A Partial Solution of Possibilistic Marginal Problem

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Abstract

A possibilistic marginal problem will be introduced in a way analogous to probabilistic framework, to address the question of whether or not a common extension exists for a given set of marginal distributions. Similarities and differences between possibilistic and probabilistic marginal problems will be demonstrated, concerning necessary and sufficient conditions and sets of all solutions. Finally, the operators of composition will be introduced and we will show how to use them for finding a T-product extension.

Keywords. Marginal problem, possibility distributions, triangular norm, conditioning, conditional independence, extension.

1 Introduction

The marginal problem — which addresses the question of whether or not a common extension exists for a given set of marginal distributions — is one of the most challenging problem types of probability theory. The challenges lie not only in its applicability to various problems of statistics, but also to a wide range of relevant theoretical problems. One of them is the problem of finding sufficient conditions for the existence of a solution to this problem.

If an extension exists, it is usually not unique, i.e., the problem has an infinite number of solutions. Therefore the problem of existence of an extension is usually solved together with the problem of choosing an — in a sense — optimal representative from within the set of all possible solutions.

Nevertheless, in the last thirty years new mathematical tools have emerged as alternatives to probability theory. They are used in situations whose nature of uncertainty does not meet the requirements of probability theory, or those in which probabilistic criteria are too strict (e.g., additivity). On the other hand, probability theory has always served as a source of inspiration for the development of these nonprobabilistic calculi and they have been continually confronted with probability theory and mathematical statistics from various points of view.

In this paper we will introduce a possibilistic marginal problem analogous to the probabilistic framework, i.e., in a somewhat more general way than in [3, 4]. We will demonstrate the similarities and differences with probabilistic marginal problems concerning necessary and sufficient conditions, sets of solutions and so-called product solutions. In the last section we will recall the definition of composition operators for possibility distributions introduced in [11] and show how to use them for solving the possibilistic marginal problem under specific conditions.

2 Basic Notions

The purpose of this section is to give, as briefly as possible, an overview of basic notions of De Cooman's measure-theoretical approach to possibility theory [5], necessary for understanding the paper. Special attention will be paid to conditioning, independence and conditional independence [13, 14, 15]. We will start with the notion of a triangular norm, since most notions in this paper are parametrised by it.

2.1 Triangular Norms

A *triangular norm* (or a *t-norm*) T is a binary operator on $[0, 1]$ (i.e. $T : [0, 1]^2 \to [0, 1]$) satisfying the following three conditions:

(i) *boundary condition:* for any $a \in [0, 1]$

$$T(1, a) = a;$$

(ii) *isotonicity:* for any $a_1, a_2, b \in [0, 1]$ such that $a_1 \leq a_2$

$$T(a_1, b) \leq T(a_2, b);$$

(iii) *associativity:* for any $a, b, c \in [0,1]$

$$T(T(a,b),c) = T(a,T(b,c)),$$

(iv) *commutativity:* for any $a, b \in [0,1]$

$$T(a,b) = T(b,a).$$

Let us note that isotonicity in the second coordinate is an easy consequence of (iv) and the "second boundary condition" $T(0,a) = 0$ of (i),(ii) and (iv).

A t-norm T is called *continuous* if T is a continuous function. Within this paper, we will only deal with continuous t-norms.

There exist three important continuous t-norms, which will be used in examples:

(i) *Gödel's t-norm:* $T(a,b) = \min(a,b)$;

(ii) *product t-norm:* $T(a,b) = a \cdot b$;

(iii) *Lukasziewicz's t-norm:* $T(a,b) = \max(0, a+b-1)$.

Let $x, y \in [0,1]$ and T be a t-norm. We will call an element $z \in [0,1]$ *T-inverse* of x w.r.t. y if

$$T(z,x) = T(x,z) = y. \tag{1}$$

It is obvious that if $x \leq y$ then there are no T-inverses of x w.r.t. y.

Let $x, y \in [0,1]$. The *T-residual* $y \triangle_T x$ of y by x is defined as

$$y \triangle_T x = \sup\{z \in [0,1] : T(z,x) \leq y\}.$$

The following lemma, proven in [5] (i) and [12] (ii), expresses the relationship between T-inverses and T-residuals for continuous t-norms.

Lemma 1 *(i) Let T a be continuous t-norm and let $x, y \in [0,1]$. If the equation $T(z,x) = y$ in z admits a solution, then $y \triangle_T x$ is its greatest solution.*

(ii) Let T be a continuous t-norm. Then, for any $x_1, x_2, y_1, y_2 \in [0,1]$ such that $x_2 \geq x_1$ and $y_2 \geq y_1$, the equality

$$T(x_1 \triangle_T x_2, y_1 \triangle_T y_2) = T(x_1, y_1) \triangle_T T(x_2, y_2) \tag{2}$$

is satisfied.

2.2 Possibility Measures and Distributions

Let $\mathbf{X}$ be a finite set called *universe of discourse* which is supposed to contain at least two elements. A *possibility measure* Π is a mapping from the power set $\mathcal{P}(\mathbf{X})$ of $\mathbf{X}$ to the real unit interval $[0,1]$ satisfying the following requirement: for any family $\{A_j, j \in J\}$ of elements of $\mathcal{P}(\mathbf{X})$

$$\Pi(\bigcup_{j \in J} A_j) = \max_{j \in J} \Pi(A_j)^1.$$

For any $A \in \mathcal{P}(\mathbf{X})$, $\Pi(A)$ is called the *possibility of* A. Π is called *normal* if $\Pi(\mathbf{X}) = 1$. Within this paper we will always assume that Π is normal.

For any Π there exists a mapping $\pi : \mathbf{X} \to [0,1]$, called a *distribution* of Π, such that for any $A \in \mathcal{P}(\mathbf{X})$, $\Pi(A) = \sup_{x \in A} \pi(x)$. This function is a possibilistic counterpart of a density function in probability theory. It is evident that (in the finite case) Π is normal iff there exists at least one $x \in \mathbf{X}$ such that $\pi(x) = 1$.

Let $\mathbf{X}_1$ and $\mathbf{X}_2$ denote two finite universes of discourse provided by possibility measures Π_1 and Π_2, respectively. The possibility measure Π on $\mathbf{X}_1 \times \mathbf{X}_2$ is called *T-product possibility measure* of Π_1 and Π_2 (denoted $\Pi_1 \times_T \Pi_2$) if for any $A_1 \in \mathcal{P}(\mathbf{X}_1)$ and $A_2 \in \mathcal{P}(\mathbf{X}_2)$

$$\Pi(A_1 \times A_2) = T(\Pi(A_1), \Pi(A_2)),$$

or, equivalently, for the corresponding possibility distributions for any $(x_1, x_2) \in \mathbf{X}_1 \times \mathbf{X}_2$

$$\pi(x_1, x_2) = T(\pi_1(x_1), \pi_2(x_2)). \tag{3}$$

Now, let us consider an arbitrary possibility measure Π defined on a product universe of discourse $\mathbf{X} \times \mathbf{Y}$. The *marginal possibility measure* on $\mathbf{X}$ is defined by the equality

$$\Pi_X(A) = \Pi(A \times \mathbf{Y})$$

for any $A \subset \mathbf{X}$, and the respective *marginal possibility distribution* by the corresponding expression

$$\pi_X(x) = \max_{y \in \mathbf{Y}} \pi(x,y) \tag{4}$$

for any $x \in \mathbf{X}$.

Let us consider a finite *basic space* Ω, provided by a possibility measure Π_Ω with distribution π_Ω. A mapping $X : \Omega \longrightarrow \mathbf{X}$ is called *possibilistic variable*[2] in $\mathbf{X}$. The *induced* (or *transformed*) possibility measure Π_X on $\mathbf{X}$ is determined by

$$\Pi_X(A) = \Pi_\Omega(X^{-1}(A))$$

[1]max must be substituted by sup if $\mathbf{X}$ is not finite.

[2]This definition corresponds to that introduced by De Cooman in [5], but it is simplified due to the assumption that possibility measures are defined on power sets instead of general ample fields.

for any $A \in \mathcal{P}(\mathbf{X})$ and its distribution is

$$\pi_X(x) = \max_{\omega : X(\omega)=x} \pi_\Omega(\omega)$$

for any $x \in \mathbf{X}$.

Example 1 Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\mathbf{X} = \{x_1, x_2\}$. Let Π_Ω be defined by its distribution π_Ω as follows:

$$\pi_\Omega(\omega_1) = 1, \qquad \pi_\Omega(\omega_2) = 0.5, \qquad \pi_\Omega(\omega_3) = 0.2,$$

Then $X : \Omega \longrightarrow \mathbf{X}$ defined by $X(\omega_1) = X(\omega_2) = x_1$ and $X(\omega_3) = x_2$ is possibilistic variable, with induced distribution

$$\pi_X(x_1) = 1, \qquad \pi_X(x_2) = 0.2. \qquad \diamond$$

A mapping $h : \mathbf{X} \to [0, 1]$ is called *fuzzy variable*, i.e. fuzzy variable is a special case of possibilistic variable. The set of all fuzzy variables on $\mathbf{X}$ will be denoted by $\mathcal{G}(\mathbf{X})$.

2.3 Conditioning

Let T be a t-norm on $[0, 1]$. For any possibility measure Π on $\mathbf{X}$ with distribution π, we define the following binary relation on $\mathcal{G}(\mathbf{X})$. For h_1 and h_2 in $\mathcal{G}(\mathbf{X})$ we say that h_1 and h_2 are (Π, T)-*equal almost everywhere* (and write $h_1 \overset{(\Pi,T)}{=} h_2$) if for any $x \in X$

$$T(h_1(x), \pi(x)) = T(h_2(x), \pi(x)).$$

This notion is very important for the definition of *conditional possibility distribution* which is defined (in accordance with [5]) as *any* solution of the equation

$$\pi_{XY}(x, y) = T(\pi_Y(y), \pi_{X|_T Y}(x|_T y)), \qquad (5)$$

for any $(x, y) \in \mathbf{X} \times \mathbf{Y}$. Continuity of a t-norm T guarantees the existence of a solution of this equation. This solution is not unique (in general), but the ambiguity vanishes when almost-everywhere equality is considered. We are able to obtain a representative of these conditional possibility distributions (if T is a continuous t-norm) by taking the residual

$$\pi_{X|_T Y}(x|_T \cdot) \overset{(\Pi_Y,T)}{=} \pi_{XY}(x, \cdot) \triangle_T \pi_Y(\cdot), \qquad (6)$$

i.e., the greatest solution of the equation (5) (cf. Lemma 1).

Let us mention that, if we use a product t-norm, we will obtain Dempster's rule of conditioning [6], Lukasziewicz' t-norm corresponds to "Lukasziewicz' " rule of conditioning [8], Gödel's t-norm leads to Hisdal's rule of conditioning [9] and the choice of Gödel's t-norm together with (6) gives the modification of Hisdal's rule proposed by Dubois and Prade [7]. For a more detailed study of the conditioning rules based on continuous t-norms, the reader is referred to the paper by De Baets et al. [1].

As mentioned in the preceding paragraph, this way of conditioning brings a unifying view on several conditioning rules, i.e., its importance from the theoretical viewpoint is obvious. On the other hand, its practical meaning is not substantial. Although De Cooman [5] claims that conditional distributions are never used *per se*, there exist situations in which it is necessary to be careful and to choose an appropriate representative of the set of solutions (cf. Example 5).

There exist other approaches to conditioning introduced Walley and De Cooman [16] and by Bouchon-Meunier et al. in [2]. These approaches is completely different from ours, since the first one is based on natural extensions and in the latter conditional possibility measures are computed directly for particular conditional events and not from joint and marginal distributions. For more details see the cited works.

2.4 Independence

Two variables X and Y (taking their values in $\mathbf{X}$ and $\mathbf{Y}$, respectively) are *possibilistically T-independent* [5] if for any $F_X \in X^{-1}(\mathcal{P}(\mathbf{X}))$, $F_Y \in Y^{-1}(\mathcal{P}(\mathbf{Y}))$,

$$\begin{aligned}
\Pi(F_X \cap F_Y) &= T(\Pi(F_X), \Pi(F_Y)), \\
\Pi(F_X \cap F_Y^C) &= T(\Pi(F_X), \Pi(F_Y^C)), \\
\Pi(F_X^C \cap F_Y) &= T(\Pi(F_X^C), \Pi(F_Y)), \\
\Pi(F_X^C \cap F_Y^C) &= T(\Pi(F_X^C), \Pi(F_Y^C)),
\end{aligned}$$

where A^C denotes the complement of A.

From this definition it immediately follows that the independence notion is parameterised by T. More specifically, it means that if X and Y are min-independent, they need not be, for example, product-independent. This fact is reflected in some definitions and assertions that follow.

From the perspective of the next paragraph, the following theorem, an immediate consequence of Proposition 2.6. of the above-mentioned paper [5], is of great importance.

Theorem 1 *Let us assume that a t-norm T is continuous. Then the following propositions are equivalent.*

(i) X and Y are T-independent.

(ii) For any $x \in \mathbf{X}$ and $y \in \mathbf{Y}$

$$\pi_{XY}(x, y) = T(\pi_X(x), \pi_Y(y)).$$

(iii) For any $x \in \mathbf{X}$ and $y \in \mathbf{Y}$

$$\begin{aligned} T(\pi_X(x), \pi_Y(y)) &= T(\pi_{X|_T Y}(x|_T y), \pi_Y(y)) \\ &= T(\pi_{Y|_T X}(y|_T x), \pi_X(x)). \end{aligned}$$

This theorem claims that the notion of independence defined by De Cooman is equivalent (for $T = \min$) to Zadeh's notion of noninteractivity [17] and, in a sense, also to Hisdal's notion of independence [9] — if the equality sign is substituted by almost-everywhere equality.

2.5 Conditional Independence

In light of these facts, we defined the conditional possibilistic independence in the following way in [12]: Given a possibility measure Π on $\mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ with the respective distribution $\pi(x, y, z)$, variables X and Y are *possibilistically conditionally T-independent* given Z (in symbols $I_T(X, Y|Z)$) if, for any pair $(x, y) \in \mathbf{X} \times \mathbf{Y}$,

$$\pi_{XY|_T Z}(x, y|_T \cdot) \overset{(\Pi_Z, T)}{=} T(\pi_{X|_T Z}(x|_T \cdot), \pi_{Y|_T Z}(y|_T \cdot)). \quad (7)$$

Let us stress again that we do not deal with the pointwise equality but with the *almost everywhere equality*, in contrast to the conditional noninteractivity introduced by Fonck [8]. The following theorem is a "conditional counterpart" of Theorem 1.

Theorem 2 *For a continuous t-norm T, the following propositions are equivalent:*

(i) X and Y are T-independent given Z.

(ii) For any $x \in \mathbf{X}$, $y \in \mathbf{Y}$ and $z \in \mathbf{Z}$

$$\pi_{X|_T YZ}(x|_T y, z) \overset{(\Pi_{YZ}, T)}{=} \pi_{X|_T Z}(x|_T z). \quad (8)$$

Theorem 2 unifies the notions of conditional noninteractivity [8] and that of conditional independence in Hisdal's sense [8, 3], as well as in Dempster's [8, 4] and "Lukasziewicz' " [8] (if we substitute Gödel's t-norm by product and Lukasziewicz' t-norms, respectively) in such a way that *pointwise* equalities are substituted by *almost everywhere* equalities.

In [13, 14, 15] we have shown that by adopting the conditional independence notion (7), we will obtain most of the previously presented conditional independence relations. Formal properties (so-called semigraphoid and graphoid ones) of this measure-theoretic approach to conditional independence correspond, in general, to those possessed by stochastic conditional independence. This fact allowed us to introduce Markov properties and factorisation of possibility distribution and to find the relationships between them. For more details, the reader is referred to [13, 14, 15].

3 Possibilistic Marginal Problem

Let $\{X_i\}_{i \in N}$ be a finite system of finitely-valued variables with values in $\{\mathbf{X}_i\}_{i \in N}$. We will deal with possibility distributions on the Cartesian-product space

$$\mathbf{X} = \times_{i \in N} \mathbf{X}_i,$$

and distributions on its subspaces

$$\mathbf{X}_K = \times_{i \in K} \mathbf{X}_i$$

for $K \subset N$.

Using the procedure of marginalisation (4) we can always uniquely restrict a possibility distribution π defined on $\mathbf{X}$ to the distribution π_K defined on $\mathbf{X}_K$ for $K \subset N$ (for $K = \emptyset$ let us set $\pi_K \equiv 1$). However, the opposite process, the procedure of an *extension* of a system of distributions π_{K_i}, $i = 1, \ldots, m$ defined on $\mathbf{X}_{K_i}$ to a distribution π_K on $\mathbf{X}_K$ ($K = K_1 \cup \cdots \cup K_m$), is not unique (if it exists) and can be done in many ways.

Let us demonstrate this fact with two simple examples.

3.1 Two Simple Examples

Example 2 Let $\mathbf{X}_1 = \mathbf{X}_2 = \{0, 1\}$ and let π_1 and π_2 be defined by Table 1.

Table 1: Example 2 — given marginal distributions

X_1	0	1
π_1	1	.7

X_2	0	1
π_2	.5	1

Our task is to find a two-dimensional possibility distribution π satisfying these marginal constraints. It is easy to realize that any possibility distribution from Table 2 such that $\alpha, \beta \in [0, 0.5]$ and $\max(\alpha, \beta) = 0.5$ is a solution to this problem. ◇

Table 2: Example 2 — set of extensions

π X_2	0	1
$X_1 = 0$	α	1
$X_1 = 1$	β	.7

Example 3 can be found in [3] in a slightly more general form. Let $\mathbf{X}_1 = \mathbf{X}_2 = \mathbf{X}_3 = \{0, 1\}$, $K_1 = \{1, 3\}, K_2 = \{2, 3\}$ and let π_{13} and π_{23} be defined as expressed by Table 3.

Table 3: Example 3 — given marginals

π_{13} X_3	0	1
$X_1 = 0$	.4	1
$X_1 = 1$	1	.7

π_{23} X_3	0	1
$X_2 = 0$	.2	1
$X_2 = 1$	1	.4

Let us look for a three-dimensional possibility distribution having these distributions as its marginals. The result can be any distribution from within the set of distributions contained in Table 4, where, $\alpha, \beta \in [0, 0.2], \gamma, \delta \in [0, 0.4]$ and $\max(\alpha, \beta) = 0.2, \max(\gamma, \delta) = 0.4$. ◇

Table 4: Example 3 — set of extensions

π X_3	0		1	
X_2	0	1	0	1
$X_1 = 0$	α	.4	1	γ
$X_1 = 1$	β	1	.7	δ

3.2 Definition

The possibilistic marginal problem can be (analogous to probability theory) understood as follows: Let us assume that $\mathbf{X}_i$, $i \in N$, $1 \leq |N| < \infty$ are finite universes of discourse, $\mathcal{K}$ is a system of nonempty subsets of N and

$$\mathcal{S} = \{\pi_K, K \in \mathcal{K}\}$$

is a family of possibility distributions, where each π_K is a distribution on a product space

$$\mathbf{X}_K = \times_{i \in K} \mathbf{X}_i.$$

The problem we are interested in is the existence of an *extension*, i.e. a distribution π on $\mathbf{X}$ whose marginals are distributions from $\mathcal{S}$; or, more generally, the set

$$\mathcal{P} = \{\pi(x) : \pi(x_K) = \pi_K(x_K), K \in \mathcal{K}\}$$

is of interest.

Let us stress that the introduced problem is different from those solved by De Campos and Huete in [3, 4]. They defined the marginal problem in a somewhat different way: Let π_{13} and π_{23} be two possibility distributions of X_1, X_3 and X_2, X_3, respectively. Then the distribution π of X_1, X_2, X_3 has to satisfy:

1. X_1 and X_2 must be independent, given X_3, i.e. $I(X_1, X_2|X_3)$ (where I is one of the independence relations studied in [3, 4]) holds for the distribution π.
2. Marginal distribution of X_1, X_3 must be preserved, i.e. $\pi(x_1, x_3) = \pi_{13}(x_1, x_3)$.
3. Marginal distribution of X_2, X_3 must be preserved, i.e. $\pi(x_2, x_3) = \pi_{23}(x_2, x_3)$.

They realised that the requirement of the conditional independence I_H (i.e. "not modifying the information" for Hisdal's conditioning rule [3])[3] may cause that these three conditions need not be, in some cases, satisfied simultaneously (in particular, in Example 3). Since our concept of conditional independence is not so strict (pointwise equality is substituted by almost everywhere equality), this situation cannot occur if *any* continuous t-norm is considered.

Because of these problems, De Campos and Huete suggested that the possibility distribution should satisfy the conditional independence constraint and the first of the marginal ones; for more details see [3]. This approach seems to be somewhat off the mark, since in the marginal problem the *primary* task is to *preserve marginals* and (conditional) independence is just a tool that helps us to find a unique solution (if it exists).

Therefore, the question of the existence of an extension will be the focus of our attention in the following paragraph.

3.3 Necessary and Sufficient Conditions

Let us note that we will not be able to find any three-dimensional distribution with prescribed two-dimensional marginals in Example 3 if these marginals do not satisfy quite a natural condition called a *projectivity* (or *compatibility*) condition. We will say (in a general case) that two possibility distributions π_I and π_J (defined on $\mathbf{X}_I$ and $\mathbf{X}_J$) are *projective* if they have common marginals, i.e. if

$$\pi_I(x_{I \cap J}) = \pi_J(x_{I \cap J}).$$

This condition is clearly necessary but it is not sufficient, as demonstrated in Example 4.

Example 4 Let $\mathbf{X}_1 = \mathbf{X}_2 = \mathbf{X}_3 = \{0, 1\}$ and consider π_{12}, π_{13} and π_{23} from Table 5.

Table 5: Example 4 — given marginals

π_{12} X_2	0	1
$X_1 = 0$	1	0
$X_1 = 1$	0	1

π_{13} X_3	0	1
$X_1 = 0$	1	0
$X_1 = 1$	0	1

π_{23} X_3	0	1
$X_2 = 0$	0	1
$X_2 = 1$	1	0

Although these three distributions are projective (more exactly, $\pi_{12}(x_1) \equiv \pi_{13}(x_1) \equiv 1$, $\pi_{12}(x_2) \equiv$

[3]It is, in fact, a pointwise version of (8) for Gödel's t-norm.

$\pi_{23}(x_2) \equiv 1$ and $\pi_{13}(x_3) \equiv \pi_{23}(x_3) \equiv 1$), a three-dimensional possibility distribution π having them as its marginals does not exist. It follows from the fact that it should be equal to zero for any combination of values x_1, x_2 and x_3 (as expressed by Table 6), because of the zero marginals, but simultaneously the maximum value of e.g. $\pi(0,0,0)$ and $\pi(0,0,1)$ should be equal to 1. ◇

Table 6: Example 4 — "extension"

X_3	0		1	
X_2	0	1	0	1
$X_1 = 0$	0	0	0	0
$X_1 = 1$	0	0	0	0

In the probabilistic framework, projectivity is a necessary condition for the existence of an extension, too, and becomes a sufficient condition if the index sets of the marginals can be ordered in such a way that it satisfies a special property called the running intersection property (see e.g. [10]). At the end of the next section we will recall this notion and prove an analogous result in the possibilistic framework.

3.4 Sets of Extensions

If a solution of a possibilistic marginal problem exists, it is (usually) not unique, as we have already seen in Examples 2 and 3. This fact is completely analogous to the probabilistic framework. However, contrary to the probabilistic marginal problem, the set of extensions of a set of possibility distributions is (generally) not convex. This means that if we have two solutions of the marginal problem π and ρ, their linear combination $\sigma = \alpha \cdot \pi + (1-\alpha) \cdot \rho$ for $\alpha \in (0,1)$ needn't be a solution to this problem. On the other hand, the set of solutions is closed under maximisation, i.e. distribution τ defined by the equality $\tau(x) = \max(\pi(x), \rho(x))$ for any $x \in \mathbf{X}$ is again a solution to that problem. Let us illustrate these two facts with the following simple example.

Example 2 *(Continued)* We have already realized that possibility distributions

$$\pi(0,0) = 0.5, \quad \pi(0,1) = 1, \quad \pi(1,0) = 0.2, \quad \pi(0,1) = 0.7,$$

$$\rho(0,0) = 0.1, \quad \rho(0,1) = 1, \quad \rho(0,0) = 0.5, \quad \rho(0,1) = 0.7$$

are solutions of the respective marginal problem, but their linear combinations

$$\sigma(0,0) = 0.1 + 0.4\alpha, \quad \sigma(0,1) = 1, \quad \sigma(1,0) = 0.5 - 0.3\alpha, \quad \sigma(0,1) = 0.7$$

are not, since $\sigma_Y(0) = \max(0.1 + 0.4\alpha, 0.5 - 0.3\alpha) < 0.5$ for $\alpha \in (0,1)$. On the other hand, distribution

$$\tau(0,0) = 0.5, \quad \tau(0,1) = 1, \quad \tau(1,0) = 0.5, \quad \tau(0,1) = 0.7$$

is clearly a solution of that possibilistic marginal problem. ◇

3.5 T-product Extensions

It is evident that it is difficult to handle the whole set of extensions and therefore an additional requirement is necessary to enable us to choose one representative of this set. The most natural requirement seems to be that of (conditional) independence.

There exists a special class of solutions to a marginal problem, namely the class of T-product distributions, defined in Paragraph 2.2. If K_1 and K_2 are disjoint, the resulting distribution is just a T-product[4] of the given distributions, i.e.,

$$\tilde{\pi}(x_{K_1 \cup K_2}) = \tilde{\pi}(x_{K_1}, x_{K_2}) = T(\pi_1(x_{K_1}), \pi_2(x_{K_2})). \tag{9}$$

For different t-norms we obtain different T-product extensions, as can be seen from the following example.

Example 2 *(Continued)* For Gödel's, product and Lukasziewicz' t-norms we get

$$\alpha_G = 0.5, \quad \alpha_p = 0.5, \quad \alpha_L = 0.5,$$
$$\beta_G = 0.5, \quad \beta_p = 0.35, \quad \beta_L = 0.2,$$

respectively. Nevertheless, not all two-dimensional possibility distributions satisfying the above-mentioned constraints can be obtained as T-product distributions (for a suitable t-norm T). For example, there does not exist a t-norm T such that

$$\alpha_T = 0.1, \qquad \beta_T = 0.5$$

are T-products of $\pi_Y(0)$ and $\pi_X(0)$ and $\pi_X(1)$, respectively. This distribution violates both (i) and (ii) of the definition of a t-norm, nevertheless it is an extension of both π_X and π_Y. ◇

It follows from Theorem 1 that the equality (9) holds iff X_{K_1} and X_{K_2} are T-independent.

The generalization of a T-product extension to a general set of marginal distributions with pairwise disjoint index sets is straightforward.

[4] Although it is not expressed explicitly, we have to keep in mind that distributions $\tilde{\pi}$ are parameterised by T.

If the index sets are not disjoint, the situation is somewhat more complicated. Let us assume π_1 and π_2 be projective distributions of X_{K_1} and X_{K_2}, respectively, $K_1 \cap K_2 \neq \emptyset$. Then the T*-product extension* of these distributions can be defined by the equality

$$\tilde{\pi}(x_{K_1 \cup K_2}) = T(\pi_1(x_{K_1}), \pi_2(x_{K_2}) \triangle_T \pi_2(x_{K_1 \cap K_2})), \quad (10)$$

or, equivalently by

$$\tilde{\pi}(x_{K_1 \cup K_2}) = T(\pi_1(x_{K_1}) \triangle_T \pi_1(x_{K_1 \cap K_2}), \pi_2(x_{K_2})).$$

Example 3 *(Continued)* Considering marginal distributions π_{12} and π_{23} from Table 3 we will obtain for Gödel's, product and Lukasziewicz' t-norms:

$$\begin{array}{lll} \alpha_G = 0.2, & \alpha_p = 0.08, & \alpha_L = 0, \\ \beta_G = 0.2, & \beta_p = 0.2, & \beta_L = 0.2, \\ \gamma_G = 0.4, & \gamma_p = 0.4, & \gamma_L = 0.4, \\ \delta_G = 0.4, & \delta_p = 0.28, & \delta_L = 0.1. \end{array}$$

Nevertheless, also in this case there exist distributions having π_{13} and π_{23} as their marginals, which cannot be expressed by the equation (10) for any continuous t-norm T, e.g. the distribution with

$$\alpha = 0.2, \quad \beta = 0.1, \quad \gamma = 0.3, \quad \delta = 0.4. \quad \diamond$$

Let us note that it is not possible to use an arbitrary solution of the equation (5) in the definition of the distribution $\tilde{\pi}$ if we want this distribution to be an extension of both its marginals. This is demonstrated by the following counterexample.

Example 5 Let $\mathbf{X}_1 = \mathbf{X}_2 = \mathbf{X}_3 = \{0, 1\}$ and $K_1 = \{1, 2\}, K_2 = \{2, 3\}$. Let π_{12} and π_{23} be defined by Table 7.

Table 7: Example 5 — distributions π_{12} and π_{23}

π_{12} X_2	0	1
$X_1 = 0$	0	0
$X_1 = 1$	1	0

π_{23} X_3	0	1
$X_2 = 0$	1	1
$X_2 = 1$	0	0

Since the marginal of π_{23} on $\mathbf{X}_2$ is

$$\pi_2(0) = 1, \qquad \pi_2(1) = 0,$$

we will obtain that generally (for any choice of a t-norm)

$$\begin{array}{rcl} \pi_{3|_T 2}(i|_T 0) & = & 1, \\ \pi_{3|_T 2}(i|_T 1) & \in & [0, 1]. \end{array}$$

If we used this set of conditional possibility distributions for definition of another "T-product" extension

$$\tilde{\rho}(x_1, x_2, x_3) = T(\pi_{12}(x_1, x_2), \pi_{3|_T 2}(x_3|_T x_2)),$$

Table 8: Example 5 — set of distributions $\tilde{\rho}$

X_3	0		1	
X_2	0	1	0	1
$X_1 = 0$	0	α	0	β
$X_1 = 1$	1	γ	1	δ

we would obtain distributions whose values are in Table 8 where $\alpha, \beta, \gamma, \delta \in [0, 1]$ and by simple marginalization we finally get their marginals $\tilde{\rho}_{12}$ (see Table 9), which evidently differ (in general) from π_{12}. $\diamond$

Table 9: Example 5 — set of marginals $\tilde{\rho}_{12}$

X_2	0	1
$X_1 = 0$	0	$\max(\alpha, \beta)$
$X_1 = 1$	1	$\max(\gamma, \delta)$

The following lemma expresses the relationship between T-product extensions and conditional independence.

Lemma 2 *Let T be a continuous t-norm and π_1 and π_2 be projective possibility distributions of X_{K_1} and X_{K_2}, respectively. Then the distribution π of $X_{K_1 \cup K_2}$*

$$\begin{aligned} \pi(x_{K_1 \cup K_2}) &= \\ &= T(\pi_1(x_{K_1}), \pi_2(x_{K_2}) \triangle_T \pi_2(x_{K_1 \cap K_2})) \quad (11) \\ &= T(\pi_1(x_{K_1}) \triangle_T \pi_1(x_{K_1 \cap K_2}), \pi_2(x_{K_2})), \end{aligned}$$

if and only if $X_{K_1 \setminus K_2}$ and $X_{K_2 \setminus K_1}$ are conditionally independent, given $X_{K_1 \cap K_2}$.

Proof. Using associativity and commutativity of T, Lemma 1 and projectivity of π_1 and π_2, we have

$$\begin{aligned} &\pi(x_{K_1 \cup K_2}) \\ &= T(\pi(x_{K_1 \cup K_2 \setminus (K_1 \cap K_2)}|_T x_{K_1 \cap K_2}), \pi(x_{K_1 \cap K_2})) \\ &= T(T(\pi(x_{K_1 \setminus K_2}|_T x_{K_1 \cap K_2}), \pi(x_{K_2 \setminus K_1}|_T x_{K_1 \cap K_2})), \\ &\quad \pi(x_{K_1 \cap K_2})) \\ &= T(\pi_1(x_{K_1 \setminus K_2}|_T x_{K_1 \cap K_2}), \\ &\quad T(\pi_2(x_{K_2 \setminus K_1}|_T x_{K_1 \cap K_2}), \pi_2(x_{K_1 \cap K_2}))) \\ &= T(\pi_1(x_{K_1 \setminus K_2}|_T x_{K_1 \cap K_2}), \\ &\quad T(\pi_2(x_{K_2}) \triangle_T \pi_2(x_{K_1 \cap K_2}), \pi_2(x_{K_1 \cap K_2}))) \\ &= T(\pi_1(x_{K_1 \setminus K_2}|_T x_{K_1 \cap K_2}), \\ &\quad T(\pi_2(x_{K_1 \cap K_2}), \pi_2(x_{K_2 \setminus K_1}) \triangle_T \pi_2(x_{K_1 \cap K_2}))) \\ &= T(T(\pi_1(x_{K_1 \setminus K_2}|_T x_{K_1 \cap K_2}), \pi_1(x_{K_1 \cap K_2})), \\ &\quad \pi_2(x_{K_2}) \triangle_T \pi_2(x_{K_1 \cap K_2})) \\ &= T(\pi_1(x_{K_1}), \pi_2(x_{K_2}) \triangle_T \pi_2(x_{K_1 \cap K_2})), \end{aligned}$$

where the second equality holds if and only if $X_{K_1 \setminus (K_1 \cap K_2)}$ and $X_{K_2 \setminus (K_1 \cap K_2)}$ are conditionally independent given $X_{K_1 \cap K_2}$. The second equality in (11) is satisfied due to the fact that π_{K_1} and π_{K_2} are projective. $\square$

A generalisation of this approach to a more general system $\mathcal{S}$ of marginal possibility distributions will be at the centre of our attention in the next section (more precisely, in its last paragraph).

4 Operators of Composition

Operators of composition introduced in [11] are based on a generalisation of the above-mentioned idea. Considering a continuous t-norm T, two subsets K_1, K_2 of $\{1, \ldots, N\}$ (not necessarily disjoint) and two normal possibility distributions $\pi_1(x_{K_1})$ and $\pi_2(x_{K_2})$,[5] we define the *operator of right composition* of these possibilistic distributions by the expression

$$\begin{aligned}&\pi_1(x_{K_1}) \rhd_T \pi_2(x_{K_2}) \\ &\quad = T\left(\pi_1(x_{K_1}), \pi_2(x_{K_2}) \triangle_T \pi_2(x_{K_1 \cap K_2})\right),\end{aligned}$$

and analogously the *operator of left composition* by the expression

$$\begin{aligned}&\pi_1(x_{K_1}) \lhd_T \pi_2(x_{K_2}) \\ &\quad = T\left(\pi_1(x_{K_1}) \triangle_T \pi_1(x_{K_1 \cap K_2}), \pi_2(x_{K_2})\right).\end{aligned}$$

It is evident that both $\pi_1 \rhd_T \pi_2$ and $\pi_1 \lhd_T \pi_2$ are (generally different) possibility distributions of variables $(X_i)_{i \in K_1 \cup K_2}$.

Now, we will present two lemmata proven in [11], expressing basic properties of these operators.

Lemma 3 *Let T be a continuous t-norm and $\pi_1(x_{K_1})$ and $\pi_2(x_{K_2})$ be two distributions. Then*

$$(\pi_1 \rhd_T \pi_2)(x_{K_1}) = \pi_1(x_{K_1})$$

and

$$(\pi_1 \lhd_T \pi_2)(x_{K_2}) = \pi_2(x_{K_2}).$$

Lemma 4 *Consider two distributions $\pi_1(x_{K_1})$ and $\pi_2(x_{K_2})$. Then*

$$(\pi_1 \rhd_T \pi_2)(x_{K_1 \cup K_2}) = (\pi_1 \lhd_T \pi_2)(x_{K_1 \cup K_2})$$

for any continuous t-norm T iff

$$\pi_1(x_{K_1 \cap K_2}) = \pi_2(x_{K_2 \cap K_1}).$$

4.1 Generating sequences

In this section we will show how to apply the operators iteratively. Consider a sequence of distributions $\pi_1(x_{K_1}), \pi_2(x_{K_2}), \ldots, \pi_m(x_{K_m})$ and the expression

$$\pi_1 \rhd_T \pi_2 \rhd_T \ldots \rhd_T \pi_m.$$

Before beginning a discussion of its properties, we have to explain how to interpret it. Though we did not mention it explicitly, the operator $\rhd_T$ (as well as $\lhd_T$) is neither commutative nor associative.[6] Therefore, generally

$$(\pi_1 \rhd_T \pi_2) \rhd_T \pi_3 \neq \pi_1 \rhd_T (\pi_2 \rhd_T \pi_3).$$

For this reason, let us note that in the part that follows, we always apply the operators from left to right, i. e.

$$\begin{aligned}&\pi_1 \rhd_T \pi_2 \rhd_T \pi_3 \rhd_T \ldots \rhd_T \pi_m \\ &\quad = (\ldots((\pi_1 \rhd_T \pi_2) \rhd_T \pi_3) \rhd_T \ldots \rhd_T \pi_m).\end{aligned}$$

This expression defines a multidimensional distribution of $X_{K_1 \cup \ldots \cup K_m}$. Therefore, for any permutation $i_1, i_2, \ldots, i_m$ of indices $1, \ldots, m$ the expression

$$\pi_{i_1} \rhd_T \pi_{i_2} \rhd \ldots \rhd_T \pi_{i_m}$$

determines a distribution of the same family of variables, however, for different permutations these distributions can differ from one another. In the following paragraph we will deal with special generating sequences (or their special permutations), which seem to possess the most advantageous properties.

4.2 Perfect sequences

An ordered sequence of possibility distributions $\pi_1, \pi_2, \ldots, \pi_m$ is said to be *perfect* if

$$\begin{aligned}\pi_1 \rhd_T \pi_2 &= \pi_1 \lhd_T \pi_2, \\ \pi_1 \rhd_T \pi_2 \rhd_T \pi_3 &= \pi_1 \lhd_T \pi_2 \lhd_T \pi_3, \\ &\vdots \\ \pi_1 \rhd_T \cdots \rhd_T \pi_m &= \pi_1 \lhd_T \cdots \lhd_T \pi_m.\end{aligned}$$

The notion of T-perfectness suggests that a sequence perfect with respect to one t-norm needn't be perfect with respect to another t-norm, analogous to (conditional) T-independence. Let us demonstrate it on the following simple example.

Example 6 Let $\mathbf{X}_1 = \mathbf{X}_2 = \{0, 1\}$ and π_1, π_2 and π_3 on $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_1 \times \mathbf{X}_2$ be defined by Table 10. Sequence π_1, π_2, π_3 is min-perfect, since

$$\pi_1 \rhd_{T_G} \pi_2 = \min(\pi_1, \pi_2) = \pi_1 \lhd_{T_G} \pi_2$$

and

$$\pi_1 \rhd_{T_G} \pi_2 \rhd_{T_G} \pi_3 = \min(\pi_1, \pi_2) = \pi_3 = \pi_1 \lhd_{T_G} \pi_2 \lhd_{T_G} \pi_3,$$

but not, for example, product-perfect, since

$$\pi_1 \rhd_{T_p} \pi_2 \rhd_{T_p} \pi_3 = \pi_1 \cdot \pi_2 \neq \pi_3 = \pi_1 \lhd_{T_p} \pi_2 \lhd_{T_p} \pi_3.$$

$\diamond$

[5] Let us stress that for the definition of these operators we do not require projectivity of distributions π_1 and π_2.

[6] Counterexamples can be found in [11].

Table 10: Distributions forming min-perfect sequence

X_1	0	1
π_2	1	.5

X_2	0	1
π_2	1	.5

π_3 X_2	0	1
$X_1 = 0$	1	.5
$X_1 = 1$	.5	.5

The following two lemmata, proven in [11], will be used for proofs of further assertions.

Lemma 5 *Let T be a continuous t-norm. The sequence $\pi_1, \pi_2, \ldots, \pi_m$ is T-perfect, if and only if the pairs of distributions $(\pi_1 \triangleright_T \cdots \triangleright_T \pi_{k-1})$ and π_k are projective for all $k = 2, 3, \ldots, m$.*

Lemma 6 *Let T be a continuous t-norm and $\pi_1, \pi_2, \ldots, \pi_m$ be a generating sequence of low-dimensional possibility distributions. Then $\pi_1 \triangleright_T \cdots \triangleright_T \pi_m$ is an extension of $\pi_1 \triangleright_T \cdots \triangleright_T \pi_k$ for all $k = 1, \ldots, m-1$.*

The following characterisation theorem expresses one of the most important results concerning perfect sequences. It says they compose into multidimensional distributions that are extensions of all the distributions from which the joint distribution is composed.

Theorem 3 *The sequence $\pi_1, \pi_2, \ldots, \pi_m$ is perfect iff all the distributions $\pi_1, \pi_2, \ldots, \pi_m$ are marginal to distribution $\pi_1 \triangleright_T \pi_2 \triangleright_T \ldots \triangleright \pi_m$.*

Proof. Let $\pi_1, \pi_2, \ldots, \pi_m$ be a perfect sequence of possibility distributions of $X_{K_1}, X_{K_2}, \ldots, X_{K_m}$, respectively. Let us consider an arbitrary $k \in \{1, \ldots, m-1\}$ and denote $\rho_k = \pi_1 \triangleright_T \cdots \triangleright_T \pi_k$. Since, due to the perfectness of $\pi_1, \ldots, \pi_k$,

$$\rho_k = \pi_1 \triangleleft_T \cdots \triangleleft_T \pi_k,$$

it is evident that ρ_k is an extension of π_k on $\mathbf{X}_{K_1 \cup \cdots \cup K_k}$. From this fact and from Lemma 6 we will immediately obtain that $\pi_1 \triangleright_T \cdots \triangleright_T \pi_m$ is an extension of π_k, too.

Let for all $i = 1, \ldots, m$, π_i be marginal distributions of $\pi_1 \triangleright_T \cdots \triangleright_T \pi_m$. Let us consider an arbitrary $i \in \{1, \ldots, m\}$. Projectivity must hold for π_i and $\pi_1 \triangleright_T \cdots \triangleright_T \pi_{i-1}$ as the latter distribution is also a marginal of $\pi_1 \triangleright_T \cdots \triangleright_T \pi_m$ (cf. Lemma 6) . Therefore, from Lemma 5 we immediately obtain that the sequence $\pi_1, \ldots, \pi_m$ of possibility distributions is perfect, which completes the proof. □

Now, we can approach formulation of the result concerning sufficient conditions for existence of an extension of the given set of low-dimensional distributions, as we promised in Paragraph 3.3. Before doing that, we need to recall what the running intersection property means and what the assertion is of the lemma concerning the relationship between this property and perfectness.

A sequence of sets $K_1, K_2, \ldots, K_n$ is said to meet *running intersection property* (RIP) if

$$\forall i = 2, \ldots, n \;\; \exists j (1 \leq j < i) \;\; (K_i \cap (K_1 \cup \ldots \cup K_{i-1})) \subseteq K_j.$$

Lemma 7 *If $\pi_1, \pi_2, \ldots, \pi_m$ is a sequence of pairwise projective low-dimensional distributions such that $K_1, \ldots, K_m$ meets RIP, then this sequence is T-perfect for any continuous t-norm T.*

Proof. Let us prove the assertion using induction. For $i = 2$

$$\pi_1 \triangleright_T \pi_2 = \pi_1 \triangleleft_T \pi_2$$

follows from Lemma 4. To get

$$\pi_1 \triangleright_T \ldots \triangleright_T \pi_i = \pi_1 \triangleleft_T \ldots \triangleleft_T \pi_i$$

for a general $i > 2$ we need a projectivity of π_i and $\pi_1 \triangleright_T \ldots \triangleright_T \pi_{i-1}$. According to RIP there is $j < i$ such that

$$K_i \cap (K_1 \cup \ldots \cup K_{i-1}) \subset K_j.$$

Using the inductive assumption, the theorem holds for $i-1$, and therefore π_j, which is projective with π_i, is a marginal of $\pi_1 \triangleright_T \ldots \triangleright_T \pi_{i-1}$ for an arbitrary continuous t-norm T. Hence, π_i must also be projective with $\pi_1 \triangleright_T \ldots \triangleright_T \pi_{i-1}$ and therefore, due to Lemma 4 and the inductive assumption,

$$\pi_1 \triangleright_T \ldots \triangleright_T \pi_i = \pi_1 \triangleleft_T \ldots \triangleleft_T \pi_i.$$

for any continuous T. □

Therefore we can conclude:

Theorem 4 *Let $\mathcal{S} = \{\pi_K, K \in \mathcal{K}\}$ be a system of pairwise projective low-dimensional possibility distributions. If there exists a permutation $i_1, \ldots, i_m$ of indices $1, \ldots, m$ such that $K_{i_1}, \ldots, K_{i_m}$ meets RIP, then, for any continuous T, there exists a T-product extension*

$$\pi_{i_1} \triangleright_T \pi_{i_2} \triangleright \ldots \triangleright_T \pi_{i_m}$$

of these distributions.

Proof of this theorem is an immediate consequence of Theorem 3 and Lemma 7.

5 Conclusions

We have introduced a possibilistic marginal problem analogous to a probabilistic one, (i.e. in a more general way than it was done by De Campos and Huete

[3, 4]). We discussed necessary and sufficient conditions, which appeared to be very similar to those found in the probabilistic framework. On the other hand, sets of all solutions are generally not convex (in contrast to the probabilistic framework). A lot of attention was paid to T-product extensions — distributions that can be obtained from the marginals by adopting a (conditional) independence requirement. We found a sufficient condition under which they exist and described the apparatus for their construction.

Nevertheless, we have shown that there are still many problems that remain to be solved. One of them is the problem of a characterization of the sets of all solutions. Another question is, in fact, closely connected with the first one: how to solve a possibilistic marginal problem if the T-product extension does not exist. And the third task may be to study the relationship between T-product extensions and properties of some suitable measures of entropy in the possibilistic framework.

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