A note on the Dutch Book method

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Abstract

The paper presents a rather general proof of the so called Dutch Book argument and shows how it may also be applied to yield the corresponding analogs of probability functions for various non-standard propositional logics for example modal, intuitionistic, and paraconsistent logics.

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1 Introduction and Notation

The purpose of this modest note is to explicitly point out the existence of a generic presentation of the basic Dutch Book argument and consider the application of this 'method' to alternate, non-Tarskian, semantics. This presentation is already implicit (and *somewhere* probably also explicit) in even elementary textbooks concerned with solving systems of linear equations and inequalities, as well as in foundational studies of rational belief and avoidance of sure loss such as Walley's monumental [9]. The applications we present to non-Tarskian semantics would however appear to be new, with the exception of a specific earlier example due to Jaffray, [4] (see also Regoli, [7]).

Before formally stating these results we need some notation. Let L be a finite propositional language and SL the set of sentences of L, say using the standard connectives $\vee, \wedge, \neg, \rightarrow$. As usual we use $p, q, ...$ etc for the propositional variables of L and $\theta, \phi, ...$ etc for elements of SL.

Let $\mathbb{V}$ be a subset of the set of all functions from SL into $\{0,1\}$. We should think of $\mathbb{V}$ as the set of 'possible worlds', where for $V \in \mathbb{V}$ and $\theta \in SL$, θ is *true/false* in V if $V(\theta) = 1/0$. To start with we shall assume that $\mathbb{V}$ is finite.

Let $\mathbb{B}$ be the set of all functions from SL into $[0,1]$. We are thinking of the elements of $\mathbb{B}$ as possible subjective *belief functions* on SL. In other words for $\theta \in SL$ and $B \in \mathbb{B}$ we should think of $B(\theta)$ as a measure of an agent's 'belief' (on the scale $[0,1]$, 1 signifying maximum possible belief etc) to θ 'being true' in the *actual world*, V ($V \in \mathbb{V}$).

One particularly relevant example of such a set $\mathbb{V}$ is the set $\mathbb{V}^T$ of (classical) valuations on SL, that is functions $V : SL \longrightarrow \{0,1\}$ satisfying the Tarski truth conditions that for $\theta, \phi \in SL$

(T1) $V(\neg\theta) = 1 \iff V(\theta) = 0,$
(T2) $V(\theta \wedge \phi) = 1 \iff V(\theta) = 1 \ \&\ V(\phi) = 1,$
(T3) $V(\theta \vee \phi) = 0 \iff V(\theta) = 0 \ \&\ V(\phi) = 0,$
(T4) $V(\theta \rightarrow \phi) = 0 \iff V(\theta) = 1 \ \&\ V(\phi) = 0.$

In this case $\mathbb{V}^T$ is finite since L is finite and every $V \in \mathbb{V}^T$ is determined by its values on the propositional variables alone.

Theorem 1 *[2],[3]*

The function $B \in \mathbb{B}$ is a probability function, that is satisfies that for all $\theta, \phi \in SL$,

(P1) If $\models \theta$ then $B(\theta) = 1$
(P2) If $\models \neg(\theta \wedge \phi)$ then $B(\theta \vee \phi) = B(\theta) + B(\phi)$

if and only if there does not exist a Dutch Book against B. That is, there do not exist some $\theta_1, \theta_2, ..., \theta_m \in SL$ and $s_1, s_2, ..., s_m \in \mathbb{R}$ such that for all $V \in \mathbb{V}^T$,

$$\sum_{i=1}^{m} s_i(V(\theta_i) - B(\theta_i)) < 0.$$

The relevance of the term 'Dutch Book' here is that if we identify $B(\theta)$ with one's limiting willingness to bet that θ is true in the actual world V (V a member of the set $\mathbb{V}^T$ of *possible worlds*) in the sense that for *any* stake s, positive or negative, we are willing to pay out[1]

[1] The s may be negative here, in which case paying out $sB(\theta)$ should be thought of as accepting $-sB(\theta)$ and gaining s should be thought of as losing $-s$.

$sB(\theta)$ to receive s if $V(\theta)=1$ and nothing if $V(\theta)=0$ then the expression on the left of the inequality in this theorem is easily seen to be one's profit (negative profit = loss) from simultaneously entering into such bets on the θ_i, $i=1,2,...,r$, at stakes s_i. Thus having a Dutch Book means that if one accepted these bets, or *gambles*, then one would be bound to make a loss no matter what the true state, V, of the world was.

The upshot of this then is that if we identify one's belief with the limit of one's willingness to bet then ones beliefs can be rational, in the sense of avoiding any Dutch Book, if and only if they determine a probability function.

Theorem 1 applies to the classical situation where the semantics are given by Tarski's conditions (T1)-(T4) and 'worlds' are equated with valuations $V \in \mathbb{V}^T$. However these are certainly not the only sorts of 'worlds' which are possible. In non-standard logics it is customary to consider semantics for languages containing additional features, for example modalities, and where the connectives have alternate interpretations. Within the context of these semantics we can still consider beliefs as reflecting limiting willingness to bet on what holds in the 'true world'. Furthermore, just as in the classical case, avoidance of a Dutch Book (the *Dutch Book method* of our title) provides a criterion of rationality for judging such belief functions.

In section 3 we shall give examples of applying this method to various non-standard logics. Firstly however we need to formulate a suitably general proof of Theorem 1.

2 The General Theorem

The usual proof of Theorem 1 (see for example [6]) proceeds, in the right to left direction, by showing that from any failure of (P1) or (P2) one can directly construct a Dutch Book. However one can give an alternate proof, which, although somewhat less elementary, is much more general and in a way makes the Dutch Book argument seem much less 'special'. The general result is the following, where a 'Dutch Book against B' is defined as in Theorem 1 but with the set of possible worlds $\mathbb{V}$ in place of $\mathbb{V}^T$.

Theorem 2 *For $\mathbb{V}$ finite a function $B \in \mathbb{B}$ does not permit a Dutch Book if and only if B is a convex combination of the functions in $\mathbb{V}$, i.e. $B=\sum_{V\in\mathbb{V}} a_V V$ where the $a_V \geq 0$ and $\sum_{V\in\mathbb{V}} a_V = 1$.*

Proof. Suppose on the contrary that B was a convex combination $\sum_{V\in\mathbb{V}} a_V V$ of functions in $\mathbb{V}$ but that for some $\theta_i \in SL$ and $s_i \in \mathbb{R}$, $i=1,...,m$,

$$\sum_{i=1}^{m} s_i(V(\theta_i)-B(\theta_i))<0$$

for all $V \in \mathbb{V}$. In that case

$$\sum_{V\in\mathbb{V}} a_V \left\{ \sum_{i=1}^{m} s_i(V(\theta_i)-B(\theta_i)) \right\} < 0.$$

But changing the order of summation gives

$$\sum_{i=1}^{m} \left\{ s_i(\sum_{V\in\mathbb{V}} a_V V(\theta_i)) - (\sum_{V\in\mathbb{V}} a_V) s_i B(\theta_i) \right\} < 0,$$

i.e.

$$\sum_{i=1}^{m} \{ s_i B(\theta_i) - s_i B(\theta_i) \} < 0,$$

a contradiction.

The converse is a special case of a well known result from linear algebra but for the sake of completeness we shall give the details. So suppose that B is not a convex combination of elements from $\mathbb{V}$. Then we can find sentences $\theta_1,...,\theta_m \in SL$ such that $B \restriction \{\theta_1,...,\theta_m\}$ (i.e. B restricted to the set $\{\theta_1,...,\theta_m\}$) is not a convex combination of the functions $V \restriction \{\theta_1,...,\theta_m\}$ for $V \in \mathbb{V}$ (otherwise by taking a suitably convergent subsequence as the $\theta_1,...,\theta_m$ extend through all of SL we could contradict our initial assumptions). This means that $< B(\theta_1),...,B(\theta_m) >$ is not in the closed convex set Y of vectors of the form

$$\sum_{V\in\mathbb{V}} a_V < V(\theta_1),...,V(\theta_m) >$$

where the $a_V \geq 0$, and $\sum_{V\in\mathbb{V}} a_V = 1$. Hence by a well known result on convex sets (see the appendix for a proof) there must be a vector $\vec{s} = < s_1,...,s_m > \in \mathbb{R}^m$ such that

$$(< v_1,...,v_m > - < B(\theta_1),...,B(\theta_m) >) \cdot \vec{s} < 0$$

for all $< v_1,...,v_m > \in Y$. In particular then substituting $< V(\theta_1),...,V(\theta_m) >$ for $< v_1,...,v_m >$ for V ranging over $\mathbb{V}$ shows that we have here a Dutch Book for B. ■

The standard Dutch Book argument for belief as probability now follows if we take $\mathbb{V}=\mathbb{V}^T$. For then to avoid a Dutch Book B must be of the form $\sum_{V\in\mathbb{V}^T} a_V V$ with $\sum_{V\in\mathbb{V}^T} a_V = 1$, $a_V \geq 0$. But each $V \in \mathbb{V}^T$ is uniquely determined by the (unique) *atom* α_V, that is sentence of the form $\pm p_1 \wedge \pm p_2 \wedge ... \wedge \pm p_n$, such that $V(\alpha_V)=1$. So $B(\alpha_V)=a_V$ and for $\theta \in SL$,

$$\begin{aligned} B(\theta) &= \sum_{V \in \mathbb{V}^T} a_V V(\theta) \\ &= \sum\{\alpha_V \mid V \in \mathbb{V}^T, \alpha_V \models \theta\}, \\ &\quad \text{since } V(\theta)=1 \iff \alpha_V \models \theta, \\ &= \sum\{\ B(\alpha_V) \mid V \in \mathbb{V}^T, \alpha_V \models \theta\} \quad (1) \end{aligned}$$

and it is well known (see [6]) that every probability function on SL arises in this way, and conversely.

Of course one could argue here that this was a pretty long winded proof of the 'Dutch Book Theorem'! The advantage of this derivation however is that we have in Theorem 2 a very general Dutch Book argument which can be applied to many other situations where our notion of 'truth' is not simply confined to be Tarskian, i.e. satisfying (T1-4). At the same time it highlights the dependence of belief as probability on the classical, Tarskian, interpretation of truth.

A second nice property of this proof is that it gives as a simple corollary a generalization of a theorem of Lehman, [5]. The generalization concerns the possibility of forming a Dutch Book when the function B is only *partial* (so in the definition of a Dutch Book the $\theta_1, ..., \theta_m$ are also required to be in the domain of B). In the following theorem $\mathbb{V}$ is, as usual, a subset of the set of all functions from SL to $\{0,1\}$.

Corollary 3 *Let B be a partial function from SL into $[0,1]$. Then B does not permit a Dutch Book if and only if B has an extension to a function in $\mathbb{B}$ which does not permit a Dutch Book, equivalently has an extension to a convex combination of functions in $\mathbb{V}$.*

Proof. Clearly the implication follows from right to left by Theorem 2. In the other direction suppose B is partial and does not permit a Dutch Book. For $V \in \mathbb{V}$ let V^- be the restriction of V to the domain of B. Clearly B does not permit a Dutch Book with respect to the set of 'valuations' $\{V^- \mid V \in \mathbb{V}\}$ so by an immediate adaption of Theorem 2 B is a convex combination $\sum_{V \in \mathbb{V}} a_V V^-$. The required extension of B to a total function is $\sum_{V \in \mathbb{V}} a_V V$. ■

Up to now we have required $\mathbb{V}$ to be finite. However in view of the proof of theorem 2 it is should be clear that we can drop that restriction provided that we require only that B be locally a convex combination of the functions in $\mathbb{V}$. Precisely,

Corollary 4 *For $\mathbb{V}$ possibly infinite, a function $B \in \mathbb{B}$ does not permit a Dutch Book if and only if for every finite $\Gamma \subset SL$ $B \restriction \Gamma$ is a convex combination of the functions in $\{V \restriction \Gamma \mid V \in \mathbb{V}\}$.*

3 Applications to Non-Tarskian Truth

We now turn to considering some other applications of Theorem 2 to alternate notions of 'truth'. As we shall see the vital, and often missing, ingredient turns out to be an argument corresponding to (1) above for characterizing those $B \in \mathbb{B}$ which are convex combinations of the $V \in \mathbb{V}$.

Our first example is a very simple modification of the classical Dutch Book Theorem given above. Namely, if we replace $\mathbb{V}^T$ by a non-empty subset, $\mathbb{V}^-$ say, of this set, effectively then saying that only certain worlds (valuations) are possible, then any $B \in \mathbb{B}$ avoiding a Dutch Book must be a convex combination of these valuations, equivalently must be a probability function on SL which assigns zero probability to the atoms α_V for $V \in \mathbb{V}^T - \mathbb{V}^-$ (and conversely).

For a second example, which is an already well known result due to Jaffray, [4], (see also [7]) suppose that our possible worlds might not yet be fully created (maybe it's still only the fifth day, or maybe these are worlds in a book where some details have not been made explicit) so that in any world our total knowledge might consist of knowing the truth of some non-contradictory $\theta \in SL$ (and hence all logical consequences of θ) where θ is not necessarily an atom. So it may be that not all questions of classical truth and falsity are determined. In that case if we were to identify true with 'known true' in the classical sense (with falsity equated with not true) then our set of valuations, denoted $\mathbb{V}^D$ would consist of $\{\ V_\theta \mid \theta \in SL\ \}$ where for $\theta \in SL$ V_θ is the function from SL to $\{0,1\}$ such that

$$V_\theta(\phi) = 1 \iff \theta \models \phi.$$

In this case by results of Shafer the functions $B \in \mathbb{B}$ which are convex combinations of functions in $\mathbb{V}^D$ (equivalently by Theorem 2 avoid a Dutch Book) are precisely the *Dempster-Shafer belief functions* on SL (see [8] or, in the notation of this note, [6]). That is functions $B \in \mathbb{B}$ such that for all $\theta, \phi, \theta_1, ..., \theta_m \in SL$

(DS1) If $\models \theta$ then $B(\theta) = 1,\ B(\neg\theta) = 0$,
(DS2) If $\models (\theta \leftrightarrow \phi)$ then $B(\theta) = B(\phi)$,
(DS3) $B(\bigvee_{i=1}^m \theta_i) \geq \sum_S (-1)^{|S|-1} B(\bigwedge_{i \in S} \theta_i)$,

where in (DS3) S ranges over the non-empty subsets of $\{1, 2, ..., m\}$.

Again we might note that as with probability the hard part of this Dutch Book result is in Shafer's derivation of the equivalence of convex combinations of functions in $\mathbb{V}^D$ with Dempster-Shafer belief functions.

For our next examples we shall generalize the conclusions following Theorem 2 to certain logics containing, possibly as derived connectives, conjunction and disjunction satisfying (T2) and (T3). Precisely let $\mathcal{L}$ be a finite propositional language with at least the connectives $\wedge, \vee$ (and possibly also modalities). Let $S\mathcal{L}$ be the set of sentences of $\mathcal{L}$ and suppose that the logic carries with it a notion of a valuation (world, interpretation) $V : S\mathcal{L} \longrightarrow \{0, 1\}$. Let $\mathbb{V}^{\mathcal{L}}$ be the set of such valuations of $\mathcal{L}$. As usual for $\theta \in S\mathcal{L}$, $\Gamma \subseteq S\mathcal{L}$ define

$$\Gamma \models_{\mathcal{L}} \theta \quad \Longleftrightarrow \quad \begin{array}{l} \forall V \in \mathbb{V}^{\mathcal{L}}, \text{ if } V(\phi) = 1 \text{ for all } \phi \in \Gamma \\ \text{then } V(\theta) = 1. \end{array}$$

As usual we write $\models_{\mathcal{L}} \theta$ (corresponding to $\Gamma = \emptyset$) if θ is an $\mathcal{L}$-tautology, i.e. $V(\theta) = 1$ for all $V \in \mathbb{V}^{\mathcal{L}}$, and write $\theta \models$ if θ is an $\mathcal{L}$-contradiction, i.e. $V(\theta) = 0$ for all $V \in \mathbb{V}^{\mathcal{L}}$. Let $\mathbb{B}^{\mathcal{L}}$ be the set of functions from $S\mathcal{L}$ into $[0, 1]$.

Theorem 5 *Let $B \in \mathbb{B}^{\mathcal{L}}$ and suppose that for $V \in \mathbb{V}^{\mathcal{L}}$, (T2) and (T3) hold. Then $B \restriction \Gamma$ is a convex combination of functions in $\{V \restriction \Gamma \mid V \in \mathbb{V}^{\mathcal{L}}\}$ for every finite $\Gamma \subset SL$ if and only if B satisfies that for all $\theta, \phi \in S\mathcal{L}$,*

$(\mathcal{L}1)$ If $\models_{\mathcal{L}} \theta$ then $B(\theta) = 1$,
if $\theta \models_{\mathcal{L}}$ then $B(\theta) = 0$,
$(\mathcal{L}2)$ If $\theta \models_{\mathcal{L}} \phi$ then $B(\theta) \leq B(\phi)$,
$(\mathcal{L}3)$ $B(\theta \vee \phi) + B(\theta \wedge \phi) = B(\theta) + B(\phi)$.

Proof. Clearly ($\mathcal{L}$1-3) hold for $V \in \mathbb{V}^{\mathcal{L}}$ and hence for any convex combination of such functions.

To prove the other direction let Γ be a finite subset of $S\mathcal{L}$ which is closed under subformulae. Now for each $\emptyset \neq \Delta \subset \Gamma$ set

$$a_\Delta = B\left(\bigwedge_{\theta \in \Delta} \theta\right) - B\left(\bigwedge_{\theta \in \Delta} \theta \wedge \bigvee_{\phi \in \Gamma - \Delta} \phi\right)$$

and pick, if possible, a $V_\Delta \in \mathbb{V}^{\mathcal{L}}$ such that

$$V_\Delta\left(\bigwedge_{\theta \in \Delta} \theta\right) = 1, \quad V_\Delta\left(\bigwedge_{\theta \in \Delta} \theta \wedge \bigvee_{\phi \in \Gamma - \Delta} \phi\right) = 0.$$

Notice that by $(\mathcal{L}2)$ $a_\Delta \geq 0$ since

$$\bigwedge_{\theta \in \Delta} \theta \wedge \bigvee_{\phi \in \Gamma - \Delta} \phi \models_{\mathcal{L}} \bigwedge_{\theta \in \Delta} \theta.$$

Notice also that if no such V_Δ exits then

$$\bigwedge_{\theta \in \Delta} \theta \models_{\mathcal{L}} \bigwedge_{\theta \in \Delta} \theta \wedge \bigvee_{\phi \in \Gamma - \Delta} \phi \models_{\mathcal{L}} \bigwedge_{\theta \in \Delta} \theta,$$

so by $(\mathcal{L}2)$ $a_\Delta = 0$.

For $\Delta = \emptyset$ let

$$a_\Delta = 1 - B\left(\bigvee_{\phi \in \Gamma} \phi\right)$$

and pick, if possible, $V_\Delta \in \mathbb{V}^{\mathcal{L}}$ such that $V(\phi) = 0$ for all $\phi \in \Gamma$. Notice that by $(\mathcal{L}1)$ such a V_Δ does exist if $a_\Delta > 0$.

Finally for $\Delta = \Gamma$ define

$$a_\Delta = B\left(\bigwedge_{\theta \in \Gamma} \theta\right)$$

and pick, if possible $V_\Delta \in \mathbb{V}^{\mathcal{L}}$ such that $V(\theta) = 1$ for all $\theta \in \Gamma$. Again by $(\mathcal{L}1)$ such a V_Δ does exist if $a_\Delta > 0$.

Then, by induction on the size, $|\Gamma|$, of Γ,

$$\begin{aligned} \sum_{\Delta \subseteq \Gamma, a_\Delta > 0} a_\Delta = \sum_{\Delta \subseteq \Gamma} a_\Delta = & \\ & \sum_{\emptyset \neq \Delta \subset \Gamma} \left\{ B\left(\bigwedge_{\theta \in \Delta} \theta\right) - B\left(\bigwedge_{\theta \in \Delta} \theta \wedge \bigvee_{\phi \in \Gamma - \Delta} \phi\right) \right\} \\ & +1 - B\left(\bigvee_{\phi \in \Gamma} \phi\right) + B\left(\bigwedge_{\theta \in \Gamma} \theta\right) \\ = \quad & 1 \end{aligned}$$

In more detail, this result is true for $|\Gamma| = 1$ whilst for $|\Gamma| > 1$, say $\Gamma = \Omega \cup \{\chi\}$ where $\chi \notin \Omega$, the term

$$\sum_{\emptyset \neq \Delta \subset \Gamma} \left\{ B\left(\bigwedge_{\theta \in \Delta} \theta\right) - B\left(\bigwedge_{\theta \in \Delta} \theta \wedge \bigvee_{\phi \in \Gamma - \Delta} \phi\right) \right\}$$

can, using $(\mathcal{L}2)$, be written as

$$B(\chi) - B\left(\chi \wedge \bigvee_{\phi \in \Omega} \phi\right) + B\left(\bigwedge_{\theta \in \Omega} \theta\right) - B\left(\bigwedge_{\theta \in \Omega} \theta \wedge \chi\right)$$

$$+ \sum_{\emptyset \neq \Delta \subset \Omega} \left\{ B\left(\chi \wedge \bigwedge_{\theta \in \Delta} \theta\right) - B\left(\chi \wedge \bigwedge_{\theta \in \Delta} \theta \wedge \bigvee_{\phi \in \Omega - \Delta} \phi\right) \right\}$$

$$+\sum_{\emptyset\neq\Delta\subset\Omega}\left\{B\left(\bigwedge_{\theta\in\Delta}\theta\right)-B\left(\bigwedge_{\theta\in\Delta}\theta\wedge(\chi\vee\bigvee_{\phi\in\Omega-\Delta}\phi)\right)\right\}.$$

Using ($\mathcal{L}2$) and ($\mathcal{L}3$) on this final term gives

$$B\left(\bigwedge_{\theta\in\Delta}\theta\wedge(\chi\vee\bigvee_{\phi\in\Omega-\Delta}\phi)\right)=$$
$$B\left(\bigwedge_{\theta\in\Delta}\theta\wedge\chi\right)+B\left(\bigwedge_{\theta\in\Delta}\theta\wedge\bigvee_{\phi\in\Omega-\Delta}\phi\right)$$
$$-B\left(\bigwedge_{\theta\in\Delta}\theta\wedge\chi\wedge\bigvee_{\phi\in\Omega-\Delta}\phi\right)$$

Making this substitution and cancelling terms gives the original expression with Ω in place of Γ and the result follows by induction.

Similarly, by induction on the size of Γ again, we have that for $\psi\in\Gamma$,

$$B(\psi)=B\left(\textstyle\bigwedge_{\theta\in\Gamma}\theta\right)$$
$$+\textstyle\sum_{\psi\in\Delta\subset\Gamma}\left\{B\left(\bigwedge_{\theta\in\Delta}\theta\right)-B\left(\bigwedge_{\theta\in\Delta}\theta\wedge\bigvee_{\phi\in\Gamma-\Delta}\phi\right)\right\}$$

But this righthandside equals

$$\sum_{\psi\in\Delta\subseteq\Gamma}a_\Delta=\sum_{\psi\in\Delta\subseteq\Gamma,\ a_\Delta>0}a_\Delta=\sum_{\Delta\subseteq\Gamma,\ a_\Delta>0}a_\Delta V_\Delta(\psi)$$

and hence, on Γ, B is, as required, the convex combination $\sum_{a_\Delta>0}a_\Delta V_\Delta$ of elements of $\mathbb{V}^{\mathcal{L}}$. ■

Theorem 5 applies to a number of well known propositional logics, for example the standard modal logics K, T, S_4, S_5 etc. and intuitionistic logic. It also applies to certain paraconsistent logics in which conjunction and disjunction retain their classical interpretation. To expand on this point, one common way to specify a paraconsistent logic (for example in [1]) is to specify a semantics based on a variation of the usual Tarskian interpretation of the connectives. So, for example, in Batens' *Negation Glut* logic $\mathcal{N}$ the set of valuations $\mathbb{V}^{\mathcal{N}}$ are those functions $V: SL\longrightarrow\{0,1\}$ satisfying (T2-4) and, in place of (T1),

(N1) $\quad V(\theta)=0\ \Rightarrow\ V(\neg\theta)=1,$

the resulting consequence relation being defined, as expected, by

$$\Gamma\models_{\mathcal{N}}\theta\quad\iff\quad\forall V\in\mathbb{V}^{\mathcal{N}},\text{ if }V(\phi)=1\text{ for all }\phi\in\Gamma\text{ then }V(\theta)=1.$$

In these semantics one might interpret the negation in $\neg\theta$ as asserting that that there were reasons to reject θ, thus allowing the possibility of taking $\neg\theta$ to be true even though one had already assigned θ truth value 1. Corollary 4 now says that according to these semantics (i.e. interpretation of the connectives) it is not possible to construct a Dutch Book against $B\in\mathbb{B}$ just if for every finite subset Γ of SL B restricted to Γ, $B\restriction\Gamma$, is a convex combination of functions in $\{V\restriction\Gamma|V\in\mathbb{V}^{\mathcal{N}}\}$, equivalently by Theorem 5[2] if

($\mathcal{N}1$) If $\models_{\mathcal{N}}\theta$ then $B(\theta)=1$,
($\mathcal{N}2$) If $\theta\models_{\mathcal{N}}\phi$ then $B(\theta)\leq B(\phi)$,
($\mathcal{N}3$) $B(\theta\vee\phi)+B(\theta\wedge\phi)=B(\theta)+B(\phi)$.

It is worth observing that the conditions (DS1) and (DS2) for a Dempster-Shafer belief function could be replaced, by analogy to ($\mathcal{L}1$), ($\mathcal{L}2$), with the conditions

(1) If $\models_{DS}\theta$ then $B(\theta)=1$,
if $\theta\models_{DS}$ then $B(\theta)=0$,
(2) If $\theta\models_{DS}\phi$ then $B(\theta)\leq B(\phi)$,

since, with the obvious meaning, $\models_{DS}$ equals classical logical consequence. Note however that the V_θ in $\mathbb{V}^D$ do not necessarily satisfy condition (T3). In turn then Jaffray's result hints that perhaps an analog of Theorem 5 might also be provable without assuming (T3).

4 Conclusion

In this short note we have pointed out that the 'Dutch Book Method' is applicable not just to classical propositional logic, where it yields probability as the only rational belief (as willingness to bet) but also to a range of non-standard logics with alternate notions of truth and possible worlds. For these it in turn produces a spectrum of 'rational' belief functions.

Appendix

Let $\vec{b}=<B(\theta_1),...,B(\theta_m)>$ and let $\vec{c}$ be the closest point in Y to $\vec{b}$. Since Y is closed convex there is a unique such point and since $\vec{b}\notin Y$, $\vec{b}\neq\vec{c}$. Let $\vec{e}=\frac{1}{2}(\vec{b}+\vec{c})$ and let H be the plane of vectors $\vec{x}$ such that

$$(\vec{x}-\vec{e})\cdot(\vec{b}-\vec{e})=0.$$

Then Y cannot intersect H, since if it did, at $\vec{a}$ say, then there would be a point on the line between $\vec{a}$ and $\vec{c}$ in Y and closer to $\vec{b}$ than $\vec{c}$. Hence Y must lie

[2] In these semantics there are no $\mathcal{N}$-contradictions.

entirely on the opposite side of H from $\vec{b}$ and

$$\vec{b} \cdot (\vec{e} - \vec{b}) < 0 < \vec{v} \cdot (\vec{e} - \vec{b})$$

for all $\vec{v} \in \mathbb{V}$, giving

$$(\vec{v} - \vec{b}) \cdot (\vec{b} - \vec{e}) < 0,$$

as required.

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