$

c') *For every finite set F, every set of comparisons $\mathcal{S}_F = \{(Y_j, X_j) \in \hat{\mathcal{R}}, j \in F\}$, $\mathcal{S}_F \cap \mathcal{R}' \neq \emptyset$, and all positive real numbers y_j, $j \in F$,*

$$\sum_{j \in F} y_j (X_j - Y_j) \neq 0. \tag{4}$$

Conditions b) and b') can be interpreted as coherence conditions by considering hypothetical bets in favour of each X_i versus Y_i; or in the case of events, consider bets on the more probable events and against the less probable, e.g. (Hartigan, 1983, p.20 [26]). Another point of view interprets these conditions as a *dominance principle*: if a positive linear combination of random quantities, X_i, dominates the same linear combination of others, Y_i, then it is rational to expect at least one of the X_i's to be bigger than the corresponding Y_i.

The equivalence between a) and b) can be found in (Buehler, 1976 [2]) or (Heath and Sudderth, 1972 [27], 1978 [28]). The equivalence between a) and c) can be proved by extending to random quantities the proof that Cohen (1991 [7]) has given for events. Condition b') extends to random quantities the condition given by Kraft *et al.* (1959 [34]). Condition c) has been given by Scott (1964 [53]) for a complete relation on an algebra of events and it has been studied by Cohen (1991 [7]) and Rios Insua (1992 [50]) for incomplete relations. Coletti (1990 [8]) gives a coherence condition for the existence of a *positive* almost representation in a finite set of events.

All these proofs are based on alternative theorems for linear systems of inequalities, e.g. (Kuhn and Tucker, 1956 [37]; Gale, 1960 [23]), or separating hyperplane theorems, e.g. (Holmes, 1975 [29]).

2.2. **Archimedean conditions.** When $\mathcal{F}$ is infinite, Scott's condition c'), or equivalently b'), implies only the representability of $\mathcal{R}$ in any finite subset and it is equivalent (Narens, 1974 [42]) to representability by a measure taking values in a nonstandard model of the real field. In the infinite case, representability by a finitely additive probability measure needs some further property which prevents the existence of "infinitesimal" differences between events. These properties are usually referred to as *Archimedean axioms*.

S — ***Separability*:** there exists a countable family $\mathcal{B} \subset \mathcal{F}$ such that for every pair of quantities $Y, X \in \mathcal{F}$ such that $Y \prec X$, there exists a $Z \in \mathcal{B}$ such that $Y \preceq Z \preceq X$.

A — ***Archimedean condition*:** for every pair $Y, X \in \mathcal{F}$ such that $Y \prec X$, there exists a natural number $n(X,Y) \in \mathbb{N}$ such that if

$$k\Omega - m(X - Y) = \sum_{j \in F} (X_j - Y_j)$$

for some $k, m \in \mathbb{N}$, $m \neq 0$, a finite set $F \subset \mathcal{R}$ and $\{(Y_j, X_j) \in F\}$, then

$$\frac{k}{m} > \frac{1}{n(X,Y)}.$$

Condition **S** is due to Cantor and it is a necessary and sufficient condition for a preorder, $\mathcal{R}$, to be representable by a real function. Condition **A**, due to Chateauneuf and Jaffray (1984 [5]), gives numerical bounds which involve additivity. Chateauneuf (1985 [4]) has proved that for a complete preorder $\mathcal{R}$ on an algebra of events $\mathcal{F}$, conditions **S** and **A** hold if and only if $\mathcal{R}$ is representable by a finitely additive probability. Since condition **A** implies the above condition b'), completeness can be dropped and the same proof can be used for a set of random quantities. Cohen (1991 [7]) suggested the following version of axiom **A**:

A' — *Archimedean condition 1*: for every pair $Y, X \in \mathcal{F}$ such that $Y \prec X$,

$$0 < \inf\{\lambda \colon \lambda - (X - Y) = \sum_{j \in F} (X_j - Y_j) \text{ for some } (Y_j, X_j) \in \hat{\mathcal{R}}\}$$

Cohen (1991 [7]) proved that this is a necessary and sufficient condition for representability of a relation $\mathcal{R}$ on a set of events, in the case that the family of strict comparisons, $\mathcal{R}'$, is countably generated in the sense of linear combinations.

Other similar Archimedean conditions, given in terms of numerical bounds, are due to Suppes and Zanotti (1976 [55], 1982 [56]), Lehrer (1991 [38]) and Domotor (1994 [17]).

Another kind of Archimedean axiom requires the finiteness of *standard sequences*. There are different types of standard sequences (Luce, 1967 [39]; Krantz *et al.*, 1971 [35]; Fine, 1973 [19]; Narens, 1974 [42]; Domotor, 1994 [17]) which are generalizations of the following definition: $A_1, A_2, \dots$ is a *Luce's standard sequence* relative to the event A if there are $B_i \sim A$ such that $B_i \cap A_i = \emptyset$ and $A_{i+1} = B_i \cup A_i$. It is worthwhile to emphasize that such a condition has a nice version in the conditional comparative context (Luce, 1968 [40]). All these conditions arise in an axiomatic framework (see further on) and they need other structural assumptions to be applied.

Finally, for a non-Archimedean comparative probability, several representations have been proposed: first of all, the one mentioned previously, using a measure taking values in a nonstandard model of the real field (Narens, 1974 [42]; Chuaqui and Malitz, 1983 [6]; Coletti, 1990 [8], Domotor, 1994 [17]). Another representation is based on an idea suggested by de Finetti: under a suitable set of axioms, a non-Archimedean comparative probability matches the order induced by a complete conditional probability (Coletti and Regoli, 1983a [12], 1983b [11], 1986 [13]) in the sense that

$$B \preceq A \Leftrightarrow P(B \mid A \cup B) \leq P(A \mid A \cup B).$$

2.3. **Countable additivity.** The existence of a countably additive representation needs some continuity axioms and a more structured framework. The two main assumptions are due respectively to Villegas (1964 [59]) and Giron and Rios (1980 [24]).

MC — *Monotone continuity*: Let $\mathcal{R}$ be a complete comparative probability on an algebra of events $\mathcal{A}$. It is said to be *monotonely continuous* if, given an increasing sequence of events $A_i \in \mathcal{A}$, $i \in \mathbb{N}$, with $A = \bigcup A_i$, then $A_i \preceq B$ implies $A \preceq B$.

If $\mathcal{R}$ is monotonely continuous, every representing probability measure is countably additive. Moreover, if $\mathcal{R}$ satisfies de Finetti's axioms and $\mathcal{A}$ is *atomless* (i.e. every event A can be partitioned into two events, B_1, B_2, and $\emptyset \prec B_i \prec A$), then a countably additive representation exists (and it is unique). In Villegas (1964 [59]), $\mathcal{A}$ is a σ-algebra. Chateauneuf and Jaffray (1984 [5]) remarked that an algebra is sufficient for these results.

If **MC** is added to Scott's condition and there exists at least one event which is atomless, then the relation $\mathcal{R}$ is representable by a countably additive probability measure (Chuaqui and Malitz, 1983 [6]).

Giron and Rios (1980 [24]) gave a continuity condition for a comparative prevision defined on a well structured set of random quantities. Let Ω be a compact

Hausdorff topological space; let $\mathcal{C}$ be the space of all real-valued continuous functions, equipped with the topology of uniform convergence. Let $\mathcal{F} \subset \mathcal{C}$ be a convex set containing all nonnegative functions. Consider a complete transitive relation $\mathcal{R}$ on $\mathcal{F}$ which is *linear*, i.e., $Y \preceq X$ if and only if $\lambda Y + (1-\lambda)H \preceq \lambda X + (1-\lambda)H$, for every H and $\lambda > 0$. $\mathcal{R}$ is said to be *continuous* if, whenever a sequence X_n converges uniformly in $\mathcal{F}$ to X, then $Y \preceq X_n \preceq Z$ implies $Y \preceq X \preceq Z$. Under all these hypotheses, $\mathcal{R}$ is representable as the expected value of some countably additive probability defined on the Boolean subsets of Ω. Such a probability is unique if $\mathcal{F} = \mathcal{C}$.

3. Families of compatible probabilities

Let $\mathcal{F}$ be a set of random quantities and let $\mathcal{R}$ be a consistent comparative prevision on $\mathcal{F}$. Let $\mathcal{P}$ be the set of all finitely additive probabilities on the algebra, $\mathcal{A}_{\mathcal{F}}$, spanned by $\mathcal{F}$. The family of compatible probabilities is the convex set Γ given by

$$\Gamma = \{P \in \mathcal{P} \colon P(X) \geq P(Y), (Y, X) \in \mathcal{R}\}.$$

Γ is the weak*-closure of the convex hull of its extreme points. If $\mathcal{A}_{\mathcal{F}}$ is finite and generated by n atoms, then Γ has a finite number of extreme points $\{Q_s \in \mathbb{R}^n \colon s = 1, \ldots, m\}$, and it is simply the convex hull of $\{Q_s\}_s$, e.g. (Walley, 1991, Chapter 3 [61]).

If $\mathcal{R}$ satisfies one of the continuity axioms given in the previous section (either in the algebra or in the linear space), then Γ contains only countably additive probabilities (Villegas, 1964 [59]; Giron and Rios, 1980 [24]).

3.1. Unicity.

Let a comparative prevision be consistent. When is its (almost) representation unique? A general necessary and sufficient criterion for unicity can be deduced from the bounds of the family of (almost) representing previsions, such as the one given by Cohen (1991 [7]) (see next subsection). Other explicit necessary and sufficient criteria have been given for finite set of events by Cohen (1991 [7]) and Fishburn (1986 [21]). The following is in Fishburn (1986 [21]):

When a set of events generates a finite set of n atoms a_j, $i = 1, 2, \ldots, n$, a positive probability representation of a comparative probability is unique if and only if there are $n-1$ pairs of events (B_i, A_i), with $B_i \sim A$ and $B_i \cap A_i = \emptyset$, such that the corresponding $n-1$ equations,

$$\sum_{a_j \in A_i} p_j - \sum_{a_j \in B_i} p_j = 0; \quad i = 1, 2, \ldots, n-1,$$

are linearly independent.

Most of the conditions given in the literature are only sufficient for the unicity of a representation. They arise in the axiomatic, foundational approach to comparative probability (see the next section); they usually require a complete relation on an algebra; moreover they involve the logical structure of the set of events and they need other structural conditions in order to imply representability. In particular, given a complete relation on an atomless algebra, if a representation exists then it is unique (Lehrer, 1991 [38]).

3.2. Precision and robustness.

Since, in general, the family of compatible probabilities, Γ, contains more than one probability, in applications an analysis of both the precision and the robustness of inferences is necessary. A natural tool to measure precision or robustness is based on the bounds of the compatible previsions. The following bounds for the compatible previsions can be found in Cohen (1991

[7]) and Walley (1991 [61], 1996 [62]). For every $Z \in \mathcal{K}_{\mathcal{F}}$ define

$$P^*(Z) = \inf\{\alpha \colon (\alpha - Z) \geq \sum_{i=1}^{k} r_i(X_i - Y_i);\ \text{for some } Y_i \preceq X_i, r_i \geq 0\},$$

and

$$P_*(Z) = \sup\{\alpha \colon (Z - \alpha) \geq \sum_{i=1}^{k} r_i(X_i - Y_i);\ \text{for some } Y_i \preceq X_i, r_i \geq 0\}.$$

If $\mathcal{R}$ is consistent, then a real number $p \in [P_*(Z), P^*(Z)]$ if and only if there is a finitely additive probability $P \in \Gamma$ such that $P(Z) = p$.

If C is an event and $P_*(C) > 0$ then $P(C) > 0$ for every $P \in \Gamma$. In that case, bounds for compatible conditional expectations are given by

$$P_*(Z \mid C) = \sup\{a \colon C(Z - a) \geq \sum_{i=1}^{k} r_i(X_i - Y_i);\ \text{for some } Y_i \preceq X_i, r_i \geq 0\},$$

and

$$P^*(Z \mid C) = \inf\{a \colon C(a - Z) \geq \sum_{i=1}^{k} r_i(X_i - Y_i);\ \text{for some } Y_i \preceq X_i, r_i \geq 0\}.$$

Contributions to robust Bayesian inference for comparative probability and prevision can be found in Giron and Rios (1980 [24]) and related discussion, and in Regoli (1994b [46], 1996b [48]). Robust Bayesian tools, deduced from Moment Problem Theory (Berger, 1994 [1]; Kemperman, 1987 [31]; Salinetti, 1994 [51]) are available for this kind of analysis as indicated in (Regoli, 1996b [48]). Morever, to measure the robustness of a particular updated expected value, the above conditional bounds can be used. If the cardinality of $\mathcal{R}$ is not too large, these bounds can be found by minimizing and maximizing the updated extreme points of Γ: this technique allows sequential updating, but the computations are not always feasible; they are feasible for particular types of comparative assessments, such as the identification of some *almost uniform partitions* (see later the axiom **AU** due to Savage; the formula for the bounds is in (Regoli, 1996b [48])).

3.3. **Extensions.** Given a comparative prevision $(\mathcal{R}, \mathcal{R}')$ on a family $\mathcal{F}$, we say that a comparative prevision $(\mathcal{T}, \mathcal{T}')$ on a family $\mathcal{G}$ is an *extension* of $(\mathcal{R}, \mathcal{R}')$ if $\mathcal{F} \subset \mathcal{G}$, $\mathcal{R} \subset \mathcal{T}$ and $\mathcal{R}' \subset \mathcal{T}'$. Note that this definition includes the case $\mathcal{R} = \mathcal{T}$ and $\mathcal{R}' \neq \mathcal{T}'$, which is the one obtained by specifying additional strict comparisons. Every order induced by an arbitrary $P \in \Gamma$ is a complete extension of the given $\mathcal{R}$ to the linear space $\mathcal{K}_{\mathcal{F}}$.

If $\mathcal{R}$ is consistent, we consider the set of all comparisons which are implied by the given ones, and call it $\mathcal{R}_w$. It is the intersection of all consistent comparative prevision extensions of $\mathcal{R}$. The following properties can be deduced from Cohen (1991 [7]).

If $\mathcal{R}$ is consistent, then $(Y, X) \in \mathcal{R}_w$ (i.e. $Y \preceq_w X$) if and only if

$$X - Y = \sum_{j \in F} y_j(X_j - Y_j) \text{ for some finite set } (Y_j, X_j) \in \hat{\mathcal{R}} \text{ and } y_j \geq 0 \qquad (5)$$

When $\mathcal{A}_{\mathcal{F}}$ is finite, the following procedure extends a consistent comparative prevision to a strongly consistent one. In fact, defining $\mathcal{R}_w^* = \mathcal{R}_w \setminus \mathcal{R}_w^T$, $(\mathcal{R}_w, \mathcal{R}_w^*)$ is a strongly consistent extension of $\mathcal{R}$. Moreover, if $(\mathcal{R}, \mathcal{R}')$ is strongly consistent then $\mathcal{R}' \subset \mathcal{R}_w^*$ and $Y \prec^* X$ if and only if equation (5) holds and it contains some strict comparison $Y_j \prec X_j$.

Another procedure for extension uses directly the above bounds of Γ: given two random variables, X and Y, define $Z = X - Y$. Then $Y \preceq X$ is compatible with (or implied by) the given $\mathcal{R}$ if and only if $P^*(Z) \geq 0$ (respectively $P_*(Z) \geq 0$).

4. Axiomatic approach

The literature dealing with the foundational point of view of comparative probability tries to point out those qualitative aspects which are basic to the probability concept. A weakened form of de Finetti's (1931 [14]) axioms is usually assumed.

Let $\mathcal{A}$ be an algebra of events, represented as an algebra of subsets of a sample space Ω. Let $\mathcal{R}$ be a nontrivial binary relation defined on $\mathcal{A}$, written as $B \preceq A$. The triplet $(\Omega, \mathcal{A}, \mathcal{R})$ is called a *comparative probability structure* (for short CPS) if it satisfies the following axioms:

- $\mathcal{R}$ is a *weak order* $\mathcal{A}$: i.e. it is a reflexive, transitive, complete binary relation.
- $\mathcal{R}$ is *nonnegative*: $\forall A \in \mathcal{A}$, $\emptyset \preceq A$.
- $\mathcal{R}$ is *additive*: if $A \cap C = B \cap C = \emptyset$, then $B \preceq A \Leftrightarrow B \cup C \preceq A \cup C$.

In this case the relation $\sim$ is an equivalence relation; the only asymmetric relation considered is $\mathcal{R}^* = \mathcal{R} - \mathcal{R}^T$, that is, $B \prec A$ if $B \preceq A$ and not $A \preceq B$. De Finetti proposed these properties as an axiomatic qualitative basis for the theory of probability. (In the original version, rather than nonnegativity, he required positivity, i.e., $\forall A \in \mathcal{A}$, $\emptyset \prec A$.)

Without some additional structural properties, these axioms do not guarantee consistency with a numerical probability model (neither strong nor weak). In fact Kraft *et al.* (1959 [34]) have given two examples: in the first, the sample space consists of five atoms and the CPS is consistent but not strongly consistent; in the second, a CPS is given on a set of six atoms and it is not consistent. De Finetti (1931 [14]) and Koopman (1940 [33]) have obtained strong consistency, with a *unique* representation in infinite algebras, by requiring the existence of a *uniform partition* on an arbitrary number of events (i.e. a partition into n events which are equally likely, for every $n \in \mathbb{N}$).

This axiom has been weakened by Savage (1954 [52]) in several ways:

AU: There exists an n-*almost uniform* partition of Ω for infinitely many $n \in \mathbb{N}$, where a partition of Ω, $\{C_i \colon i = 1, ...n\}$, is called an n-almost uniform partition if

$$\bigcup_{j=1}^{r} C_{i_j} \preceq \bigcup_{j=1}^{r+1} C_{k_j}, \quad \forall r < n; \quad \forall i_j, k_j \in \{1, \ldots, n\}. \tag{6}$$

F: Relation $\mathcal{R}$ is *fine* in a set Ω if, $\forall B \succ \emptyset$, there is a partition of Ω into a finite number of events, each of which is not more probable than B.

AEP: For every $n \in \mathbb{N}$, there exists a partition of Ω into n *almost equivalent* events, where two events, A_1 and A_2, are almost equivalent if $A_j \preceq A_i \cup C_i$, for every $C_i \succ \emptyset$ with $C_i \cap A_i = \emptyset$, $i, j = 1, 2$.

Properties **AU** and **F** are equivalent. Property **AU** is a necessary and sufficient condition for the existence of a *unique, nonatomic* finitely additive probability which almost represents $\mathcal{R}$. Property **AEP** implies property **AU** and for a CPS on a σ-algebra, the two properties are equivalent (Savage, 1954 [52]; Niiniluoto, 1972 [43]; Wakker, 1981 [60]). Moreover, if almost equivalence implies equivalence, the almost representation is a (strong) representation. Another partition axiom, called *trisplittability* by Narens (1974 [42]), implies existence and unicity of the representation; in this case there are no atoms.

In order to consider finite algebras, or, more generally, models containing atoms, structural solvability axioms have been introduced. Luce (1967 [39]), Suppes (1974 [54]), Krantz *et al.* (1971 [35]) and Van Lier (1989 [57]) require that the algebra is rich in equally likely "copies" of suitable events; they couple this requirement with some standard-scale conditions to obtain a *unique* representation. Coletti and Regoli (1986 [13]) get the same consequence by requiring that $\mathcal{R}$ be fine in some event A and satisfy a scale condition: namely $\Omega \in \mathcal{A}_n$ for some n where $\mathcal{A}_n = \{\bigcup_{i\in F} E_i : F \text{ finite}, \exists A_i \in \mathcal{A}_{n-1}, E_i \preceq A_i\}$ and $\mathcal{A}_0 = \{A\}$. Essentially, these properties characterize the CPS representable by a finitely additive probability with a nonnull continuous component.

Neglecting the problem of consistency with a probability measure, the axiomatic approach has been carried on as an autonomous theory (Keynes, 1921 [32]; Fine, 1977 [20]; Walley and Fine, 1979 [63]). This case is usually referred to as *nonadditive CPS*. From this point of view, some improvement has been made in developing the concepts of conditional and joint probability and independence (next sections).

Interest in comparative probability axioms weaker than CPS has been shown from the decision theoretic point of view: Fishburn (1986 [21]) reviews some of these aspects and, in particular, discusses the problem of representation by an "imprecise numerical probability" (Koopman, Good, Suppes, Walley and Fine). More recently, Wong *et al.* (1991 [64]), Regoli (1994a [47]), Fishburn (1996 [22]), and Capotorti, Coletti and Vantaggi (1998 [3]) deal with comparative relations representable by some "nonadditive probabilities" such as Dempster-Shafer belief functions or convex capacities. In particular, the last paper shows that comparative structures on a set of events do not distinguish between belief functions and convex capacities, or between lower probabilities and 0-monotone capacities.

5. Comparative conditional probability

Conditional probability can be considered basic in subjective probability theory: unconditional probability is just a simple particular case. It is even more natural to consider comparative conditional probability judgments as basic in the theory of comparative probability, e.g. (Koopman, 1940 [33]).

A *comparative conditional probability* is a binary relation on a set of pairs of events, $\mathcal{E} \times \mathcal{F}$. We denote by $(A, B) \preceq (C, D)$ or $A \mid B \preceq C \mid D$ the assertion that "*A given B is not more probable than C given D*". A comparative conditional probability is *consistent* if there exists a numerical conditional probability, P, such that

$$A \mid B \preceq C \mid D \Rightarrow P(A \mid B) \leq P(C \mid D).$$

Classically this problem has been studied by assuming that $\mathcal{E}$ is an algebra, $\mathcal{F} \subset \mathcal{E}$ (typically $\mathcal{F}$ is an ideal of $\mathcal{E}$), and $\preceq$ is a complete relation on $\mathcal{E} \times \mathcal{F}$. In such a case the representation P is the conditional probability derived by ratio from a finitely additive probability P defined on the algebra $\mathcal{E}$ and such that $P(F) > 0$ for all $F \in \mathcal{F}$.

In this framework, conditions for consistency are given by Koopman (1940 [33]), Luce (1968 [40]), Krantz *et al.* (1971 [35]), Suppes and Zanotti (several papers: e.g. 1982 [56]).

Neglecting the completeness of the relation, Buehler (1976 [2]) has given a coherence condition in order to apply Bayes' theorem to conditional preferences. Given finite sets $\mathcal{X}$ and $\mathcal{T}$, he considers two families of conditional comparisons on real-valued functions defined on $\mathcal{X} \times \mathcal{T}$. In one family, the comparisons are conditional given some value $\theta \in \mathcal{T}$ and in the other family they are conditional given some value $x \in \mathcal{X}$. The coherence condition is necessary and sufficient for the relations to be compatible with a joint finitely additive probability, P, on the subsets of $\mathcal{X} \times \mathcal{T}$, such that its conditional previsions $P(h(x,\theta) \mid \theta)$ and $P(h(x,\theta) \mid x)$, represent respectively the two given families of conditional comparisons.

Via coherence conditions, Coletti, Gilio and Scozzafava (1993 [10]), Coletti (1994 [9]) and Vicig (1995 [58]) treat the problem of consistency of a comparative conditional probability on an arbitrary set of pairs of events: the relation is represented by de Finetti's coherent conditional probability and allows conditioning on events which may have probability 0. Their work also deals with the search for algorithms which could be implemented in an automatic support system.

6. Independence and joint comparative probability

Starting with Fine (1973 [19]), the concepts of independence and joint comparative probability have been studied with emphasis on the nonadditive case. The concept of independence in a CPS is characterized by some version of the following property (Fine, 1973 [19]; Kaplan and Fine, 1977 [30]; Luce and Narens, 1978 [41]):

$$\text{if } A \perp B, C \perp D, A \sim C \text{ then } B \preceq D \Rightarrow A \cap B \preceq C \cap D.$$

where the symbol $A \perp B$ means A *is independent of* B.

Even assuming consistency of the CPS, this property is not sufficient for the existence of a multiplicative representation by a probability measure P, i.e.,

$$A \perp B \Rightarrow P(A \cap B) = P(A)P(B).$$

To obtain sufficient conditions, solvability and Archimedean conditions are required by Luce and Narens (1978 [41]). Regoli (1996a [49]) obtains a multiplicative probability representation, via a coherence condition on a set of logically independent events.

Finally, the existence of general joint comparative probability structures for given CPS marginals is investigated by Kaplan and Fine (1977 [30]); from their natural definitions of joint CPS several interesting results follow:

(1) There are pairs of CPS marginals which cannot be combined into any joint CPS.
(2) There is a class of CPS (strictly containing the consistent CPS) which can be combined with every other CPS into a joint CPS.
(3) A given CPS admits an arbitrary number of *independent* identically distributed repetitions (i.e. n jointly independent copies of the given CPS, for every natural number n) if and only if it is consistent with a finitely additive probability.

Sufficient conditions for the consistency of independent identical CPS are also in Luce and Narens (1978 [41]).

References

[1] J. O. Berger. An overview of robust Bayesian analysis. *Test*, 3:5–124, 1994. with discussion.
[2] R. J. Buehler. Coherent preferences. *The Annals of Statistics*, 4:1051–1064, 1976.
[3] A. Capotorti, G. Coletti, and B. Vantaggi. Non additive ordinal relations representable by lower or upper probabilities. *Kibernetika*, 34:79–90, 1998.
[4] A. Chateauneuf. On the existence of a probability measure compatible with a total preorder on a Boolean algebra. *Journal of Mathematical Economics*, 14:43–52, 1985.
[5] A. Chateauneuf and J. Y. Jaffray. Archimedean qualitative probabilities. *Journal of Mathematical Psychology*, 28:191–204, 1984.
[6] R. Chuaqui and J. Malitz. Preorderings compatible with probability measures. *Transactions of the American Mathematical Society*, 279:811–824, 1983.
[7] M. A. Cohen. Necessary and sufficient conditions for existence and uniqueness of weak qualitative probability structures. *Journal of Mathematical Psychology*, 35:242–259, 1991.
[8] G. Coletti. Coherent qualitative probability. *Journal of Mathematical Psychology*, 34:297–310, 1990.

[9] G. Coletti. Coherent numerical and ordinal probabilistic assessments. *IEEE Transactions on Systems, Man and Cybernetics*, 12:1747–1754, 1994.
[10] G. Coletti, A. Gilio, and R. Scozzafava. Comparative probability for conditional events: a new look through coherence. *Theory and Decision*, 35:237–258, 1993.
[11] G. Coletti and G. Regoli. Non-Archimedean qualitative probability and realizability. *Riv. Mat. Sci. Econ. Soc.*, 6:79–99, 1983.
[12] G. Coletti and G. Regoli. Sul confronto tra eventi di probabilità nulla. *Rend. Ist. Mat. Trieste*, XV:19–31, 1983.
[13] G. Coletti and G. Regoli. Realizations of qualitative probabilities. *Boll. Unione Mat. Ital., VI Ser C*, 5:95–112, 1986.
[14] B. de Finetti. Sul significato soggettivo della probabilità. *Fundamenta Mathematicae*, 17:298–329, 1931.
[15] B. de Finetti. La prévision, ses lois logiques, ses sources subjectives. *Annales de l'Institut Henri Poincaré*, 7:1–68, 1937. English translation in (Kyburg and Smokler, 1964).
[16] B. de Finetti. *Teoria della Probabilità*. Einaudi, 1970. English translation: *Theory of Probability*, Vol. 1 (1974), Vol. 2 (1975), Wiley, New York.
[17] Z. Domotor. Qualitative probabilities revisited. In *Patrick Suppes: Scientific Philosopher*, volume 1 of *Synthese Library 233*, pages 197–238. Kluwer, Dordrecht, 1994.
[18] L. Dubins. Finitely additive conditional probabilities, conglomerability and disintegration. *Annals of Probability*, 3:89–99, 1975.
[19] T. L. Fine. *Theories of Probability*. Academic Press, New York, 1973.
[20] T. L. Fine. An argument for comparative probability. In R. E. Butts and J. Hintikka, editors, *Basic Problems in Methodology and Linguistics*, pages 105–111. Reidel, Dordrecht, 1977.
[21] P. C. Fishburn. The axioms of subjective probability. *Statistical Science*, 1:335–358, 1986. with discussion.
[22] P. C. Fishburn. Finite linear qualitative probability. *Journal of Mathematical Psychology*, 40:64–77, 1996.
[23] D. Gale. *The Theory of Linear Economic Models*. McGraw Hill, New York, 1960.
[24] F. J. Giron and S. Rios. Quasi-Bayesian behaviour: a more realistic approach to decision making? In J. M. Bernardo, M. H. De Groot, D. V. Lindley, and A. F. M. Smith, editors, *Bayesian Statistics*, pages 17–38. Valencia University Press, Valencia, 1980. with discussion.
[25] I. J. Good. *Probability and the Weighing of Evidence*. Griffin, London, 1950.
[26] J. A. Hartigan. *Bayes Theory*. Springer-Verlag, New York, 1983.
[27] D. C. Heath and W. D. Sudderth. On a theorem of de Finetti, oddsmaking, and game theory. *Annals of Mathematical Statistics*, 6:2072–2077, 1972.
[28] D. C. Heath and W. D Sudderth. On finitely additive priors, coherence, and extended admissibility. *The Annals of Statistics*, 6:333–345, 1978.
[29] R. B. Holmes. *Geometric Functional Analysis and its Applications*. Springer-Verlag, New York, 1975.
[30] M. Kaplan and T. L. Fine. Joint orders in comparative probability. *Annals of Probability*, 5:161–179, 1977.
[31] J. K. B. Kemperman. Geometry of the moment problem. In *Proceedings of Symposia in Applied Mathematics*, volume 37, pages 16–53. 1987.
[32] J. M. Keynes. *A Treatise on Probability*. Macmillan, New York, 1921.
[33] B. O. Koopman. The bases of probability. *Bulletin of the Americal Mathematical Society*, 46:763–774, 1940. Reprinted in (Kyburg and Smokler, 1964).
[34] C. H. Kraft, J. Pratt, and A. Seidenberg. Intuitive probability on finite sets. *Annals of Mathematical Statistics*, 30:408–430, 1959.
[35] D. H. Krantz, R. D. Luce, P. Suppes, and A. Tversky. *Foundations of Measurement*, volume 1. Academic Press, New York, 1971.
[36] P. H. Krauss. Representation of conditional probability measures on Boolean algebras. *Acta Math. Acad. Sci. Hungar.*, 19:229–241, 1968.
[37] H. V. Kuhn and A. W. Tucker. Linear inequalities and related systems. *Annals of Mathematical Studies*, 38, 1956.
[38] E. Lehrer. On a representation of a relation by a measure. *Journal of Mathematical Economics*, 20:107–118, 1991.
[39] R. D. Luce. Sufficient conditions for the existence of a finitely additive probability measure. *Annals of Mathematical Statistics*, 38:780–786, 1967.
[40] R. D. Luce. On the numerical representation of qualitative conditional probability. *Annals of Mathematical Statistics*, 39:481–491, 1968.
[41] R. D. Luce and L. Narens. Qualitative independence in probability theory. *Theory and Decision*, pages 225–239, 1978.

[42] L. Narens. Minimal conditions for additive conjoint measurement and qualitative probability. *Journal of Mathematical Psychology*, 11:404–430, 1974.

[43] I. Niiniluoto. A note on fine and tight qualitative probabilities. *Annals of Mathematical Statistics*, 43:1581–1591, 1972.

[44] E. Regazzini. De Finetti's coherence and statistical inference. *The Annals of Statistics*, 15:845–864, 1987.

[45] E. Regazzini. Subjective probability. In *Encyclopedia of Statistical Sciences*, volume 9, pages 55–64. Wiley, New York, 1988.

[46] G. Regoli. Qualitative probabilities in an elicitation process. In *Atti XXXVII Riunione Scientifica SIS*, pages 153–165, San Remo, April 1994.

[47] G. Regoli. Rational comparisons and numerical representation. In S. Rios, editor, *Decision Theory and Analysis: Trends and Challenges*, pages 113–126. Kluwer, Boston, 1994.

[48] G. Regoli. Comparative probability and robustness. In J. O. Berger, B. Betro, E. Moreno, L. R. Pericchi, F. Ruggeri, G. Salinetti, and L. Wasserman, editors, *Bayesian Robustness*, number 29 in IMS Lecture notes, Monograph Series, pages 327–336. 1996.

[49] G. Regoli. Stochastic independence and comparative probability assessments. In *Proceedings of IPMU '96*, volume I, pages 49–53. Granada, 1996.

[50] D. Rios Insua. On foundation of decision making under partial information. *Theory and Decision*, pages 83–100, 1992.

[51] G. Salinetti. Discussion of "an overview of robust bayesian analysis" by j. o. berger. *Test*, 3:5–124, 1994.

[52] L. J. Savage. *The Foundations of Statistics*. Wiley, New York, 1954.

[53] D. Scott. Measurement structures and linear inequalities. *Journal of Mathematical Psychology*, 1:233–247, 1964.

[54] P. Suppes. The measurement of belief. *Journal of the Royal Statistical Society, Series B*, 36:160–175, 1974. with discussion.

[55] P. Suppes and M. Zanotti. Necessary and sufficient condition for a unique measure strictly agreeing with a qualitative probability ordering. *J. Philos. Logic*, 3:431–438, 1976.

[56] P. Suppes and M. Zanotti. Necessary and sufficient qualitative axioms for conditional probability. *Zeitschrift für Wahrscheinlichkeitsrechnung und Verwandte Gebiete*, 60:163–169, 1982.

[57] L. Van Lier. A simple sufficient condition for the unique representability of a finite qualitative probability by a probability measure. *Journal of Mathematical Psychology*, 33:91–98, 1989.

[58] P. Vicig. Conditional and comparative probabilities in artificial intelligence. In G. et al. Coletti, editor, *Mathematical Models for Handling Partial Knowledge in Artificial Intelligence*, pages 271–280. Plenum Press, New York, 1995.

[59] C. Villegas. On qualitative probability σ-algebras. *Annals of Mathematical Statistics*, 35:1787–1796, 1964.

[60] P. Wakker. Agreeing probability measures for comparative probability structures. *The Annals of Statistics*, 9:658–662, 1981.

[61] P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.

[62] P. Walley. Measures of uncertainty in expert systems. *Artificial Intelligence*, 83:1–58, 1996.

[63] P. Walley and T. Fine. Varieties of modal (classificatory) and comparative probability. *Synthese*, 41:321–374, 1979.

[64] S. K. M. Wong, Y. Y. Yao, P. Bollmann, and H. C. Burger. Axiomatization of qualitative belief structures. *IEEE Transactions on Systems, Man and Cybernetics*, 21:726–734, 1991.