Dempster's Rule of Combination in Modal Logic

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Abstract

A modal logic interpretation of Dempster's rule of combination is developed. It is shown that the model of modal logic, obtained by combining two finite models of modal logic, induces plausibility and belief measures, which are, in fact, the orthogonal sums of the measures, corresponding to the original models.

Keywords: belief measure, Dempster's rule of combination, modal logic, multivalued mapping, orthogonal sum, plausibility measure.

1 Introduction

Modal logic is an extension of classical propositional logic that provides two additional modalities, representing the meaning of possibility and necessity. A model of modal logic is a triplet consisting of a set of possible worlds, a binary relation on this set of worlds, called accessibility relation, and a value assignment function, which assigns truth values to each atomic proposition in each possible world.

The development of Dempster–Shafer theory was initiated by Dempster in 1967. He showed that any multivalued mapping propagates a probability measure defined over subsets of one universe into a system of upper and lower probabilities over subsets of another universe [4]. Shafer reinterpreted these upper and lower probabilities by identifying them as plausibility and belief measures [17]. In [22], we have reworked the modal logic interpretations of Dempster–Shafer theory, established in [5, 6], by employing notions and concepts from multivalued analysis. The latter work is closely related to that of Ruspini and Pearl [15, 11, 16]. Pearl [11] has presented the "probability of provability" interpretation of Dempster-Shafer theory whereas Ruspini [15] has studied a similar problem by using the modal approach. Ruspini's considerations are based on a form of epistemic logic, but Harmanec *et al.* use a more general system of modal logic and also address the completeness of the interpretation. Smets [18, 19], has also presented the probability of provability interpretation of Dempster–Shafer theory and has shown how the conditioning rules can be derived. Many other interpretations of the mathematical theory of evidence have been proposed. Kohlas and Monney [8, 9], have designed hints as a model of Dempster's orginal approach to evidence theory. Also, Dempster's rule of combination has been defined in its most general form for combining arbitrary hints, not only independent ones. A quite different approach to evidence theory has been proposed by Provan [12], based on former work of Laskey and Lehner [10], which noted a relation between evidence theory and the well-known technigue of assumption-based truth maintenance systems (ATMS). In [20], the Transferable Belief Model has been described as a model for representing quantified beliefs based on belief functions.

Many existing modal logic interpretations of evidence theory, [5, 6, 7, 13, 14, 22], have been developed within the framework of models of modal logic in which in any possible world one and only one atomic proposition is true. We have proposed in [23], a modal logic interpretation of evidence theory in a very general setting, without imposing any restrictions on the accessibility relation and on the value assignment function; i.e. within the framework of multivalued models in which in any possible world an arbitrary number (possibly zero) of atomic propositions can be true.

In [4], Dempster also solved the problem of combining several independent sources of information using a multivalued approch. In this work, we develop a modal logic interpretation of Dempster's rule of combination within the framework of multivalued models of modal logic, again following Dempster's approach. We build a model of modal logic combination of two given finite models, such that the multivalued mapping associated with it is exactly the one used in [4].

2 Dempster–Shafer theory and multivalued mappings

2.1 Multivalued mappings

Let us briefly recall some basic concepts from the theory of multivalued mappings [1, 2]. A *multivalued mapping* F from a universe X into a universe Y associates to each element x of X a subset $F(x)$ of Y. The *domain* of F, denoted $\mathrm{dom}(F)$, and the range of F, denoted $\mathrm{rng}(F)$, are defined as

$$\mathrm{dom}(F) = \{x \mid x \in X \ \wedge \ F(x) \neq \emptyset\}$$
$$\mathrm{rng}(F) = \bigcup_{x \in X} F(x).$$

F is called *non-void* if $\mathrm{dom}(F) = X$. The *inverse* of F is the multivalued mapping F^- from Y into X that associates to each y of Y the subset $F^-(y) = \{x \mid x \in X \ \wedge \ y \in F(x)\}$ of X, i.e. $y \in F(x)$ is equivalent to $x \in F^-(y)$. The *direct image* of a subset A of X under F is defined as:

$$F(A) = \bigcup_{x \in A} F(x). \tag{1}$$

Given a subset B of Y, *the inverse and superinverse images* of B under F can be considered:

(i) the inverse image:

$$F^-(B) = \{x \mid x \in X \ \wedge \ F(x) \cap B \neq \emptyset\};$$

(ii) the superinverse image:

$$F^+(B) = \{x \mid x \in \mathrm{dom}(F) \ \wedge \ F(x) \subseteq B\}.$$

One easily verifies that $F^-(\emptyset) = F^+(\emptyset) = \emptyset$, $F^-(Y) = F^+(Y) = \mathrm{dom}(F)$. Notice that the inverse image of a subset A of X under F^- is the same as the direct image of A under F, i.e. $(F^-)^-(A) = F(A)$. It is clear that the inclusion $F^+(B) \subseteq F^-(B)$ always holds.

Let us consider a second multivalued mapping G from a universe Y into a universe Z. The *composition* of F and G is the multivalued mapping $G \circ F$ from X into Z defined by, for any $x \in X$:

$$(G \circ F)(x) = G(F(x)). \tag{2}$$

Now, let G is a multivalued mapping from a universe X into a universe Z. The *cartesian product* of F and G is the multivalued mapping $F \times G$ from X into $Y \times Z$ defined by, for any $x \in X$:

$$(F \times G)(x) = F(x) \times G(x). \tag{3}$$

For the inverse mapping of the cartesian product $F \times G$ the following property holds, for any $(y, z) \in Y \times Z$ [21]:

$$(F \times G)^-(y, z) = F^-(y) \cap G^-(z). \tag{4}$$

2.2 Plausibility and belief measures as upper and lower probabilities

Considering a probability measure P on $\mathcal{P}(X)$, Dempster showed that a multivalued mapping Γ from X into Y such that $P(\mathrm{dom}(\Gamma)) > 0$ induces *upper and lower probabilities* on $\mathcal{P}(Y)$ [4]. These upper and lower probabilities can be interpreted as conditional probabilities of inverse and superinverse images given the domain of the multivalued mapping, as follows:

$$\begin{aligned} P^*(A) &= P(\Gamma^-(A) \mid \mathrm{dom}(\Gamma)) \\ P_*(A) &= P(\Gamma^+(A) \mid \mathrm{dom}(\Gamma)). \end{aligned} \tag{5}$$

In his essay on a mathematical theory of evidence [17], Shafer reinterpreted the upper and lower probabilities as degrees of plausibility and belief, emphasizing the fact that they exhibit, respectively, subadditivity and superadditivity properties, in contrast to the additivity of probability. Plausibility and belief measures come in dual pairs. For any plausibility measure Pl on $\mathcal{P}(X)$, the $\mathcal{P}(X) \to [0,1]$ mapping Bel defined by $\mathrm{Bel}(A) = 1 - \mathrm{Pl}(\mathrm{co}A)$ is a belief measure on $\mathcal{P}(X)$, called the *dual* measure of Pl.

Furthermore, in the case of a finite universe X, Shafer also showed that any plausibility measure and its dual belief measure can be expressed equivalently in terms of their basic probability assignment [17]:

$$\mathrm{Pl}(A) = \sum_{C \cap A \neq \emptyset} m(C)$$
$$\mathrm{Bel}(A) = \sum_{C \subseteq A} m(C).$$

Formally, a $\mathcal{P}(X) \to [0,1]$ mapping m is called a *basic probability assignment* on $\mathcal{P}(X)$ if $m(\emptyset) = 0$ and

$$\sum_{A \in \mathcal{P}(X)} m(A) = 1.$$

A subset F of X for which $m(F) > 0$ is called a *focal element* of m.

3 Dempster's rule of combination

In [17], Shafer asserted that Dempster's rule of combination is most accessible to intuition when it is expressed in terms of the basic probability assignment. He considered a finite universe Y, belief functions Bel_1 and Bel_2 on $\mathcal{P}(Y)$ with basic probability assignments m_1 and m_2, and the corresponding focal elements $A_1, A_2 \ldots, A_k$ and $B_1, B_2, \ldots, B_l$. It is supposed, that

$$\sum_{\substack{i,j \\ A_i \cap B_j = \emptyset}} m_1(A_i) m_2(B_j) < 1, \tag{6}$$

which provides that Bel_1 and Bel_2 do not flatly contradict each other.

(i) A $\mathcal{P}(X) \to [0,1]$ mapping m, defined by $m(\emptyset) = 0$ and

$$m(A) = \frac{\sum_{\substack{i,j \\ A_i \cap B_j = A}} m_1(A_i) m_2(B_j)}{\sum_{\substack{i,j \\ A_i \cap B_j \neq \emptyset}} m_1(A_i) m_2(B_j)}$$

for any non-empty set $A \in \mathcal{P}(Y)$ is a basic probability assignment on $\mathcal{P}(Y)$.

(ii) The belief function given by m is called the *orthogonal sum* of Bel_1 and Bel_2 and is denoted by $\mathrm{Bel}_1 \oplus \mathrm{Bel}_2$. If (6) does not hold, then the orthogonal sum $\mathrm{Bel}_1 \oplus \mathrm{Bel}_2$ does not exist.

The discribed rule combines a pair of belief functions, but by repeatedly applying it one can obviously combine any number of belief functions.

In his first paper on the upper and lower probabilities [4], Dempster also solved the problem of combining several independent sources of information. The sources considered in [4] are defined by their basic probability spaces $(X_i, \mathcal{P}(X_i), P_i)$ and multivalued mappings Γ_i from X_i into Y, for each $i \in \{1, 2, \ldots, n\}$, where n is the number of sources. If n sources are assumed independent, then the combined source $(X, \mathcal{P}(X_1) \times \ldots \times \mathcal{P}(X_n), P)$ is defined by

$$\begin{aligned} X &= X_1 \times \ldots \times X_n; \\ P(A_1 \times \ldots \times A_n) &= P_1(A_1) \ldots P_n(A_n), \end{aligned} \tag{7}$$

for any $A_1 \times \ldots \times A_n \in \mathcal{P}(X_1) \times \ldots \times \mathcal{P}(X_n)$ and the multivalued mapping Γ from X into Y is given by

$$\Gamma(x_1, \ldots, x_n) = \Gamma_1(x_1) \cap \ldots \cap \Gamma_n(x_n), \tag{8}$$

for any $(x_1, \ldots, x_n) \in X$.

4 Modal logic

Modal logic is an extension of classical propositional logic [3]. The language of modal logic consists of a set of atomic propositions, logical connectives $\wedge, \vee, \neg, \to, \leftrightarrow$, and modal operators of *possibility* $\Diamond$ and *necessity* $\Box$. The propositions of the language are of the following form:

- atomic propositions,
- if p and q are propositions, then so are $\neg p$, $p \wedge q$, $p \vee q$, $p \to q$, $p \leftrightarrow q$, $\Box p$, $\Diamond p$.

Any system of modal logic contains the axiom $\Diamond p \leftrightarrow \neg\Box\neg p$. Moreover, various systems of modal logic can be obtained by imposing additional axioms. The most important fact is that these different systems have a clear interpretation in terms of Kripke's semantics of possible worlds. Thus, the semantic analysis of a system of modal logic is performed using the notion of a *model of modal logic*, which is usually viewed as a structure $M = \langle W, R, V \rangle$ [3], where W denotes a *set of possible worlds*, R is a binary relation on W called *accessibility relation* and V is a multivalued mapping from the set of atomic propositions into W called *value assignment function*.

The interpretation of the accessibility relation R in a model of modal logic can vary significantly, but in general it may be viewed as expressing the fact that some things may be possible from the standpoint of one world and impossible from the standpoint of another. Thus, wRv means that world v is possible or is relevant to world w.

The value assignment function V associates to each atomic proposition p the set $V(p)$ of those possible worlds in which p is true. Of course, any of the sets $V(p)$ can be empty and it is possible that there exists a world $w \in W$ that does not belong to any of them. We will use $||p||^M$ to denote the *truth set* of a proposition p, i.e. the set of all worlds in which p is true:

$$||p||^M = \{w \mid w \in W \wedge V(w,p) = T\}.$$

Hence $V(p) = ||p||^M$ for any atomic proposition p. The value assignment function is inductively extended to all non-modal propositions (propositions that do not contain $\Diamond$ and $\Box$) in the usual way. The truth conditions of modal propositions are defined using the accessibility relation R, as follows:

$$\begin{aligned} w \in ||\Diamond p||^M &\Leftrightarrow (\exists v \in W)(wRv \wedge v \in ||p||^M) \\ w \in ||\Box p||^M &\Leftrightarrow (\forall v \in W)(wRv \Rightarrow v \in ||p||^M), \end{aligned}$$

for any proposition p and any world $w \in W$. Observing that the accessibility relation R can be regarded as a multivalued mapping from W into W, we have shown in [22] that the foregoing expressions for the truth sets of modal propositions can be rewritten in terms of inverse and superinverse images under R of truth sets of non-modal propositions:

$$\begin{aligned} ||\Diamond p||^M &= R^-\left(||p||^M\right) \\ ||\Box p||^M &= R^+\left(||p||^M\right) \cup \mathrm{co\,dom}(R). \end{aligned} \tag{9}$$

5 A modal logic interpretation of Dempster–Shafer theory by multivalued analysis

In this section, we recall the modal logic interpretations of Dempster–Shafer theory as established in [23]. Let us consider a universe X and associate to it propositions of the form:

$$e_A = \text{“}\textit{The true value of } e \textit{ is in } A\text{”},$$

where e is a quantity with a set of possible values X and $A \in \mathcal{P}(X)$. Next, assume that all atomic propositions are of the form $e_{\{x\}}$, for $x \in X$. Therefore, the propositions e_A are defined by the equations

$$e_\emptyset = \bigwedge_{x \in X} \neg e_{\{x\}}$$

$$e_A = \bigvee_{x \in A} e_{\{x\}}, \quad \forall A \in \mathcal{P}(X) \setminus \{\emptyset\}.$$

Consider a model $M = \langle W, R, V, P \rangle$, where W and R are as above, V is a multivalued mapping from X into W that associates to each x the set $V(x)$ of those possible worlds in which $e_{\{x\}}$ is true, i.e. $V(x) = ||e_{\{x\}}||^M$, and P is a probability measure on $\mathcal{P}(W)$, such that $P\left(||\Diamond e_X||^M\right) > 0$. Observe that a proposition $e_{\{x\}}$ is true in a world w if and only if $w \in V(x)$ and, moreover, due to (1), for any $A \in \mathcal{P}(X)$:

$$V(A) = ||e_A||^M. \tag{10}$$

In [23], we developed interpretations of plausibility and belief measures on the set of atomic propositions X induced by the model $M = \langle W, R, V, P \rangle$ that contains two multivalued attributes: the accessibility relation R, which can be regarded as a multivalued mapping from W into W, and the value assignment function V that is a multivalued mapping from X into W. We constructed a multivalued mapping Γ as a composition of R and V^-, i.e. $\Gamma = V^- \circ R$. The inverse and superinverse images of any $A \in \mathcal{P}(X)$ under Γ were expressed as

$$\begin{aligned} \Gamma^-(A) &= ||\Diamond e_A||^M \\ \Gamma^+(A) &= ||\Box \neg e_{\mathrm{co}A}||^M \cap ||\Diamond e_X||^M. \end{aligned} \tag{11}$$

Since the constructed multivalued mapping Γ established a correspondence between the set of possible worlds W and the set of atomic propositions X, then the plausibility and belief measures on X were defined as Dempster's upper and lower probabilities. Hence, any multivalued model $M = \langle W, R, V, P \rangle$ with $P\left(||\Diamond e_X||^M\right) > 0$ induces a plausibility measure Pl_M and a belief measure Bel_M on $\mathcal{P}(X)$, defined by [23]:

$$\begin{aligned} \mathrm{Pl}_M(A) &= P\left(||\Diamond e_A||^M \mid ||\Diamond e_X||^M\right) \\ \mathrm{Bel}_M(A) &= P\left(||\Box \neg e_{\mathrm{co}A}||^M \mid ||\Diamond e_X||^M\right). \end{aligned} \tag{12}$$

6 Dempster's rule of combination in modal logic

Let us consider propositions of the form e_A, for all $A \in \mathcal{P}(X)$ associated to a given universe X, and assume that all atomic propositions are of the form $e_{\{x\}}$, for $x \in X$. Next, regard finite multivalued models $M_1 = \langle W_1, R_1, V_1, P_1 \rangle$ and $M_2 = \langle W_2, R_2, V_2, P_2 \rangle$, and admit that there exists $x \in X$, such that

$$\left(P_1\left(||\Diamond e_{\{x\}}||^{M_1}\right) > 0\right) \wedge \left(P_2\left(||\Diamond e_{\{x\}}||^{M_2}\right) > 0\right). \tag{13}$$

The latter provides that there is at least one $x \in X$, for which the proposition $\Diamond e_{\{x\}}$ is probable in both given models M_1 and M_2. Obviously, the above assumption implies:

$$\left(P_1\left(||\Diamond e_X||^{M_1}\right) > 0\right) \wedge \left(P_2\left(||\Diamond e_X||^{M_2}\right) > 0\right). \tag{14}$$

It turns out that there is a simple way to build a model M, combination of the models M_1 and M_2, such that the plausibility and belief measures induced by it are, in fact, the orthogonal sums of the plausibility and belief measures corresponding to the orginal models M_1 and M_2. Let us consider a model $M = \langle W_1 \times W_2, R, V_1 \times V_2, P \rangle$, where R is constructed as follows :

$$R(w_1, w_2) = R_1(w_1) \times R_2(w_2), \tag{15}$$

for any $(w_1, w_2) \in W_1 \times W_2$ and P is given by :

$$P((w_1, w_2)) = P_1(w_1) P_2(w_2),$$

for any $(w_1, w_2) \in W_1 \times W_2$. Obviously, for any $U_1 \times U_2 \in \mathcal{P}(W_1) \times \mathcal{P}(W_2)$, it holds

$$P(U_1 \times U_2) = P_1(U_1) P_2(U_2). \tag{16}$$

Now, let us show that $P(||\Diamond e_X||^M) > 0$. Indeed, the assumption (13) implies $||\Diamond e_{\{x\}}||^{M_1} \neq \emptyset$ and $||\Diamond e_{\{x\}}||^{M_2} \neq \emptyset$. From this it follows that there is $(w_1, w_2) \in W_1 \times W_2$, such that:

$$(w_1, w_2) \in ||\Diamond e_{\{x\}}||^{M_1} \times ||\Diamond e_{\{x\}}||^{M_2}.$$

In view of (9), it is equivalent to

$$\left(R_1(w_1) \cap ||e_{\{x\}}||^{M_1} \neq \emptyset\right) \wedge \left(R_2(w_2) \cap ||e_{\{x\}}||^{M_2} \neq \emptyset\right)$$

and consequently, $R_1(w_1) \times R_2(w_2) \cap ||e_{\{x\}}||^{M_1} \times ||e_{\{x\}}||^{M_2} \neq \emptyset$. Note that, due to (3) and (10), it holds that

$$(\forall x \in X)\left(||e_{\{x\}}||^M = ||e_{\{x\}}||^{M_1} \times ||e_{\{x\}}||^{M_2}\right).$$

Then, combining the above equality with (15), we obtain

$$(\exists (w_1, w_2) \in W_1 \times W_2)\left(R(w_1, w_2) \cap ||e_{\{x\}}||^M \neq \emptyset\right).$$

Again by (9), we have that $(w_1, w_2) \in ||\Diamond e_{\{x\}}||^M$, i.e.

$$||\Diamond e_{\{x\}}||^{M_1} \times ||\Diamond e_{\{x\}}||^{M_2} \subseteq ||\Diamond e_{\{x\}}||^M.$$

Next, by (16), we find that

$$P\left(||\Diamond e_{\{x\}}||^M\right) \geq P\left(||\Diamond e_{\{x\}}||^{M_1} \times ||\Diamond e_{\{x\}}||^{M_2}\right)$$

$$= P_1(||\Diamond e_{\{x\}}||^{M_1}) P_2(||\Diamond e_{\{x\}}||^{M_2}) > 0.$$

Consequently, $P(||\Diamond e_X||^M) > 0$, since $||\Diamond e_{\{x\}}||^M \subseteq ||\Diamond e_X||^M$. Thus, the constructed model M induces a plausibility measure Pl_M and a belief measure Bel_M on $\mathcal{P}(X)$, given by (12).

Theorem 6.1 *Given finite models* $M_1 = \langle W_1, R_1, V_1, P_1\rangle$ *and* $M_2 = \langle W_2, R_2, V_2, P_2\rangle$, *such that* $(\exists x \in X)\ \left(P_1\left(||\Diamond e_{\{x\}}||^{M_1}\right) P_2\left(||\Diamond e_{\{x\}}||^{M_2}\right) > 0\right)$, *the combined model* $M = \langle W_1 \times W_2, R, V_1 \times V_2, P\rangle$ *induces a plausibility measure* Pl_M *and a belief measure* Bel_M *on* $\mathcal{P}(X)$, *which are, in fact, the orthogonal sums of the plausibility and belief measures corresponding to the models* $M_1 = \langle W_1, R_1, V_1, P_1\rangle$ *and* $M_2 = \langle W_2, R_2, V_2, P_2\rangle$, *i.e.*

$$\mathrm{Pl}_M(A) = \mathrm{Pl}_{M_1} \oplus \mathrm{Pl}_{M_2}(A)$$

$$\mathrm{Bel}_M(A) = \mathrm{Bel}_{M_1} \oplus \mathrm{Bel}_{M_2}(A).$$

Proof. As was shown in the previous section, to any model of modal logic there corresponds a multivalued mapping, resulting from the composition of the accessibility relation and the inverse of the value assignment function, that induces plausibility and belief measures on the set of atomic propositions. Let us consider the multivalued mapping $\Gamma = (V_1 \times V_2)^- \circ R$ corresponding to M and multivalued mappings $\Gamma_1 = {V_1}^- \circ R_1$ and $\Gamma_2 = {V_2}^- \circ R_2$ associated to M_1 and M_2, respectively. Hence, due to (8), it is sufficient to prove that $\Gamma(w_1, w_2) = \Gamma_1(w_1) \cap \Gamma_2(w_2)$, for any $(w_1, w_2) \in W_1 \times W_2$. The latter is easily obtained by applying (1), (2) and (4).

$$\begin{aligned}
\Gamma(w_1, w_2) &= (V_1 \times V_2)^- \left(R(w_1, w_2)\right) \\
&= \bigcup_{(u_1,u_2)\in R(w_1,w_2)} (V_1 \times V_2)^- (u_1, u_2) \\
&= \bigcup_{(u_1,u_2)\in R_1(w_1)\times R_2(w_2)} (V_1 \times V_2)^- (u_1, u_2) \\
&= \bigcup_{(u_1,u_2)\in R_1(w_1)\times R_2(w_2)} {V_1}^-(u_1) \cap {V_2}^-(u_2) \\
&= \bigcup_{u_1\in R_1(w_1)} {V_1}^-(u_1) \cap \bigcup_{u_2\in R_2(w_2)} {V_2}^-(u_2) \\
&= {V_1}^-(R_1(w_1)) \cap {V_2}^-(R_2(w_2)) \\
&= \Gamma_1(w_1) \cap \Gamma_2(w_2). \quad \square
\end{aligned}$$

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