Computing the Reliability of Complex Systems

Lev V. Utkin
Department of Computer Science
Forest Technical Academy,
St.Petersburg, Russia
lvu@utkin.usr.etu.spb.ru

Igor O. Kozine
Risoe National Laboratory
Systems Analysis Department,
Roskilde, Denmark
igor.kozine@risoe.dk

Abstract

Methods for computing the reliability of complex systems described in the current paper are grounded on partial information on system components. A tool for inferring the intervals is the natural extension and the upper and lower bounds of the characteristics to be interpreted as coherent upper and lower previsions. A generic algorithm to find a solution of the natural extension in a practically affordable way braking down the general problem into problems that are much easier to solve is described. In general this can be made at the cost of a lesser precision in the previsions of interest. It is also shown that for some particular cases the genuine, minimally coherent, solutions can be found through the algorithm developed. The second part of the paper is devoted to those cases when the reliability of components constituting a system is represented by identical interval-valued reliability characteristics. That is, all the components are characterized, for example, by probabilities to failure in the same time interval, or by mean times to failure or some others. Often namely these particular cases take place in reliability analysis practice. In this respect, based on the previous works by the authors of the current paper some new findings have been disclosed and new results obtained on particular practical cases.

Keywords. Imprecise probability theory, imprecise reliability, natural extension, previsions.

1 Introduction

Methods for computing the reliability of complex systems described in the current paper are based on partial information for system components. Since the given information is partial, we do not expect precise numbers, but we would like to know an interval of possible values of a desired reliability characteristic. A tool for inferring the intervals is the natural extension and the upper and lower bounds of the characteristics to be interpreted as coherent upper and lower previsions in the sense of P. Walley [11] and V. Kuznetsov [6].

The reliability analysis of complex systems becomes a difficult problem in case the reliability information on the components is partial and the number of components in a system is large. Even if information about the independence of components is absent (this case is easier to handle in the framework of coherent imprecise previsions), the natural extension as a linear programming problem has a large dimensionality. Nevertheless in many practical cases source reliability data on the components is homogeneous (each component is quantified by the same reliability characteristic) and there is no need to seek for a solution of the natural extension in its general form. It is possible to simplify the problem and to find the solution easier, in some cases only approximate.

The general case of system reliability calculations based on coherent upper and lower previsions was developed by L. Utkin and described in [8, 10, 3]. A particular case, when component reliabilities are characterized by coherent imprecise probabilities was developed by I. Kozine and described in [4] and [5].

The current paper describes a generic algorithm to find a solution of the natural extension in a practically affordable way by breaking down the general problem into problems that are much easier to solve. In general, this can be made at the cost of a lesser precision in the previsions of interest. It is also shown that for some particular cases the genuine, minimally coherent, solutions can be found through the algorithm developed. The second part of the paper is devoted to those cases when the reliability of components constituting a system is represented by identical interval-valued reliability characteristics. That is, all the components are characterized, for example, by probabilities to failure in the same time interval, or by mean times to failure or some others. Often namely these particular cases take place in reliability analysis practice. In this respect, based on the previous works by the authors of the current paper some new findings have been disclosed and new results obtained for particular practical cases.

2 Formulation of the problem

Consider a system consisting of n components. Let $\varphi_{ij}(x_i)$ be a function of the i-th component lifetime x_i, $j = 1, ..., m_i$. Here m_i is the number of quantitative or qualitative judgements that are related to the i-th component. According to Barlow and Proschan [1], the system lifetime is uniquely determined by the component lifetimes. Denote $\mathbf{X} = (x_1, ..., x_n)$, then there exists a function $g(\mathbf{X})$ of the component lifetimes that characterizes a system reliability characteristic. The functions $\varphi_{ij}(x_i)$ and $g(\mathbf{X})$ can be regarded as gambles, where a gamble is a real-valued function on a possibility space whose value is uncertain [11, 6].

Suppose that partial reliability information on the components is represented as a set of lower and upper previsions $\underline{a}_{ij} = \underline{M}^{'}(\varphi_{ij}(x_i))$, $\overline{a}_{ij} = \overline{M}^{'}(\varphi_{ij}(x_i))$, $i = 1, ..., n$, $j = 1, ..., m_i$. The designation of the previsions with the prime notation indicates that the previsions should be considered initial and not necessarily coherent. If, for instance, the gamble $\varphi_{ij}(x_i) = x$, then $\underline{a}_{ij}$ is the lower bound of the mean time to failure of the i-th component; or if $\varphi_{ij}(x_i) = I_{[t,\infty)}(x)$, then $\underline{a}_{ij}$ is the lower probability of the failure occurrence within $[t, \infty)$ etc.

In order to compute the coherent lower and upper previsions $\overline{\mathbf{M}}(g)$ and $\underline{\mathbf{M}}(g)$ of interest characterizing the system reliability the natural extension can be used in the following form [3]:

$$\begin{aligned} \overline{\mathbf{M}}(g) &= \sup_{\mathcal{P}} \int_{\mathbf{R}_+^n} g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X}, \\ \underline{\mathbf{M}}(g) &= \inf_{\mathcal{P}} \int_{\mathbf{R}_+^n} g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X}, \end{aligned} \tag{1}$$

subject to

$$\left.\begin{array}{c} \rho(\mathbf{X}) \geq 0,\ \int_{\mathbf{R}_+^n} \rho(\mathbf{X})\mathrm{d}\mathbf{X} = 1 \\ \underline{M}^{'}(\varphi_{ij}(\mathbf{X})) \leq \int_{\mathbf{R}_+^n} \varphi_{ij}(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X} \\ \int_{\mathbf{R}_+^n} \varphi_{ij}(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X} \leq \overline{M}^{'}(\varphi_{ij}(\mathbf{X})) \\ j = 1, ..., m_i,\ i = 1, ..., n. \end{array}\right\} \tag{2}$$

Here the minimum and maximum are taken over the set $\mathcal{P}$ of all possible n-dimensional density functions $\{\rho(\mathbf{X})\}$ satisfying conditions (2). A solution of problem (1) exists if all the constraints (2) form a joint set $\mathcal{P}$. Otherwise, if the initial interval-valued data forming the constraints are not consistent, some of the subsets of $\mathcal{P}$ are disjoint and the solution does not exist. The introduction of the assumption of the existence of a convex set $\mathcal{P}$ is necessary to obtain a solution of the problem (1) subject to (2). This assumption is equivalent to the principle of avoiding sure loss [11].

It should be noted that we cannot expect the set of initial source interval-valued data to be consistent in a stronger way than the avoiding sure loss. We cannot require from the experts or analysts to generate coherent intervals, but interval-valued assessments that avoid sure loss. In order to have the initial set of data coherent, we have to use the natural extension in a form similar to (1)-(2) where the objective functions are $\underline{a}_{ij} = \underline{\mathbf{M}}(\varphi_{ij}(x_i))$ and $\overline{a}_{ij} = \overline{\mathbf{M}}(\varphi_{ij}(x_i))$ for some specific $j = k$, and the constraints do not contain these previsions. Solving these optimization problems would make all the previsions coherent. The natural extension provides us not only with a tool to obtain new previsions coherent to the set of initial judgements, but with a tool to make the initial judgements coherent.

To find the exact solution of the general (primal) optimization problem (1) subject to (2), if n is relatively large and $m_i > 1$, is a hardly affordable task (for our knowledge a general analytical solution of problem (1)-(2) cannot be found), yet numerical iterative algorithms can be developed to obtain approximate lower and upper previsions of interest. For some particular cases, for example, when each component in a system is characterised by the same set of reliability characteristics, exact solutions can be found and all those cases revealed by the authors are addressed in section 4.

3 Algorithm for computing approximate previsions

Let the number of possible gambles, i.e. functions g_k, be equal to N, where $N \geq m_i$ for any $i = 1, ..., n$. Then we offer the following algorithm to approximately compute any imprecise previsions of interest for an arbitrary system.

1. Take $k = 1$.
2. For each component, the coherent previsions $\underline{\mathbf{M}}(g_k(x_i))$ and $\overline{\mathbf{M}}(g_k(x_i))$ of the gamble g_k are computed from previsions $\overline{M}^{'}(\varphi_{ij}(x_i))$ and $\underline{M}^{'}(\varphi_{ij}(x_i))$, $j = 1, ..., m_i$. As a result we obtain the lower and upper previsions of the gamble g_k for each component

$$\begin{aligned} \underline{\mathbf{M}}(g_k(x_i)) &= \inf_{\mathcal{P}_i} \int_{\mathbf{R}_+} g_k(x_i)\rho(x_i)\mathrm{d}x_i, \\ \overline{\mathbf{M}}(g_k(x_i)) &= \sup_{\mathcal{P}_i} \int_{\mathbf{R}_+} g_k(x_i)\rho(x_i)\mathrm{d}x_i, \end{aligned}$$

subject to

$$\underline{M}^{'}(\varphi_{ij}(x_i)) \leq \int_{\mathbf{R}_+} \varphi_{ij}(x_i)\rho(x_i)\mathrm{d}x_i,$$

$$\int_{\mathbf{R}_+} \varphi_{ij}(x_i)\rho(x_i)\mathrm{d}x_i \leq \overline{M}^{'}(\varphi_{ij}(x_i)),$$

$$\rho(x_i) \geq 0, \int_{\mathbf{R}_+} \rho(x_i)\mathrm{d}x_i = 1, \ j = 1, ..., m_i.$$

3. Compute the coherent lower $\underline{\mathbf{M}}(g_k(z))$ and upper $\overline{\mathbf{M}}(g_k(z))$ previsions of the system for the same gamble g_k assuming that all components are characterized by the calculated previsions of the gamble g_k (previous Step 2). Here z is the lifetime of the system

$$\underline{\mathbf{M}}(g_k(z)) = \inf_{\mathcal{R}^{(k)}} \int_{\mathbf{R}_+^n} g_k(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X},$$

$$\overline{\mathbf{M}}(g_k(z)) = \sup_{\mathcal{R}^{(k)}} \int_{\mathbf{R}_+^n} g_k(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X},$$

subject to

$$\underline{\mathbf{M}}(g_k(x_i)) \leq \int_{\mathbf{R}_+^n} g_k(x_i)\rho(x_i)\mathrm{d}x_i \leq \overline{\mathbf{M}}(g_k(x_i)),$$

$$\rho(\mathbf{X}) \geq 0, \int_{\mathbf{R}_+^n} \rho(\mathbf{X})\mathrm{d}\mathbf{X} = 1, \ i = 1, ..., n.$$

4. Take $k = k + 1$. If $k \leq N$, then go to Step 2, otherwise - to Step 5.

5. As a result of the previous steps, we obtain a set of lower and upper previsions $\underline{\mathbf{M}}(g_k(z))$, $\overline{\mathbf{M}}(g_k(z))$, $k = 1, ..., N$, for the system. Now the set of these previsions acts as the constraints in the optimization problem (1), and the system previsions of interest $\underline{M}^*(g(z))$ and $\overline{M}^*(g(z))$ are computed. The asterisk in the notation is used to distinguish between the genuine solution of the problem and its approximation

$$\underline{M}^*(g(z)) = \inf_{\mathcal{R}} \int_{\mathbf{R}_+} g(z)\rho(z)\mathrm{d}z,$$

$$\overline{M}^*(g(z)) = \sup_{\mathcal{R}} \int_{\mathbf{R}_+} g(z)\rho(z)\mathrm{d}z,$$

subject to

$$\underline{\mathbf{M}}(g_k(z)) \leq \int_{\mathbf{R}_+} g_k(z)\rho(z)\mathrm{d}z \leq \overline{\mathbf{M}}(g_k(z)),$$

$$\rho(z) \geq 0, \int_{\mathbf{R}_+} \rho(z)\mathrm{d}z = 1, \ k = 1, ..., N.$$

The proposed algorithm is easy to fulfil through the use of the duality theorem and a proper program has been created by the authors to make the algorithm run. It is proven below that the previsions computed through the algorithm are coherent, and the proof is based on the fact that any wider interval $[\underline{M}^*(g), \overline{M}^*(g)]$ in comparison with the genuine coherent interval $[\underline{\mathbf{M}}(g), \overline{\mathbf{M}}(g)]$ avoids sure loss.

One more positive feature of the algorithm is worth mentioning. Step 3 implies the calculation of system reliability characteristics. In many cases there will not be a need to solve this optimization problem since a number of analytical solutions have been obtained by the authors for calculating interval-valued previsions of interest at system level. All the formulas revealed and allowing to avoid solving the general problem under Step 3 can be found in section 4 of the current paper.

Let us show that the proposed algorithm produces an interval of previsions which is wider then the minimally coherent interval or equal to it. In fact, the theorem below proves that any solution obtained with the offered algorithm is a coherent approximation which in some special cases can coincide with the exact solution.

Theorem 1 *If the lower and upper previsions $\underline{M}^*(g)$ and $\overline{M}^*(g)$ are computed according to the above algorithm, then*

$$[\underline{\mathbf{M}}(g), \overline{\mathbf{M}}(g)] \subseteq [\underline{M}^*(g), \overline{M}^*(g)],$$

where $\underline{M}(g)$ and $\overline{M}(g)$ are the genuine solutions of the problem (1) subject to (2).

Proof. Let $\mathcal{P}_{ij}$ be a set of all densities satisfying the j-th constraint for the i-th component and $\mathcal{P}_i = \bigcap_{j=1}^{m_i} \mathcal{P}_{ij}$ be a set of all densities satisfying all constraints for the i-th component, $i = 1, ..., n$. Then a set of all densities $\mathcal{P}$ satisfying all constraints for all components is determined as

$$\mathcal{P} = \mathcal{P}_1 \cap ... \cap \mathcal{P}_n = \bigcap_{i=1}^{n} \mathcal{P}_i = \bigcap_{i=1}^{n} \bigcap_{j=1}^{m_i} \mathcal{P}_{ij}.$$

Suppose that coherent previsions $\underline{\mathbf{M}}(g_k(x_i))$ and $\overline{\mathbf{M}}(g_k(x_i))$ for a particular k and for the i-th component produce constraints corresponding to a set of densities $\mathcal{R}_i^{(k)}$. ($\underline{\mathbf{M}}(g_k(x_i))$ and $\overline{\mathbf{M}}(g_k(x_i))$ are the outcome of Step 2 of the algorithm) It can be concluded that $\mathcal{P}_i \subseteq \mathcal{R}_i^{(k)}$. Indeed, since $\underline{\mathbf{M}}(g_k(x_i))$ and $\overline{\mathbf{M}}(g_k(x_i))$ were defined on the set $\mathcal{P}_i$, the inverse problem (defining a set $\mathcal{P}_i^{'}$ based on $\underline{\mathbf{M}}(g_k(x_i))$ and $\overline{\mathbf{M}}(g_k(x_i))$) cannot produce the set $\mathcal{P}_i^{'}$ smaller than $\mathcal{P}_i$, hence $\mathcal{P}_i \subseteq \mathcal{R}_i^{(k)}$. Then $\mathcal{R}_i^{(k)} = \mathcal{P}_i \cup \Delta\mathcal{R}_i^{(k)}$, where $\Delta\mathcal{R}_i^{(k)}$ is the complement of $\mathcal{P}_i$ to $\mathcal{R}_i^{(k)}$. By using the approximate algorithm, we obtain at the k-th stage a set of densities $\mathcal{R}^{(k)}$ for the system determined as

$$\mathcal{R}^{(k)} = \bigcap_{i=1}^{n} \mathcal{R}_i^{(k)} = \bigcap_{i=1}^{n} (\mathcal{P}_i \cup \Delta\mathcal{R}_i^{(k)})$$

$$= \left(\bigcap_{i=1}^{n} \mathcal{P}_i \right) \cup \Delta\mathcal{R}^{(k)} = \mathcal{P} \cup \Delta\mathcal{R}^{(k)},$$

where $\Delta\mathcal{R}^{(k)}$ is an additional set of densities which gives extension of the interval of previsions obtained by means of approximate algorithm. Then the final set of densities

for computing the lower and upper system previsions is

$$\mathcal{R} = \bigcap_{k=1}^{N} \mathcal{R}^{(k)} = \bigcap_{k=1}^{N} (\mathcal{P} \cup \Delta\mathcal{R}^{(k)})$$

$$= \mathcal{P} \cup \bigcap_{k=1}^{N} \Delta\mathcal{R}^{(k)} = \mathcal{P} \cup \Delta\mathcal{R},$$

where the set $\Delta\mathcal{R}$ determines the error of the approximate algorithm.

It follows from the last expression that $\mathcal{P} \subseteq \mathcal{R}$ and $[\underline{\mathbf{M}}(g), \overline{\mathbf{M}}(g)] \subseteq [\underline{M}^*(g), \overline{M}^*(g)]$. ■

If the system components are independent, then the proof of the theorem is similar under the condition that all the sets considered in the above proof consist of densities represented as the product of marginal densities.

It is possible that for some k $\Delta\mathcal{R}^{(k)} = \emptyset$ or for some $k \neq l$ $\Delta\mathcal{R}^{(k)} \cap \Delta\mathcal{R}^{(l)} = \emptyset$, then $\Delta\mathcal{R} = \emptyset$ meaning that the solution produced by the algorithm is coherent, i.e. coinciding with the solution of the general problem (1) - (2). The larger number of different gambles $g_k(x)$ is involved, the greater chance that $\Delta\mathcal{R}$ is empty and the solution is coherent. As a matter of fact, this is quite likely to obtain minimally coherent previsions with the algorithm offered in some practical cases. The theorem below demonstrates one case where the exact solution of the natural extension can be obtained. Informally, the theorem states if a system reliability characteristic which is going to be calculated is the same as one of the characteristics quantifying a component reliability in the system, then one obtains exact (not approximate) solution of the natural extension. For example, if one is interested to know the mean time to failure of a system and at least for one component the mean time to failure (precise or imprecise) is known, then the solution found through the algorithm is the genuine solution of problem (1) - (2). Formally, this is posed as follows:

Theorem 2 *Suppose that n components are characterized by the lower and upper previsions of a set of gambles*

$$\Phi = \{\varphi_{ij}(x), j = 1, ..., m_i, i = 1, ..., n\}.$$

Here m_i is the number of judgements about the ith component. Let Ψ be a subset of the set Φ such that all gambles belonging to Ψ are different. Denote L the cardinality of Ψ, i.e. the number of its components. Then by taking $N = L$ and $g_k(x) = \varphi_{ij}(x) \in \Psi$ at the k-th stage of the algorithm, the solution of problem (1) - (2) coincides with the solution obtained by the algorithm.

Proof. In order to prove the theorem it is enough to prove that $\Delta\mathcal{R} = \emptyset$. Let us consider the jth initial lower and upper prevision of the ith component $[\underline{M}^{'}(\varphi_{ij}(x_i)), \overline{M}^{'}(\varphi_{ij}(x_i))]$, where gamble $\varphi_{ij}(x_i) \in \Psi$. As it is assumed by the algorithm, these previsions must avoid sure loss and be not necessarily coherent. In applying the natural extension to these previsions one obtains the coherent previsions $\underline{\mathbf{M}}(g_k(x))$ and $\overline{\mathbf{M}}(g_k(x))$ for which it is always valid that

$$\underline{\mathbf{M}}(g_k(x_i)) \geq \underline{M}^{'}(\varphi_{ij}(x_i)),\ \overline{\mathbf{M}}(g_k(x_i)) \leq \overline{M}^{'}(\varphi_{ij}(x_i)),$$

and $\mathcal{R}_i^{(k)} \subseteq \mathcal{P}_{ij}$. Hence

$$\bigcap_{j=1}^{m_i} \mathcal{R}_i^{(j)} \subseteq \bigcap_{j=1}^{m_i} \mathcal{P}_{ij}.$$

Since all gambles corresponding to the i-th component belong to the set Ψ, then it follows from the last expression that

$$\bigcap_{k=1}^{N} \mathcal{R}_i^{(k)} \subseteq \bigcap_{j=1}^{m_i} \mathcal{R}_i^{(j)} \subseteq \bigcap_{j=1}^{m_i} \mathcal{P}_{ij}.$$

This implies that

$$\mathcal{R} = \bigcap_{k=1}^{N} \mathcal{R}^{(k)} = \bigcap_{k=1}^{N} \bigcap_{i=1}^{n} \mathcal{R}_i^{(k)}$$

$$= \bigcap_{i=1}^{n} \bigcap_{k=1}^{N} \mathcal{R}_i^{(k)} \subseteq \bigcap_{i=1}^{n} \bigcap_{j=1}^{m_i} \mathcal{P}_{ij} = \mathcal{P}.$$

Since $\mathcal{P} \subseteq \mathcal{R}$ (see the proof of Theorem 1), then $\mathcal{P} = \mathcal{R}$ and $\Delta\mathcal{R} = \emptyset$. ■

4 Systems with identical gambles

4.1 General statements

This section addresses the reliability of a system consisting of components whose quantification is restricted by the same reliability characteristics. For example, all components constituting the system are characterized by probabilities to failure within the same time interval, or by mean times to failure etc. The problem of computing the lower and upper previsions of the system can be stated as follows.

Let a system consist of n components, $f(x_i)$ be a function of the i-th component lifetime x_i, and $f(z) = f(g(\mathbf{X}))$ be a function of the system lifetime z. Suppose that partial information is represented as a set of the lower and upper previsions $\underline{a}_i = \underline{M}^{'}(f(x_i))$, $\overline{a}_i = \overline{M}^{'}(f(x_i))$, $i = 1, ..., n$. We seek for the lower and upper previsions $\underline{\mathbf{M}}(f(g(\mathbf{X})))$ and $\overline{\mathbf{M}}(f(g(\mathbf{X})))$ of the system.

Let us introduce some assumptions:

1. All available previsions avoid sure loss, in the sense that all constraints define a non-empty set of probability densities $\mathcal{P} = \bigcap_{i=1}^{n} \mathcal{P}_i \neq \emptyset$, and are not necessarily coherent.

2. The gambles x_i are restricted by the interval $[0, T]$, that is $\inf x_i = 0$ and $\sup x_i = T$. Then

$$\mathbf{X} \in D = \prod_{i=1}^{n} [0, T] \subset \mathbf{R}^n.$$

In particular, we can assume that $T \to \infty$.

3. The system structure is monotonic. (The structure function $g(x_1, ..., x_n) = g(\mathbf{X})$ is monotonic if g is increasing in each argument [1]) The system has p minimal paths $P_1, ..., P_p$ containing $m_1, ..., m_p$ components, respectively, and k minimal cut sets $K_1, ..., K_k$. (A minimal cut set is defined as the set of minimum number of components that, when failed, guarantees the failure of the system [7]. A minimal path set is defined as the set of minimum number of components needed to guarantee the success state of the system.). Its lifetime $g(\mathbf{X})$ is given by

$$g(\mathbf{X}) = \max_{1 \le j \le p} \min_{i \in P_j} x_i = \min_{1 \le j \le k} \max_{i \in K_j} x_i.$$

In practice one can encounter three cases of the state of knowledge on components independence: (1) the components are independent, (2) dependent, and (3) indeterminacy whether the components are independent or not.

The first case dominates in reliability applications. The second case is difficult to implement as data on the degree of dependence is usually absent and difficult to obtain. The third one cannot be handled in the framework of the conventional reliability theory. We will confine ourselves by generalizing to imprecise previsions the first and third case. It is quite clear, the fewer constraints are imposed, the more imprecise previsions of interest are. The judgement of independence will allow obtaining more precise previsions. It is regarded as a strong structural judgement. Omitting this judgement assumes the absence of strong structural constraints which in some cases can be considered a more credible model of the state of affairs. (It should be noted that a weak structural assumption on the components independence does implicitly always exist and it is referred to as logical independence [2] and [5])

Let us consider first the case of complete ignorance whether the components are independent or not. This case is easier to handle with the coherent imprecise previsions.

4.2 Absence of information about components independence

The natural extension in its primal form (1)-(2) is difficult to solve. The dual representation of the natural extension is often more practical for inferring desirable analytical expressions.

It has been proven [6, 3] that in the absence of information about independence of components the natural extension in its dual form can be written as follows:

$$\begin{aligned} \overline{\mathbf{M}}(f) &= \inf_{c, c_i, d_i} \left(c + \sum_{i=1}^{n} (c_i \overline{a}_i - d_i \underline{a}_i) \right), \\ \underline{\mathbf{M}}(f) &= -\overline{\mathbf{M}}(-f), \end{aligned}$$

subject to $c_i \in \mathbf{R}^+$, $d_i \in \mathbf{R}^+$, $c \in \mathbf{R}$, and

$$f(g(\mathbf{X})) \le c + \sum_{i=1}^{n} (c_i - d_i) f(x_i), \ \forall \mathbf{X} \in D.$$

Since the function $g(\mathbf{X})$ is not decreasing, then for the monotone non-decreasing function $f(x_i)$, there holds

$$f(g(\mathbf{X})) = g(f(x_1), ..., f(x_n)). \tag{3}$$

Indeed, by using the representation of the function $g(X)$ through minimal cuts or minimal paths, we can write

$$\begin{aligned} g(f(x_1), ..., f(x_n)) &= \max_{1 \le j \le p} \min_{i \in P_j} f(x_i) \\ &= f\left(\max_{1 \le j \le p} \min_{i \in P_j} x_i \right) = f(g(\mathbf{X})). \end{aligned}$$

Denote $z_i = f(x_i)$. Then the above optimization problem is

$$\begin{aligned} \overline{\mathbf{M}}(g) &= \inf_{c, c_i, d_i} \left(c + \sum_{i=1}^{n} (c_i \overline{a}_i - d_i \underline{a}_i) \right), \\ \underline{\mathbf{M}}(g) &= -\overline{\mathbf{M}}(-g), \end{aligned}$$

subject to $c_i \in \mathbf{R}^+$, $d_i \in \mathbf{R}^+$, $c \in \mathbf{R}$, and $\forall z_i \in [\inf f(x_i), \sup f(x_i)]$,

$$g(z_1, ..., z_n) \le c + \sum_{i=1}^{n} (c_i - d_i) z_i,$$

Thus we have the problem which was thoroughly studied in [5, 8, 9, 10]. In particular, for a system with **components connected in series** the lower and upper previsions of are

$$\underline{\mathbf{M}}(g) = \max \left(0, \sum_{i=1}^{n} \underline{a}_i - (n-1) \sup f(x_i) \right),$$

$$\overline{\mathbf{M}}(g) = \min_{i=1,...,n} \overline{a}_i.$$

For **components connected in parallel**

$$\underline{\mathbf{M}}(g) = \max_{i=1,...,n} \underline{a}_i,$$

$$\overline{\mathbf{M}}(g) = \min \left(\sum_{i=1}^{n} \overline{a}_i, \sup f(x_i) \right).$$

For a **system of an arbitrary structure**

$$\begin{aligned}\underline{\mathbf{M}}(g) &\geq \max_{1\leq j\leq p}\max\left(0, L_j\right),\\ L_j &= \sum_{i\in P_j}\underline{a}_i-(m_j-1)\sup f(x_i)\end{aligned}$$

$$\overline{\mathbf{M}}(g)\leq \min_{1\leq j\leq k}\min\left(\sum_{i\in K_j}\overline{a}_i, \sup f(x_i)\right).$$

Example 1 *Consider a parallel system consisting of* 3 *components. The lower and upper mean times to failure of the components are known:* $\underline{a}_1=10$, $\overline{a}_1=12$, $\underline{a}_2=8$, $\overline{a}_2=14$, $\underline{a}_3=12$, $\overline{a}_3=13$. *Here* $f(x_i)=x_i$ *and* $g(x_1,x_2,x_3)=\max(x_1,x_2,x_3)$, $\sup f(x_i)\to\infty$. *Then the system lower and upper mean times to failure are*

$$\underline{\mathbf{M}}(g)=\max(\underline{a}_1,\underline{a}_2,\underline{a}_3)=\max(10,8,12)=12,$$

$$\overline{\mathbf{M}}(g)=\overline{a}_1+\overline{a}_2+\overline{a}_3=12+14+13=39.$$

Example 2 *Consider a series system consisting of* 2 *components. The lower and upper operating probabilities of the components within the time interval* $[0,6]$ *are:* $\underline{a}_1=0.3$, $\overline{a}_1=0.4$, $\underline{a}_2=0.8$, $\overline{a}_2=0.9$. *Here* $f(x_i)=I_{[6,\infty]}(x_i)$ *is the indicator function taking the value of* 1 *for* $x_i\geq 6$ *and* 0 *for* $x_i<6$, $g(x_1,x_2)=\min(x_1,x_2)$, $\sup f(x_i)=1$. *Then the system lower and upper operating probabilities in the time interval* $[0,6]$ *are*

$$\begin{aligned}\underline{\mathbf{M}}(g)&=\max\left(0,\underline{a}_1+\underline{a}_2-1\right)\\ &=\max\left(0,0.3+0.8-1\right)=0.1,\end{aligned}$$

$$\overline{\mathbf{M}}(g)=\min(\overline{a}_1,\overline{a}_2)=\min(0.4,0.9)=0.4.$$

4.3 Independent components

Below we keep the same notation of $z_i=f(x_i)$.From (1)-(2) and the condition of the components independence the natural extension for the same gamble as the components' is of the form:

$$\underline{\mathbf{M}}(g)=\inf_{\mathcal{P}}G,\ \overline{\mathbf{M}}(g)=\sup_{\mathcal{P}}G,$$

$$G=\int_0^T\cdot\cdot\int_0^T g(z_1,...,z_n)\rho_1(z_1)\cdot\cdot\rho_n(z_n)\mathrm{d}z_n\cdot\cdot\mathrm{d}z_n,$$

subject to

$$\begin{aligned}\rho_i(z) &\geq 0,\ \int_{\mathbf{R}_+^n}\rho_i(z)\mathrm{d}z=1,\\ \underline{a}_i &\leq \int_{\mathbf{R}_+^n} z\rho_i(z)\mathrm{d}z\leq\overline{a}_i,\ i=1,...,n.\end{aligned}$$

It is well known that in the case of independent components in a system, there is a function h linking the system reliability $H(t)$ in interval $[0,t]$ and component reliabilities $H_i(t)$ in the same interval. For example, for series systems there holds $H(t)=h(H_1(t),...,H_n(t))=\prod_{i=1}^n H_i(t)$, for parallel systems $H(t)=h(H_1(t),...,H_n(t))=1-\prod_{i=1}^n(1-H_i(t))$. This implies that the above optimization problem can be rewritten

$$\begin{aligned}\underline{\mathbf{M}}(g) &= \inf_{\mathcal{P}}\int_0^T(f(t))'h(H_1(t),...,H_n(t))\mathrm{d}t,\\ \overline{\mathbf{M}}(g) &= \sup_{\mathcal{P}}\int_0^T(f(t))'h(H_1(t),...,H_n(t))\mathrm{d}t,\end{aligned}$$

subject to

$$\underline{a}_i\leq\int_0^T(f(t))'H_i(t)\mathrm{d}t\leq\overline{a}_i,\ i=1,...,n.$$

Here the minimum and maximum are taken over the set $\mathcal{P}$ of all possible distribution functions $H_i(t)$ satisfying the constraints of the problem. Basing on this representation of the natural extension the following analytical expressions have been inferred [3].

System with components in series.

If we know the lower $\underline{a}_i$ and upper $\overline{a}_i$ mean times to failure of the components, then the lower and upper mean times to failure of a series system are computed as follows:

$$\underline{\mathbf{M}}(g)=\frac{1}{T^{n-1}}\prod_{i=1}^n\underline{a}_i,\ \overline{\mathbf{M}}(g)=\min_{i=1,...,n}\overline{a}_i.$$

Similarly, if the lower $\underline{a}_i$ and upper $\overline{a}_i$ m-th moments of time to failure of the components are known, then the coherent imprecise m-th moments of time to failure of a series system are computed as follows:

$$\underline{\mathbf{M}}(g)=\frac{1}{(T^{n-1})^m}\prod_{i=1}^n\underline{a}_i,\ \overline{\mathbf{M}}(g)=\min_{i=1,...,n}\overline{a}_i.$$

If we know the lower $\underline{a}_i$ and upper $\overline{a}_i$ operating probabilities before time t of components, then the lower and upper operating probabilities before time t of a series system are:

$$\underline{\mathbf{M}}(g)=\prod_{i=1}^n\underline{a}_i,\ \overline{\mathbf{M}}(g)=\prod_{i=1}^n\overline{a}_i.$$

System with components in parallel.

If we know the lower $\underline{a}_i$ and upper $\overline{a}_i$ mean times to failure of components, then the lower and upper mean times to failure of a parallel system are computed as follows:

$$\underline{\mathbf{M}}(g)=\max_{i=1,...,n}\underline{a}_i,\ \overline{\mathbf{M}}(g)=T-T\prod_{i=1}^n\left(1-\frac{\overline{a}_i}{T}\right).$$

Similarly, if we know the lower $\underline{a}_i$ and upper $\overline{a}_i$ m-th moments of time to failure of components, then the lower and upper m-th moments of time to failure of a parallel system are computed as follows:

$$\underline{\mathbf{M}}(g) = \max_{i=1,\dots,n} \underline{a}_i,\ \overline{\mathbf{M}}(g) = T^m - T^m \prod_{i=1}^{n} \left(1 - \frac{\overline{a}_i}{T^m}\right).$$

If we know the lower $\underline{a}_i$ and upper $\overline{a}_i$ operating probabilities before time t of components, then the lower and upper operating probabilities before time t of a parallel system are computed as follows:

$$\underline{\mathbf{M}}(g) = 1 - \prod_{i=1}^{n} (1 - \underline{a}_i)\,,\ \overline{\mathbf{M}}(g) = 1 - \prod_{i=1}^{n} (1 - \overline{a}_i)\,.$$

Example 3 *Let us consider a system consisting of* 2 *independent components connected in series. The information about the reliability of the components is the following:*

1. *First component: lower and upper mean times to failure* $\underline{a}_{11} = \underline{M}^{'}(x_1) = 13$, $\overline{a}_{11} = \overline{M}^{'}(x_1) = 14$, *lower and upper second moments* $\underline{a}_{12} = \underline{M}^{'}(x_1^2) = 160$, $\overline{a}_{12} = \overline{M}^{'}(x_1^2) = 170$, *lower and upper probabilities of failure after time* 14 *hours* $\underline{a}_{13} = \underline{M}^{'}(I_{[14,\infty]}(x_1)) = 0.3$, $\overline{a}_{13} = \overline{M}^{'}(I_{[14,\infty]}(x_1)) = 0.4$.
2. *Second component: lower and upper second moments* $\underline{a}_{21} = \underline{M}^{'}(x_2^2) = 120$, $\overline{a}_{21} = \overline{M}^{'}(x_2^2) = 130$, *lower and upper probabilities of failure after time* 6 *hours* $\underline{a}_{22} = \underline{M}^{'}(I_{[12,\infty]}(x_2)) = 0.8$, $\overline{a}_{22} = \overline{M}^{'}(I_{[12,\infty]}(x_2)) = 0.9$.

How to find the lower and upper probabilities of the system failure after time 13 *hours? This problem is difficult enough to be solved directly through the use of either primal general optimization problem in the form (1) subject to (2) or its dual representation. Therefore, the proposed algorithm can be used.*

Stage k=1. Let $g_1(x) = x$ *(mean time to failure). Then*

1. *first component:* $\underline{\mathbf{M}}(g_1) = 13$, $\overline{\mathbf{M}}(g_1) = 13.02$.
2. *second component:* $\underline{\mathbf{M}}(g_1) = 9.6$, $\overline{\mathbf{M}}(g_1) = 11.3$.
3. *system:* $\underline{\mathbf{M}}(g_1) = 0$, $\overline{\mathbf{M}}(g_1) = 11.3$.

Stage k=2. Let $g_2(x) = x^2$ *(second moment). Then*

1. *first component:* $\underline{\mathbf{M}}(g_2) = 169$, $\overline{\mathbf{M}}(g_2) = 170$.
2. *second component:* $\underline{\mathbf{M}}(g_2) = 120$, $\overline{\mathbf{M}}(g_2) = 130$.
3. *system:* $\underline{\mathbf{M}}(g_2) = 0$, $\overline{\mathbf{M}}(g_2) = 130$.

Stage k=3. Let $g_3(x) = I_{[12,\infty]}(x)$ *(probability). Then*

1. *first component:* $\underline{\mathbf{M}}(g_3) = 0$, $\overline{\mathbf{M}}(g_3) = 0.4$.
2. *second component:* $\underline{\mathbf{M}}(g_3) = 0.8$, $\overline{\mathbf{M}}(g_3) = 0.9$.
3. *system:* $\underline{\mathbf{M}}(g_3) = 0$, $\overline{\mathbf{M}}(g_3) = 0.36$.

Stage k=4. Let $g_4(x) = I_{[14,\infty]}(x)$ *(probability). Then*

1. *first component:* $\underline{\mathbf{M}}(g_4) = 0.3$, $\overline{\mathbf{M}}(g_4) = 0.4$.
2. *second component:* $\underline{\mathbf{M}}(g_4) = 0$, $\overline{\mathbf{M}}(g_4) = 0.28$.
3. *system:* $\underline{\mathbf{M}}(g_4) = 0$, $\overline{\mathbf{M}}(g_4) = 0.112$.

Thus, we have got the system lower and upper previsions for the four different gambles. In the following stage these previsions at system level will act as initial data to calculate the reliability characteristic of interest: the probability of failure in the time interval $[13, \infty)$.

Stage k=5. Let us break this stage into steps in order to see how the number of previsions effect the precision of the characteristic to be calculated. If only g_1 *is used for computing the system reliability measure, then* $\underline{P}(13 \le z < \infty) = 0$ *and* $\overline{P}(13 \le z < \infty) = 0.87$, *where* z *is the system lifetime. If we use* g_1, g_2, *then* $\underline{P} = 0$ *and* $\overline{P} = 0.77$. *If we use* g_1, g_2, g_3, *then* $\underline{P} = 0$ *and* $\overline{P} = 0.36$. *If we use* g_1, g_2, g_3, g_4, *then* $\underline{P} = 0$ *and* $\overline{P} = 0.03$. *Therefore, the final results for the series system are* $\underline{P}(13 \le z < \infty) = 0$ *and* $\overline{P}(13 \le z < \infty) = 0.03$.

Example 4 *Let us consider a parallel system consisting of the same independent components as the system in Example 3. Then* $\underline{\mathbf{M}}(g_1) = 13$, $\overline{\mathbf{M}}(g_1) = 24.32$, $\underline{\mathbf{M}}(g_2) = 169$, $\overline{\mathbf{M}}(g_2) = 300$, $\underline{\mathbf{M}}(g_3) = 0.8$, $\overline{\mathbf{M}}(g_3) = 0.94$, $\underline{\mathbf{M}}(g_4) = 0.3$, $\overline{\mathbf{M}}(g_4) = 0.568$. *The final results for the parallel system: the lower probability is* 0.3 *and the upper probability is* 0.94.

5 Concluding remarks

The current paper summarizes the authors developments on the reliability of non-repairable systems the components of which are quantified by interval-valued characteristics.

The generalization of system reliability calculations to interval-valued characteristics has distinguished features that cannot be achieved in the framework of conventional reliability theory. Among those are

1. A possibility to calculate a system reliability characteristic of interest based on arbitrary sets of components reliability characteristics. The only condition imposed is the avoiding sure loss among the reliability information available on the components or groups of components.

2. The unnecessity to assume any probability distribution of time to failure.

3. Easy interval calculation of "typical" system reliability characteristics in case all components are quantified by the same characteristics.

4. The validity of all the formulas and the algorithm developed for precise reliability parameters. This fact allows us to refer to the work done as the generalization of system reliability calculations to imprecise previsions.

Despite all the advantages pointed out above, we profess that they are achieved at the cost of imprecision which rapidly increases as the number of components in a system grows. It is clear, the lesser assumptions/constraints are introduced, the lesser precision of a system reliability characteristic is. For example, the absence of the judgement on components independence inevitably results in a larger imprecision. An open question is: whether the imprecision one yields is able to make the results practical in consequent decision-making? Nevertheless, an obvious conclusion is: one will yield higher confidence to the quantified reliability of complex systems.

References

[1] R.E. Barlow and F. Proschan. *Statistical Theory of Reliability and Life Testing: Probability Models*. Holt, Rinehart and Winston, New York, 1975.

[2] B. de Finetti. *Theory of Probability*, volume 1. Wiley, London, 1974. English Translation of *Teoria delle Probabilità*.

[3] S.V. Gurov and L.V. Utkin. *Reliability of Systems under Incomplete Information*. Lubavich Publ., Saint Petersburg, 1999. in Russian.

[4] I. Kozine. Imprecise probabilities relating to prior reliability assessments. In S. Moral G. de Cooman, F.G. Cozman and P. Walley, editors, *ISIPTA '99 - Proceedings of the First International Symposium on Imprecise Probabilities and Their Applications*, pages 241–248, Zwijnaarde, Belgium, 1999.

[5] I. Kozine and Y. Filimonov. Imprecise reliabilities: experiences and advances. *Reliability Engineering and System Safety*, 67:75–83, 2000.

[6] V. P. Kuznetsov. *Interval Statistical Models*. Radio and Communication, Moscow, 1991. in Russian.

[7] S.S. Rao. *Reliability-Based Design*. McGraw-Hill, New York, 1992.

[8] L.V. Utkin. General reliability theory on the basis of upper and lower previsions. In D. Ruan, H. Ait Abderrahim, P. D'hondt, and E.E. Kerre, editors, *Fuzzy Logic and Intelligent Technologies for Nuclear Science and Industry. Proceedings of the 3rd International FLINS Workshop*, pages 36–43, Antwerp, Belgium, September 1998.

[9] L.V. Utkin and S.V. Gurov. New reliability models on the basis of the theory of imprecise probabilities. In *IIZUKA'98 - The 5th International Conference on Soft Computing and Information / Intelligent Systems*, volume 2, pages 656–659, Iizuka, Japan, October 1998.

[10] L.V. Utkin and S.V. Gurov. Imprecise reliability of general structures. *Knowledge and Information Systems*, 1(4):459–480, 1999.

[11] P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.