Conditional Upper Probabilities Assigned by a Class of Hausdorff Outer Measures

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Abstract

Coherent conditional probabilities, in the sense of de Finetti, are given by a class of Hausdorff dimensional measures. In particular the case where conditioning events have infinite Hausdorff measure is considered. The problem of the extensions of these conditional probabilities to the class of all subsets of [0,1] is investigated. Conditional upper probabilities, assigned by a class of Hausdorff dimensional outer measures, are considered and their properties are analised.

Keywords. Coherence, conditional upper probabilities, Hausdorff dimensional outer measures.

1 Introduction

Partial knowledge is a natural interpretation of conditional probability. This interpretation can be formalized in a different way in the axiomatic approach and in the subjective approach where conditional probability is defined by coherence condition. Some critical situations, that highlight as the axiomatic definition of conditional probability is not always a useful tool to represent partial knowledge, are proposed in literature ([2]). They are caused from the fact that regular conditional distributions can not be *proper* ([3], [4]). This property is always satisfied if conditional probability is defined by coherence condition. The interest for coherent conditional probabilies is motivated by the fact that they can be a usefull tool to the handling of uncertainty.

In the probabilistic approach of de Finetti a coherent conditional probability is a real function P defined on an arbitrary family of conditional events, satisfying the coherence condition.

In this approach, the treatment of uncertainty is based on the interpretation of conditional events as partial information about an experiment and coherent conditional probability as a tool to represent the level of knowledge about it. New information about the experiment can be represented by a new conditional event and the updating of knowledge is equivalent to determine coherent extensions.

The starting point of the present paper is the analyse of the role of the Radon-Nikodym derivative in the assessment of coherent conditional probabilities according to the different approaches to coherence proposed by de Finetti ([5],[6]) and by Walley ([15]).

A comparison between the axiomatic definition of conditional probability and those based on the coherence condition, has been proposed in [1], [14], [15].

The new aspect of the result (Theorem 1) proposed in this paper is that it establishes that, every time that the σ-field of the conditioning events is contained in the σ-field of the probability space and it contains all singletons, the Radon-Nikodym derivative can not be used as tool to define coherent conditional probability. This is due to the fact that one of the defining properties of the Radon-Nikodym derivative, that is to be measurable with respect to the σ-field of the conditioning events, contradicts a necessary condition for the coherence.

Non coherence of conditional probability defined by the Radon-Nikodym derivative is proved without the hypoteses that the σ-field of the conditioning events is not countably generated ([3]) or the probability space is a Lusin space and the σ-field of the conditioning events is countably generated ([4]).

Let Ω=[0,1] and let **F** and **G** be σ-fields of subsets of Ω, with **G** sub-σ-field properly contained in **F** and such that it contains all singletons of [0,1].

Let P* be a real function defined on $\mathbf{F}\times\mathbf{G}^0$, where $\mathbf{G}^0=\mathbf{G}-\varnothing$; having fixed A in **F**-**G**, if P*(A|{ω}) is equal, with probability 1, to the Radon-Nikodym derivative, then the following condition, necessary for the coherence, ([13]) P*(A|{ω})=1or 0 according to ω

belongs to A or to A^c, is not satisfied. In fact the indicator function of A, is not **G**-measurable, it does not verify a property of the Radon-Nikodym derivative and so it can not be assumed as conditional probability, according to the axiomatic definition.

Moreover, when the family of conditioning events is the partition **B** of all singletons of [0,1] and **F** is a σ-field containing **B** the functions $P^*(A|\{\omega\})$ defined for A in **F** and $\{\omega\}$ in **B**, are not *separately coherent* ([15]). In fact if for every fixed A in **F** $P^*(A|\{\omega\})$ is equal, with probability 1, to the Radon-Nikodym derivative, then the condition $P^*(\{\omega\}|\{\omega\})=1$ is not satisfied for all ω in [0,1].

In [3] *normal conditional distributions* are defined with the aim to assure to conditional distribution the property to be proper; the existence of a countably additive, normal, conditional distribution Q given a countably generated sub-σ-field **A** is proved (see [3] theorem 2).

This result is not sufficient to resolve the problem of coherence of regular conditional distribution. In fact, if the conditioning event is a countable set, coherence required that conditional probability is finitely additive, but not countably additive and the existence of a normal conditional distribution Q (finitely additive but not countably additive) given **A** is not assured for every countably generated sub-σ-field **A** (see example 4.3 of [1]).

Analysis done points out the necessity to introduce different tools to define coherent conditional probabilities.

The aim of this paper it to assess coherent conditional probabilities by a class of Hausdorff dimensional outer measures. In particular, when the conditioning events have infinite Hausdorff measure, coherent conditional probability is given by a 0-1 valued finitely additive probability.

The results proposed are the following:

-Coherent conditional probabilities in the sense of de Finetti are assigned by a class of Hausdorff dimensional outer measures when **F** is the σ-field of the borelian sets of [0,1] and **G** is an additive subclass of **F** such that every H belonging to **G,** has positive and finite Hausdorff s-dimensional measure, where s is the Hausdorff dimension of H.

-some properties of the extensions P(.|H) of these functions to the class of all subsets of [0,1] are proved.

-The restrictions $P(A|\{\omega\})$ with A in **F** and $\{\omega\}$ in **B** are separately coherent.

2 Different approaches to conditional probabilities

In this section we compare two different approaches to coherent conditional probability, one proposed by de Finetti ([5], [6]) and the other by Walley ([15]), with the axiomatic definition, of conditional probability, ([2]), based on the Radon-Nikodym derivative.

Let Ω=[0,1] and let **F** and **G** be two sub σ-fields of subsets of Ω with **G** properly contained in **F** and such that it contains all singletons of [0,1]. Let P be a probability measure on **F**; having fixed A in **F** let ν be a finite measure on **G** defined by $\nu(H)=P(A\cap H)$ with $H\in$ **G**. Let $P[A||\mathbf{G}]_\omega$ be the Radon-Nikodym derivative of ν with respect to P, that is a function, defined on Ω, **G**-measurable, integrable with respect to P and satisfying the functional equation $P(A\cap H)=\int_H P[A||\mathbf{G}]_\omega dP$ with $H\in$ **G**. This function is unique up to a set of P-measure zero and it is called *version* of the conditional probability.

Theorem 1. *Let P* be a real function defined on* $\boldsymbol{C}=\boldsymbol{F}\times\boldsymbol{G}^0$; *having fixed A in **F**-**G**, consider the subfamily* $\boldsymbol{E}=\{(A,\{\omega\}) : \omega \in [0,1]\}$ *of **C**. If the restriction* P_r^* *of P* to **E** is equal to the Radon-Nikodym derivative with probability 1, that is*

$$P_r^*(A,\{\omega\})=P[A||\mathbf{G}]_\omega \qquad (1)$$

except on a subset N of [0,1] of P-measure zero, then P is not a coherent conditional probability on **C**.*

Proof. If equality (1) holds with probability 1, then we have that $P^*_r(A,\{\omega\})$ is different from the indicator function of A. In fact having fixed A in **F-G**, the indicator function of A, is not **G**-measurable, it does not verify a property of the Radon-Nikodym derivative and so it can not be assumed as conditional probability, according to the axiomatic definition. So the following condition, necessary for the coherence, $P^*_r(A,\{\omega\})=1$or 0 respectively if ω belongs to A or to A^c is not satisfied. Therefore P^*_r is not coherent on **E** and so, there is not a coherent extension P^* to $\mathbf{C}\supset\mathbf{E}$.

In Walley ([15] cap.6) coherent conditional probabilities are considered as a special case of coherent conditional previsions, that are characterised in the case where conditioning events form a partition **B** of Ω. The real numbers $\underline{P}(X|B)$ are specified for B in **B** and all gambles X in some domain H(B). Conditional previsions $\underline{P}(X|B)$, defined for B in **B** and X in H(B) are *separately coherent* when for every conditioning event B, $\underline{P}(.|B)$ is a coherent lower prevision on the domain H(B) and $\underline{P}(B|B)=1$.

When the domain H(B) is a class of events that can be regarded as a class of 0-1 valued gambles, $\underline{P}(X|B)$ is a lower conditional probability.

Let **F** be a sub σ-fields of subsets of Ω=[0,1], containing all singletons of [0,1] and **B** the partition of Ω that consists of all singletons of [0,1]. For each $\{\omega\}$

in **B** and A in **F** $\underline{P}(A|\{\omega\})$ are conditional probabilities that are *separately coherent* when for every conditioning event B, $\underline{P}(.|B)$ is a coherent (lower) probability on **F** and $P(B|B)=1$.
This last condition is not satisfied if conditional probability is defined by the Radon-Nikodym derivative.

3 Finitely additive conditional probabilities

In this section coherence, in the sense of de Finetti, of conditional probability assigned by a class of Hausdorff measures is proved. Firstly the case where conditioning events have finite Hausdorff measure is considered. Then in Example 2, conditioning events with infinite Hausdorff measure are also considered. By coherence it follows that for each countable H, the function $P^*(\cdot|H)$ must be a finitely additive but not σ-additive probability.

Let **F** and **G** be two arbitrary families of subsets of Ω, such that $\Omega\in$ **F**. If P*, defined on $\mathbf{C}\subseteq \mathbf{F}\times\mathbf{G}^0$ is coherent then ([9], [13]):

I) given any $H\in\mathbf{G}^0$ and $A_1,...,A_n\in\mathbf{F}$ such that their union belongs to **F** and $A_iA_j=\emptyset$ for $i\neq j$, the function $P^*(.|H)$ defined on **F** is such that

I) $P^*(A|H)\geq 0$, $\quad P^*(\bigcup_{k=1}^{n} A_k|H)=\sum_{k=1}^{n} P^*(A_k|H)$, $P^*(\Omega|H)=1$

II) $P^*(H|H)=1$ if $H\in \mathbf{F}\cap\mathbf{G}^0$

III) given $E\in\mathbf{F}$, $H\in\mathbf{F}$ $EH\in\mathbf{F}$ with $A\in\mathbf{G}^0$ and $EA\in\mathbf{G}^0$ then $P^*(EH|A)=P^*(E|A)P^*(H|EA)$.

From conditions I) and II) we have

II') $P^*(A|H)=1$ if $A\in$ **F**, $H\in\mathbf{G}^0$ and $H\subset A$.

Conditions I), II), III) are sufficient ([13]) for the coherence of P* on $\mathbf{C}=\mathbf{F}\times\mathbf{G}^0$ if **F** and **G** are fields of subsets of Ω with $\mathbf{G}\subseteq\mathbf{F}$.

This result holds ([9]) even if **G** is an additive subclass of **F**.

Let **F** and **G** be two fields of subsets of Ω, with $\mathbf{G}\subseteq\mathbf{F}$, P* is a *finitely additive conditional probability*, ([7]), on (**F**,**G**) if it is a real function defined on $\mathbf{F}\times\mathbf{G}^0$, such that the conditions I), II), III) hold.

A finitely additive conditional probability P* on (**F**,**G**) is *complete* on **F** if **G**=**F**.

Coherent conditional probabilities assigned by a class of Hausdorff dimensional measures are given. Let (Ω,d) be the Euclidean metric space with Ω=[0,1]. The diameter of a nonempty set U of Ω is defined as $|U|=\sup\{|x-y|: x,y\in U\}$ and if a subset A of Ω is such that $A\subset\bigcup_i U_i$ and $0<|U_i|<\delta$ for each i, the class $\{U_i\}$ is called a δ-cover of A. Let s be a non-negative number. For δ>0 we define $h^s_\delta(A)=\inf\sum_{i=1}^{\infty}|U_i|^s$, where the infimum is over all δ-covers $\{U_i\}$. The Hausdorff s-dimensional outer measure of A, denoted by $h^s(A)$, is defined as $h^s(A)=\lim_{\delta\to 0} h^s_\delta(A)$. This limit exists, but may be infinite, since $h^s_\delta(A)$ increases as δ decreases.

The Hausdorff dimension of a set A, dimA, is defined as the unique value, such that $h^s(A)=\infty$ if $0\leq s<\dim A$ and $h^s(A)=0$ if $\dim A<s<\infty$.

A subset A of Ω is called measurable with respect to the outer measure h^s if it decomposes every subset of Ω additively, that is if $h^s(E)=h^s(AE)+h^s(E-A)$ for all sets $E\subset\Omega$.

The restriction of h^s to the σ-field of h^s- measurable sets, containing the σ-field of the borelian sets, is called Hausdorff s-dimensional measure. In particular the Hausdorff 0-dimensional measure is the counting measure and the Hausdorff 1-dimensional measure is the Lebesgue measure. For more details about Hausdorff measures see, for example [8].

The coherence of the real function P* assigned by a class of Hausdorff measures is proved in the following cases.

Theorem 2. *Let Ω=[0,1] and let **F** be the σ-field of the borelian sets of [0,1] and let **G** be an additive subclass of **F** of sets such that for every H in **G** we have $0<h^s(H)<\infty$, where s is the Hausdorff dimension of H and h^s is the Hausdorff s-dimensional measure. Then the real function P*, defined on **C=F×G**, such that*

$$P^*(A|H)=\frac{h^s(AH)}{h^s(H)}$$

*is a coherent conditional probability on **C**.*

Proof. Since P* is defined on **C=F×G** with **F** σ-field of subsets of [0,1] and **G**, an additive subclass of **F**, then P* is a coherent conditional probability if and only if conditions I), II), III) hold.

The restriction of the Hausdorff s-dimensional measure to the σ-field of the borelian sets of [0,1] is a measure for every s and, by definition, we have, for any $H\in\mathbf{G}$ $P^*(\Omega|H)=P^*(H|H)=\frac{h^s(H)}{h^s(H)}=1$, then $P^*(.|H)$ verifies condition I) and II).

Moreover given $E\in\mathbf{F}$, $H\in\mathbf{F}$ $EH\in\mathbf{F}$ with $A\in\mathbf{G}$ and $EA\in\mathbf{G}$, and denoted by s the Hausdorff dimension of A and by t the Hausdorff dimension of EA, by the definition of P*, condition III) can be written as

$$\frac{h^s(EAH)}{h^s(A)}=\frac{h^s(EA)}{h^s(A)}\cdot\frac{h^t(EAH)}{h^t(EA)} \qquad (2)$$

Two cases are possible: i) s= t or ii) s > t.
If i) holds than (2) is obviously satisfied. If ii) holds than $h^s(EA)=0$ and also, by the monotony of h^s, $h^s(EAH)=0$; so equation (2) is satisfied.
In particular if all sets belonging to **G** are finite, then the previous probability can be extended in the following way

Proposition 1. *Let Ω=[0,1] and let **F** be the σ-field of all subsets of [0,1] and let **G** be an additive subclass of **F** of finite sets. Denoted by h^0 the Hausdorff 0-dimensional measure, the real function P*, defined on **C=F×G⁰**, such that*

$$P^*(A|H)=\frac{h^0(AH)}{h^0(H)}$$

*is a coherent conditional probability on **C**.*

Proof. Since the class of subsets of Ω measurable with respect to h^0 is the class of all subsets of [0,1], than P*(.|H) is a measure on **F** and, by definition, $P^*(\Omega|H)=P^*(H|H)=\frac{h^0(H)}{h^0(H)}=1$. Moreover P* verifies equation (2) since all sets of **G** have Hausdorff dimension equal to 0, then P* is a coherent conditional probability since it satisfies conditions I), II), III).

Theorem 3. *Let **F** be the σ-field of the Lebesgue-measurable subsets of [0,1] and let **G** be the class of **F** of sets that are either finite or co-countable. Denoted by h^1 the Lebesgue measure and by h^0 the counting measure, we have that the real function P* defined on $C=F\times G^0$ such that*

$$P^*(A|H)=\begin{cases}\dfrac{h^1(AH)}{h^1(H)} & H \quad co-countable\\[2ex] \dfrac{h^0(AH)}{h^0(H)} & H \quad finite\end{cases}$$

is a coherent conditional probability.

Proof. Since the class of subsets of Ω measurable with respect to h^0 is the class of all subsets of [0,1], than both h^0 either h^1 are measures on **F** and by definition, we have, for any $H\in G^0$ P*(Ω|H)=P*(H|H)= 1 then P*(.|H) verifies condition I) and II). Condition III) is olso satisfied and it can be proved as in theorem 2.

Remark. The restriction to any subclass of **C** of the function P*, defined in the previous theorem is a coherent conditional probability.

Example 1. The restriction of the function P* of Theorem 3, to $A\times A^0$, where **A** is the field of sets that are finite or cofinite is

$$P^*(A|H)=\begin{cases}\dfrac{h^0(AH)}{h^0(H)} & A \quad and \quad H \quad finite\\[2ex] 1 & A \quad and \quad H \quad cofinite\\[2ex] 0 & A \quad finite \quad H \quad cofinite\end{cases}$$

Example 2. Let (Ω,**F**,P) be a probability space where Ω=[0,1], **F** is the σ-field of Borel of Ω and P is the Lebesgue measure on **F**. Let **G** be the sub σ-field of **F** of sets that are either countable or co-countable.
Since the probability of the events of the σ-field **G** is either 0 or 1, we have that the probability of A given **G** is equal to P(A), with probability 1, if conditional probability is defined by the Radon-Nikodym derivative. That is

$$P[A\|G]_\omega=P(A)$$

except on a countable subset N of [0,1].
Given A=[a,b] with 0<a<b<1, let P* be a real function defined on $C=F\times G^0$ such that the restriction P^*_r to $E=\{(A,\{\omega\}):\omega\in[0,1]\}$ is equal, with probability 1, to the Radon-Nikodym derivative $P[A\|G]_\omega$. From theorem 1 we have that P* is not coherent on **C**.
A finitely additive conditional probability on $C=F\times G^0$ can be defined by

$$P^*(A|H)=\begin{cases}\dfrac{h^1(AH)}{h^1(H)} & H \quad co-countable\\[2ex] \dfrac{h^0(AH)}{h^0(H)} & H \quad finite\\ m(AH) & H \quad countable\end{cases}$$

where m is a 0-1 valued finitely additive (but not countably additive) probability measure.
An explicit construction of m is proposed by R.Israel ([10]).
The function P* is a coherent conditional probability since it verifies the axioms of a finitely additive probability in the sense of Dubins.
Th restriction to **F×B** of conditional probabilities P*(A|H), defined in this section, are separately coherent.

4 Extensions

The aim of this section is to extend the functions $P^*(.|H)$, defined in theorem 2 and 3, to the class of all subsets of $\Omega=[0,1]$. The natural extensions are the inner measures generated by the Hausdorff dimensional measures and the conjugate upper probabilities that are the Hausdorff dimensional outer measures.

Exstensions $\overline{P}(.|H)$ are assigned by a class of Hausdorff outer measures in the following way:

$$\overline{P}(A|H)=\frac{h^S(AH)}{h^S(H)}$$

where s is the Hausdorff dimension of H and h^s is the Hausdorff s-dimensional outer measure.

In general this implies to renounce to the addivity so these extensions are not coherent in the sense of de Finetti, but the functions $\overline{P}(.|H)$ are subadditive.

In the following result some properties of the function $\overline{P}(.|H)$ with respect to the Hausdorff dimension s of the conditioning event H, are proved.

Theorem 4. *Let $\Omega=[0,1]$ and let **F** be the σ-field of all sub sets of $[0,1]$ and let **G** be an additive subclass of **F** of sets such that for every H in **G** we have $0<h^s(H)<\infty$, where s is the Hausdorff dimension of H and h^s is the Hausdorff s-dimensional outer measure. Then for each H in **G** the real function $\overline{P}(.|H)$ defined on **F**, such that*

$$\overline{P}(A|H)=\frac{h^S(AH)}{h^S(H)}$$

verifies the following properties:

a) $0\leq\overline{P}(A|H)\leq 1$;

b) $\overline{P}(A\cup B|H)\leq\overline{P}(A|H)+\overline{P}(B|H)$

and

$\overline{P}(A\cup B|H)=\overline{P}(A|H)+\overline{P}(B|H)$) whenever A and B are positively separated, that is $d(A,B)=\inf\{d(x,y): x\in A, y\in B\}>0$;

c) for each $H\in$ **G** $\overline{P}(\cdot|H)$ is a coherent upper probability.

Proof . For each H belonginig to **G** we have, for the monotony of the Hausdorff outer measures, we have

$$0\leq\overline{P}(A|H)=\frac{h^S(AH)}{h^S(H)}\leq\frac{h^S(H)}{h^S(H)}=1;$$

Moreover, since h^s is an outer measure for every s then it is subadditive. For every s the Hausdorff outer measure h^s is a *metric outer measure* that is $h^s(A\cup B)=h^s(A)+h^s(B)$ whenever A and B are positively separated.

Property c) follows from Theorem 3.1.5. of [15].

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