Lattice structure of the families of compatible frames of discernment

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Abstract

One of the central ideas in Shafer's mathematical theory of evidence is the concept of different level of knowledge of a given phenomenon, embodied into the notion of *compatible frames of discernment*. In this work we are going to analyze the concept of family of frames from an algebraic point of view, distinguish among finite and general families and introduce the internal operation of *maximal coarsening*, originating the structure of *semimodular lattice*. We will show the equivalence between the classical independence of frames and the independence of frames as elements of a locally finite Birkhoff lattice, eventually prefiguring a solution to the conflict problem based on a *pseudo Gram-Schmidt* algorithm.

Keywords. Family of compatible frames, Birkhoff lattice, linear dependence, conflict.

1 Introduction

Together with the nature of belief functions as set-valued measures, the major concept of the *theory of evidence* ([8]) is the formalization of the idea of structured collection of representations of the external world, encoded into the notion of *family of frames*. As the following equation clearly shows, the concept of *independence* of frames is reminiscent of the idea of independent vectors of a linear space:

$$\omega_1(A_1)\cap\cdots\cap\omega_n(A_n) \neq \emptyset \leftrightarrow v_1+...+v_n \neq 0,\ \forall v_i \in V_i.$$

We are going to prove that this is more than a simple analogy: it is the symptom of a deeper similarity of these structure at the algebraic level.

In this work we will expose an analysis of the algebraic properties of the families of frames, not intended as partitions of a common finite set, but as objects obeying a small number of axioms ([8]). We will distinguish among finite and general families, describe the monoidal properties of both collections of frames and refinings, and introduce the internal operation of *maximal coarsening* originating the structure of *semimodular lattice bounded below*.

Finally, we will show the equivalence between the classical independence of frames as Boolean sub-algebras and the *external* independence of frames as elements of a locally finite Birkhoff lattice, eventually prefiguring a solution to the *conflict* problem based on a pseudo Gram-Schmidt algorithm.

Not much work has been done along this path; in [9] can be found an analysis of the collections of partitions of a given frame in the context of the hierarchical representation of belief. A wider exposition of the algebraic properties of the families of frames can instead be found in [6], where Chapter 7 is devoted to their lattice-theoretical interpretation and the meaning of the concept of independence, while Chapter 8 develops the consequences of the application of constraints to the structure of the family and deepens the properties of *Markov trees* as a useful characterization.

2 The theory of evidence

Let us call [8] the finite set of possible answers to a given problem *frame of discernment* (FOD).

Definition 1. *A* basic probability assignment *(b.p.a.) over a FOD Θ is a function $m : 2^\Theta \to [0,1]$ such that*

$$m(\emptyset) = 0,\ m(A) \geq 0\ \forall\ A \subset \Theta,\ \sum_{A\subset\Theta} m(A) = 1.$$

The elements of 2^Θ associated to non-zero values of m are called *focal elements* and their union *core*. Now suppose a b.p.a. is introduced over an arbitrary FOD.

Definition 2. *The* belief function *associated to the basic probability assignment* m *is defined as $Bel(A) = \sum_{B\subset A} m(B)$.*

Belief functions representing distinct bodies of evidence are combined by means of the *Dempster's rule* *of combination.*

Definition 3. *The* orthogonal sum $Bel_1 \oplus Bel_2$ *of two belief functions is a function whose focal elements are all the possible intersections between the combining focal elements and whose b.p.a. is given by*

$$m(A) = \frac{\sum_{i,j:A_i \cap B_j = A} m_1(A_i) m_2(B_j)}{1 - \sum_{i,j:A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j)}.$$

The normalization constant in the above expression measures the *level of conflict* between belief functions, for it represents the amount of probability they attribute to contradictory (i.e. disjoint) propositions.

New amount of evidence can allow us to make decisions (by using Dempster's rule) over more detailed domains. This argument is embodied into the notion of *refining.*

Definition 4. *Given two frames* Θ *and* Ω*, a map* $\omega : 2^\Theta \to 2^\Omega$ *is a* refining *if:*

1. $\omega(\{\theta\}) \neq \emptyset \quad \forall \theta \in \Theta$;
2. $\omega(\{\theta\}) \cap \omega(\{\theta'\}) = \emptyset \quad if\ \theta \neq \theta'$;
3. $\cup_{\theta \in \Theta} \omega(\{\theta\}) = \Omega$.

The finer frame is called *refinement* of the first one and Θ *coarsening* of Ω.

2.1 Families of compatible frames

The intuitive idea of different descriptions of a same phenomenon is described by the concept of *family of compatible frames* (see [8], pages 121-125).

Definition 5. *A non-empty collection of finite non-empty sets* $\mathcal{F}$ *is a* family of compatible frames of discernment *with refinings* $\mathcal{R}$*, where* $\mathcal{R}$ *is a non-empty collection of refinings between couples of frames in* $\mathcal{F}$*, if* $\mathcal{F}$ *and* $\mathcal{R}$ *satisfy the following requirements:*

Axiom 1. composition of refinings*: if* $\omega_1 : 2^{\Theta_1} \to 2^{\Theta_2}$ *and* $\omega_2 : 2^{\Theta_2} \to 2^{\Theta_3}$ *are in* $\mathcal{R}$*, then* $\omega_1 \circ \omega_2 \in \mathcal{R}$.

Axiom 2. identity of coarsenings*: if* $\omega_1 : 2^{\Theta_1} \to 2^\Omega$ *and* $\omega_2 : 2^{\Theta_2} \to 2^\Omega$ *are in* $\mathcal{R}$ *and* $\forall \theta_1 \in \Theta_1 \ \exists \theta_2 \in \Theta_2$ *such that* $\omega_1(\{\theta_1\}) = \omega_2(\{\theta_2\})$ *then* $\Theta_1 = \Theta_2$ *and* $\omega_1 = \omega_2$.

Axiom 3. identity of refinings*: if* $\omega_1 : 2^\Theta \to 2^\Omega$ *and* $\omega_2 : 2^\Theta \to 2^\Omega$ *are in* $\mathcal{R}$*, then* $\omega_1 = \omega_2$.

Axiom 4. existence of coarsenings*: if* $\Omega \in \mathcal{F}$ *and* $A_1, ..., A_n$ *is a disjoint partition of* Ω*, then there is a coarsening* Ω' *of* Ω *in* $\mathcal{F}$ *corresponding to this partition, i.e.* $\forall A_i$ *there exists an element of* Ω' *whose image under the appropriate refining is* A_i.

Axiom 5. existence of refinings*: if* $\theta \in \Theta \in \mathcal{F}$ *and* $n \in \mathcal{N}$ *then there exists a refining* $\omega : 2^\Theta \to 2^\Omega$ *in* $\mathcal{R}$ *and* $\Omega \in \mathcal{F}$ *such that* $\omega(\{\theta\})$ *has n elements.*

Axiom 6. existence of common refinements*: every pair of elements in* $\mathcal{F}$ *has a common refinement in* $\mathcal{F}$.

From property (6) a collection of compatible frames has many common refinements. One of these is particularly simple.

Proposition 1. *If* $\Theta_1, ..., \Theta_n$ *are elements of a family* $\mathcal{F}$ *then there exists a unique element* $\Theta \in \mathcal{F}$ *such that*

1. $\forall i \ \exists \omega_i : 2^{\Theta_i} \to 2^\Omega$ *refining;*
2. $\forall \theta \in \Theta \ \exists \theta_i \in \Theta_i$ *for* $i = 1, ..., n$ *such that*

$$\{\theta\} = \omega_1(\{\theta_1\}) \cap ... \cap \omega_n(\{\theta_n\}). \quad (1)$$

This unique FOD is called the *minimal refinement* $\Theta_1 \otimes \cdots \otimes \Theta_n$ of the collection $\Theta_1, ..., \Theta_n$ and is the simplest space in which you can compare propositions belonging to different compatible frames.

Definition 6. *Let* $\Theta_1, ..., \Theta_n$ *be compatible frames, and* $\omega_i : 2^{\Theta_i} \to 2^{\Theta_1 \otimes \cdots \otimes \Theta_n}$ *the corresponding refinings to their minimal refinement. They are said to be* independent *if*

$$\omega_1(A_1) \cap \cdots \cap \omega_n(A_n) \neq \emptyset \quad (2)$$

whenever $\emptyset \neq A_i \subset \Theta_i$ *for* $i = 1, ..., n$.

3 Motivations: measurement conflict

Belief functions carrying distinct pieces of evidence are combined by using the Dempster's rule: unfortunately, the following theorem shows that the combination is guaranteed *only for trivially interacting feature spaces.*

Theorem 1. *Let* $\Theta_1, ..., \Theta_n$ *a set of compatible FODs. Then the following conditions are equivalent:*

1. *all the possible collections of belief functions* $s_1, ..., s_n$ *defined respectively over* $\Theta_1, ..., \Theta_n$ *are combinable over their minimal refinement* $\Theta_1 \otimes ... \otimes \Theta_n$;
2. $\Theta_1, ..., \Theta_n$ *are* independent;
3. *there is a 1-1 correspondence* $\Theta_1 \otimes \cdots \otimes \Theta_n \leftrightarrow \Theta_1 \times \cdots \times \Theta_n$;
4. $|\Theta_1 \otimes \cdots \otimes \Theta_n| = \prod_{i=1}^n |\Theta_i|$.

Proof. $1 \Rightarrow 2$. We know that if $s_1, ..., s_n$ are combinable then s_i, s_j must be combinable $\forall i, j = 1, ..., n$. Hence $\omega_i(\mathcal{C}_i) \cap \omega_j(\mathcal{C}_j) \neq \emptyset \ \forall i, j$ where $\mathcal{C}_i$ is the core of s_i. Being s_i, s_j arbitrary, their cores $\mathcal{C}_i, \mathcal{C}_j$ can be any pair of subsets of Θ_i, Θ_j respectively, so that the condition can be rewritten as

$$\omega_i(A_i) \cap \omega_j(A_j) \neq \emptyset \quad \forall A_i \subset \Theta_i, A_j \subset \Theta_j.$$

$2 \Rightarrow 1$. It suffices to take $A_i = \mathcal{C}_i \ \forall i = 1, ..., n$.
$2 \Rightarrow 3$. We note that

$$\bigcap_i \omega_i(\theta_i^k) = \bigcap_i \omega_i(\theta_i^l) \Leftrightarrow \theta_i^k = \theta_i^l \quad \forall i = 1, ..., n.$$

In fact, if $\exists j$ s.t. $\theta_i^k \neq \theta_i^l$ then $\omega_i(\{\theta_i^k\}) \neq \omega_i(\{\theta_i^l\})$ for the properties of refinings: since they are disjoint, it must be $\bigcap_i \omega_i(\{\theta_i^k\}) \neq \bigcap_i \omega_i(\{\theta_i^l\})$.
Hence their number coincides with the number of the n-tuples of singletons $|\Theta_1| \times \cdots \times |\Theta_n|$, and being $\Theta_1 \otimes \cdots \otimes \Theta_n$ a minimal refinement each of these subsets must correspond to a singleton.
$3 \Rightarrow 2$. For Proposition 1, each $\theta \in \Theta_1 \otimes \cdots \otimes \Theta_n$ corresponds to a subset of the form $\bigcap_i \omega_i(\{\theta_i^k\})$. Since the singletons are $|\Theta_1| \times \cdots \times |\Theta_n|$ the above subsets are in the same number, but then they all are non-empty, so that $\Theta_1, ..., \Theta_n$ are independent.
$3 \Rightarrow 4$. Obvious.
$4 \Rightarrow 3$. $\Theta_1 \otimes \cdots \otimes \Theta_n = \{\bigcap_i \omega_i(\{\theta_i\}) \forall \theta_i \in \Theta_i\}$, so that if they are $|\Theta_1| \times \cdots \times |\Theta_n|$ they all must be non-empty and each of them can be labeled by $(\theta_1, ..., \theta_n)$. □

Any given set of belief functions is then characterized by a *level of conflict* $\mathcal{K}$. If $\mathcal{K} = \infty$ they are not combinable.
In some recent works of ours ([2], [3]), we proposed an evidential framework for the solution of the *object tracking* problem, in which, given a sequence of images of a moving articulated object we want to reconstruct the position and configuration of the body at each time instant. We represented a number of different *features* extracted from the images as belief functions and combined them to compute an estimate of the pose $\hat{q}(t) \in \tilde{\mathcal{Q}}$, where $\tilde{\mathcal{Q}}$ is a "good" finite approximation of the parameter space $\mathcal{Q}$ of the object. The feature spaces and the collection $\tilde{Q}$ form a finite subset of a family of compatible frames of discernment, where $\tilde{Q}$ is a common refinement of the collection of feature spaces. Since the measurement belief functions built from the extracted features are not always compatible, we need a method for detecting the subset we are going to apply the rule of combination to.
A basic property of the conflict level is

$$\mathcal{K}(s_1, ..., s_{n+1}) = \mathcal{K}(s_1, ..., s_n) + \mathcal{K}(s_1 \oplus ... \oplus s_n, s_{n+1}),$$

so that if $\mathcal{K}(s_i, s_j) = +\infty$ then $\mathcal{K}(s_i, s_j, s_k) = +\infty \ \forall s_k$. This suggests a bottom-up technique. First the level of conflict is computed for each pair of measurement functions (s_i, s_j), $i, j = 1, ..., n$. Then a suitable threshold is decided and a *conflict graph* is built, where every node represents a belief function while the edges indicate a conflict level below the threshold. Finally the subsets of combinabile b.f. of size $d+1$ are *recorsively* computed from those of size d, eventually detecting the most coherent set of features. This approach obviously suffers a high computational cost when large groups of functions are compatible.
A few work has been done on the problem of conflict ([4]). Murphy ([7]), for instance, presented another problem, the failure to balance multiple evidence, illustrated the proposed solutions and described their limitations.
In the final part of this paper, we are going to show how the problem of conflicting measurements could possibly find an elegant interpretation in the context of the algebraic analysis of the families of frames.

4 Axiom analysis

Consider an arbitrary finite set S and check the result of the application of the axioms of Definition 5. We must first apply Axiom $A4$, obtaining the collection of all the possible partitions of S and the refinings between each of them and the basis set. Applying $A4$ again to the new sets all the refinings among them are achieved, while no other set is add to the collection. Axioms 2 and 3 guarantee the uniqueness of the obtained maps and sets. Observe that rule $A1$ *in this situation* is redundant, for it does not add any new refining.
It is clear even at a first glance that rule $A6$ claims an existing condition but it is not *constructive*, i.e. it does not allow to generate new sets from a given initial collection. Therefore let us introduce a new axiom

Axiom 7. Existence of the minimal refinement*: for every pair of elements in $\mathcal{F}$ there exists their minimal refinement in $\mathcal{F}$, i.e. a set satisfying the conditions in Proposition* 1.

and build another set of axioms by replacing $A6$ with $A7$. Let us call $A_{1..6}$ and $A_{1..5,7}$ these two formulations.

Theorem 2. *$A_{1..6}$ and $A_{1..5,7}$ are equivalent formulations of the notion of family of frames.*

Proof. It is necessary and sufficient to prove that (i) $A7$ can be obtained by using the set of axioms $A1, ..., A6$ and (ii) that $A6$ can be obtained from $A1, ..., A5, A7$.
(i). See [8] or Proposition 1 . (ii). Each refinement of a given pair of sets Θ_1, Θ_2 can be obtained by arbitrarily refining $\Theta_1 \otimes \Theta_2$ by means of Axiom $A5$. Anyway, the minimal refinement is obviously a refinement so that $A7 \Rightarrow A6$. □

4.1 Family generated by a set

If we assume *finite and static* our knowledge of the phenomenon, the axiom $A5$ of the families of compatible frames (Definition 5) cannot be used. According to the established notation let us call $A_{1..4,7}$ the set of axioms corresponding to finite knowledge.

Definition 7. *The* subfamily generated by a collection of sets $\Theta_1, ..., \Theta_n$ *by means of a set of axioms $\mathcal{A}$ is the smallest collection of frames $\langle\Theta_1, ..., \Theta_n\rangle_{\mathcal{A}}$ containing $\Theta_1, ..., \Theta_n$ and close under the application of the axioms in $\mathcal{A}$.*

Lemma 1. *The minimal refinement of two coarsenings Θ_1, Θ_2 of a FOD S is still a coarsening of S.*

Proof. From the hypothesis S is a common refinement of Θ_1 and Θ_2, and since the minimal refinement is coarsening of every other refinement the thesis follows. □

Theorem 3. *The subfamily of compatible frames generated by the application of the restricted set of rules $A_{1..4,7}$ to a basis set S is the collection of all the disjoint partitions of S along with the appropriate refinings.*

Note that this is not necessarily true when using Axiom $A6$.

5 Monoidal structure

Let us introduce in a family of compatible frames the internal operation

$$\begin{array}{cccc} \otimes : & \mathcal{F} \times \cdots \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ & \{\Theta_1, ..., \Theta_n\} & \mapsto & \otimes_i \Theta_i \end{array} \tag{3}$$

associating to a collection of frames their minimal refinement. It is well defined, for Axiom $A7$ ensures the existence of $\otimes_i \Theta_i$ and the following result guarantees its associativity and commutativity.

5.1 Finite families as commutative monoids

Let us first consider a finite subfamily of frames $\mathcal{F}' \doteq \langle S\rangle_{A_{1..4,7}}$ for some $S \in \mathcal{F}$.

Theorem 4. *The internal operation $\otimes$ of minimal refinement satisfies the following properties:*

1. associativity*:* $\Theta_1 \otimes (\Theta_2 \otimes \Theta_3) = (\Theta_1 \otimes \Theta_2) \otimes \Theta_3, \ \forall\, \Theta_1, \Theta_2, \Theta_3 \in \mathcal{F}'$*;*
2. commutativity*:* $\Theta_1 \otimes \Theta_2 = \Theta_2 \otimes \Theta_1, \ \forall \Theta_1, \Theta_2 \in \mathcal{F}'$*;*
3. existence of unit*:* $\Theta \otimes 1 = \Theta, \ \forall \Theta \in \mathcal{F}'$*;*
4. annihilator*:* $\Theta \otimes S = S, \ \forall \Theta \in \mathcal{F}'$

where 1 is the unique frame of $\mathcal{F}'$ containing a single element. In other words, a finite family of frames of discernment $(\mathcal{F}', \otimes)$ is a finite commutative monoid with annihilator *with respect to the internal operation of minimal refinement.*

Proof. Associativity and commutativity. If we look at expression 1, associativity and commutativity follow from the analogous property of set-theoretic intersection.
Unit. Let us prove that there exists a *unique* frame in $\mathcal{F}'$ with cardinality 1. Given $\Theta \in \mathcal{F}'$ due to Axiom $A4$ (existence of coarsenings) there exists a coarsening 1_Θ of Θ and a refining $\omega_{1_\Theta \Theta} : 2^{1_\Theta} \longrightarrow 2^\Theta$. Then following rule $A1$ there exists another refining $1_\omega \in \mathcal{R}'$ such that

$$1_\omega : 2^{1_\Theta} \longrightarrow 2^S.$$

Now, considering a pair of elements of $\mathcal{F}'$, say $\Theta_1, \ \Theta_2$, following the above procedure we get two pairs set-refining $(1_{\Theta_i}, 1_{\omega_i})$ with $1_{\omega_i} : 2^{1_{\Theta_i}} \to 2^S$. But if we call 1_{θ_i} the unique element of 1_{Θ_i}

$$1_{\omega_i}(\{1_{\theta_i}\}) = S \ \forall i = 1, 2$$

so that for Axiom $A2$ (identity of coarsenings) the uniqueness follows: $1_{\omega_1} = 1_{\omega_2}, \quad 1_{\Theta_1} = 1_{\Theta_2}$.
Annihilator. If $\Theta \in \mathcal{F}'$ then obviously Θ is a coarsening of S, therefore their minimal refinement coincides with S itself. □

5.1.1 Isomorphism frames-refinings

A family of frames can be dually viewed as a set of refining maps with attached their domains and codomains (perhaps the most correct approach for it takes account of sets automatically).
We can establish the following correspondence:

$$\Theta \longleftrightarrow \omega_S^\Theta : 2^\Theta \to 2^S. \tag{4}$$

Definition 8. *Given $\omega_1 : 2^{\Theta_1} \to 2^S$ and $\omega_2 : 2^{\Theta_2} \to 2^S$ their composition induced by the operation of minimal refinement is the unique (for Axiom $A3$) refining from $\Theta_1 \otimes \Theta_2$ to S*

$$\omega_1 \otimes \omega_2 : 2^{\Theta_1 \otimes \Theta_2} \to 2^S.$$

Theorem 5. *The subcollection of the refinings of a finite family of frames $\langle S\rangle_{A_{1..4,7}}$ with codomain S*

$$\langle S\rangle_{A_{1..4,7}}^{\omega_S^{(\cdot)}} \doteq \{\omega_S^\Theta, \ \Theta \in \langle S\rangle_{A_{1..4,7}}\}$$

is a finite commutative monoid with annihilator with respect to the above operation.

Proof. Associativity and commutativity comes from the analogous property of the minimal refinement operator. The unit element is $1_\omega : 2^{1_S} \to 2^S$, in fact

$$Dom(1_\omega \otimes \omega) = 1_S \otimes Dom(\omega) = Dom(\omega) \Rightarrow 1_\omega \otimes \omega = \omega$$

for Axiom $A2$. On the other side if we consider $0_\omega : 2^S \to 2^S$ the unique refining from S onto itself (existing from Axiom $A4$ with $n = 1$) we have

$$Dom(0_\omega \otimes \omega) = S \otimes Dom(\omega) = S$$

so that 0_ω is an annihilator for $\langle S\rangle^{\omega_S^{(.)}}_{A_{1..4,7}}$. □

From the above proof it easily follows that

Proposition 2. *Given a finite family of compatible frames, the map (4) is an isomorphism between commutative monoids.*

It is interesting to note that both the existence of unit element and annihilator in a finite family of frames are consequences of the following Proposition:

Proposition 3. $(\omega_S^\Theta \circ \omega_\Theta^\Omega) \otimes \omega_S^\Theta = \omega_S^\Theta \quad \forall\, \Omega, \Theta.$

Proof. $\mathrm{Cod}(\omega_S^\Theta \circ \omega_\Theta^\Omega) = \mathrm{S}$ so the operation $\otimes$ can be applied. By noting that
$Dom(\omega_S^\Theta \circ \omega_\Theta^\Omega) = \Omega$ is a coarsening of $Dom(\omega_S^\Theta) = \Theta$ we get

$$Dom((\omega_S^\Theta \circ \omega_\Theta^\Omega) \otimes \omega_S^\Theta) =$$

$$= Dom(\omega_S^\Theta \circ \omega_\Theta^\Omega) \otimes Dom(\omega_S^\Theta) = Dom(\omega_S^\Theta) = \Theta$$

and from Axiom $A3$ the thesis follows. □

In fact, when $\Omega = 1_S$ then $\omega_S^\Theta \circ \omega_\Theta^\Omega = \omega_S^\Theta \circ \omega_\Theta^{1_S} = 1_\omega$ and we get $1_\omega \otimes \omega_S^\Theta = \omega_S^\Theta$. On the other side, when $\Theta = S$ then $\omega_S^\Theta = \omega_S^S = 0_\omega$ and we have the annihilation property $\omega_S^\Omega \otimes 0_\omega = 0_\omega$.

5.1.2 Generators

Given a monoid $\mathcal{M}$, the submonoid $\langle S\rangle$ generated by a subset S is defined as the intersection of all the submonoids of $\mathcal{M}$ containing S.

Definition 9. *The set of* generators *of a monoid $\mathcal{M}$ is a finite set S such that $\langle S\rangle = \mathcal{M}$.*

Theorem 6. *The set of generators of the finite family $\langle S\rangle_{A_{1..4,7}}$ seen as finite commutative monoid with respect to the internal operation $\otimes$ (minimal refinement) is the collections of all the binary frames. The set of generators of the finite family $\langle S\rangle^{\omega_S^{(.)}}_{A_{1..4,7}}$ is the collection of refinings from all the binary partitions of S to S itself:*

$$\langle S\rangle_{A_{1..4,7}} = \langle\{\omega_{ij} : 2^{\Theta_{ij}} \to 2^S |\, |\Theta_{ij}| = 2,\ \omega_{ij} \in \mathcal{R}\}\rangle.$$

Proof. We need to prove that all the possible partitions of S can be obtained as minimal refinement of a number of binary partitions.
Let us consider a generic partition $\Pi = \{S^1, ..., S^n\}$ and define

$$\Pi_1 = \{S^1, S^2 \cup \cdots \cup S^n\} \doteq \{A_1, B_1\}$$

$$\Pi_2 = \{S^1 \cup S^2, S^3 \cup \cdots \cup S^n\} \doteq \{A_2, B_2\} \quad \cdots$$

$$\Pi_{n-1} = \{S^1 \cup \cdots \cup S^{n-1}, S^n\} \doteq \{A_{n-1}, B_{n-1}\}.$$

It is not difficult to see that every arbitrary intersection of elements of $\Pi_1, ..., \Pi_{n-1}$ is an element of the n-ary partition Π, in fact

$$A_i \cap B_k = \emptyset,\ k \geq i \qquad A_i \cap A_k = A_i,\ k \geq i$$

$$B_i \cap B_k = B_k,\ k \geq i$$

so that $\bigcap_i A_i = A_1 = S^1$, $\bigcap_i B_i = B_{n-1} = S^n$. If both As and Bs are present the resulting intersection is $\emptyset$ whenever there exist a pair B_l, A_m with $l \geq m$. Hence the only non-empty mixed intersections in the class $\{X_1 \cap \cdots \cap X_n\}$ must necessarily be of the following kind

$$B_1, ..., B_k, A_{k+1}, ..., A_{n-1}$$

and we have

$$B_1 \cap \cdots \cap B_k \cap A_{k+1} \cap \cdots \cap A_{n-1} =$$

$$= (S^{k+1} \cup \cdots \cup S^n) \cap (S^1 \cup \cdots \cup S^{k+1}) = S^{k+1}$$

where $k+1$ goes from 2 to $n-1$. That satisfies the fundamental condition for Π to be the minimal refinement of $\Pi_1, ..., \Pi_{n-1}$ (note that the choice of the binary frames is not unique).
The second part of the proof, concerning the refining maps, comes directly from the existence of the isomorphism (4). □

5.2 General families as commutative monoids

We can ask whether a general family of frames maintain the algebraic structure of monoid. The answer is positive.

Theorem 7. *$\mathcal{F}$ is an* infinite commutative monoid without annihilator.

Proof. The proof of Theorem 4 holds for the first two points.
Existence of unit. Suppose there exist two frames $\mathbf{1} = \{1\}$ and $\mathbf{1}' = \{1'\}$ with size 1. For Axiom $A6$ they have a common refinement Θ, with $\omega_1 : 2^1 \to 2^\Theta$ and

$\omega_{1'} : 2^{1'} \to 2^{\Theta}$ refinings. But then $\omega_1(\{1\}) = \Theta = \omega_{1'}(\{1'\})$, and by Axiom $A2$ $\mathbf{1} = \mathbf{1}'$. Now, for every frame $\Omega \in \mathcal{F}$ Axiom $A4$ ensures that there exists a partition of Ω with only one element, $\mathbf{1}_\Omega$. For the above argument, $\mathbf{1}_\Omega = \mathbf{1}$: in conclusion there is only one monic frame in $\mathcal{F}$, and since it is refinement of every other frame, it is the unit element with respect to $\otimes$.
About the fourth point, supposing a frame 0_Θ exists such that $0_\Theta \otimes \Theta = 0_\Theta$ for each Θ we would have, given a refinement Ω of 0_Θ (built by means of Axiom $A5$), $0_\Theta \otimes \Omega = \Omega$ that is a contradiction. □

In a complete family whatever basis set S you choose there are refinings with codomain distinct from S, so that it is impossible to write down a correspondence among frames and maps.
Nevertheless, if we note that given two maps ω_1 and ω_2 their codomains Ω_1, Ω_2 always have a common refinement Ω, we can write

$$\omega_1 \otimes \omega_2 \doteq \omega_1' \otimes \omega_2' \tag{5}$$

where, calling w_1 (w_2) the refining map between Ω_1 (Ω_2) and Ω,

$$\omega_1' = w_1 \circ \omega_1, \; \omega_2' = w_2 \circ \omega_2$$

and $\otimes$ on the right side of equation 5 stands for the composition of refinings in the finite family $\langle \Omega \rangle_{A_{1..4,7}}$ generated by Ω. That way the operation $\otimes$ is again well-defined. The composition of refinings can be resembled in a more elegant way, showing us its validity even for general families.

Definition 10. *Given two maps $\omega_1, \omega_2 \in \mathcal{R}$, $\omega_1 : 2^{\Theta_1} \longrightarrow 2^{\Omega_1}$ and $\omega_2 : 2^{\Theta_2} \longrightarrow 2^{\Omega_2}$ we define*

$$\omega_1 \otimes \omega_2 : 2^{\Theta_1 \otimes \Theta_2} \longrightarrow 2^{\Omega_1 \otimes \Omega_2}. \tag{6}$$

This operation is well-defined, for the correspondence

$$(Dom(\omega), Cod(\omega)) \longleftrightarrow \omega \tag{7}$$

(guaranteed by Axiom $A3$) is a bijection.

Theorem 8. *The set of refinings $\mathcal{R}$ of a complete generated family of frames is a commutative monoid with respect to the internal operation* (6).

Proof. Obviously $\otimes$ is commutative and associative for the commutativity and associativity of the operation of minimal refinement among frames.
To find the unit element it suffices to note that $1_\omega = (\Theta, \Omega)$ such that

$$\Theta \otimes \Theta_1 = \Theta_1 \; \forall \Theta_1 \in \mathcal{F} \; \wedge \; \Omega \otimes \Omega_1 \; \forall \Omega_1 \in \mathcal{F}$$

but that means $\Theta = \Omega = \mathbf{1}$ so that $1_\omega : 2^{\mathbf{1}} \to 2^{\mathbf{1}}$ and 1_ω turns out to be only the identity map on $\mathbf{1}$. □

Corollary 1. *$\mathcal{R}$ is a submonoid of the* product monoid $(\mathcal{F}, \otimes) \times (\mathcal{F}, \otimes)$ *through the map* (7).

5.3 Monoidal structure of the family

Let us complete the picture of the algebraic structures including in a family of frames, by specifying the missing relations among them. Clearly, $(\langle S \rangle^{\omega}_{A_{1..4,7}}, \otimes)$ (where $\langle S \rangle^{\omega}_{A_{1..4,7}}$ is the collection of *all* the refinings of a finite family) is a monoid, too, and

Proposition 4. *$(\langle S \rangle^{\omega_s^{(\cdot)}}_{A_{1..4,7}}, \otimes)$ is a submonoid of $(\langle S \rangle^{\omega}_{A_{1..4,7}}, \otimes)$.*

Proof. Obviously $\langle S \rangle^{\omega_s^{(\cdot)}}_{A_{1..4,7}} \subset \langle S \rangle^{\omega}_{A_{1..4,7}}$ in a set-theoretic sense. We only have to prove that the internal operation of the first set is inherited from the second one. If $\omega_1 : 2^{\Theta_1} \to 2^S$ and $\omega_2 : 2^{\Theta_2} \to 2^S$ then $S \otimes S = S$ and

$$\omega_1 \otimes_{\langle S \rangle^\omega} \omega_2 : 2^{\Theta_1 \otimes \Theta_2} \to 2^{S \otimes S} = 2^S \;\equiv\; \omega_1 \otimes_{\langle S \rangle^{\omega_s^{(\cdot)}}} \omega_2$$

□

Proposition 5. *$(\langle S \rangle^{\omega}_{A_{1..4,7}}, \otimes)$ is a submonoid of $(\mathcal{R}, \otimes)$.*

Proof. It suffices to prove that $\langle S \rangle^{\omega}_{A_{1..4,7}}$ is closed with respect to composition operator (6). But then, given two maps ω_1, ω_2 whose domains and codomains are both coarsening of S $Dom(\omega_1 \otimes \omega_2) = Dom(\omega_1) \otimes Dom(\omega_1)$ and $Cod(\omega_1 \otimes \omega_2) = Cod(\omega_1) \otimes Cod(\omega_1)$ are still coarsenings of S. □

Proposition 6. *$(\langle S \rangle^{\Theta}_{A_{1..4,7}}, \otimes)$ is a submonoid of $(\mathcal{F}, \otimes)$.*

Proof. Obvious, for the finite family is strictly included into the complete one and $\langle S \rangle^{\Theta}_{A_{1..4,7}}$ is closed with respect to $\otimes$, i.e. if Θ_1 and Θ_2 are coarsenings of S then their minimal refinement is still a coarsening of S. □

The following diagram summarize the relations among the monoidal structures associated to a family of compatible frames $(\mathcal{F}, \mathcal{R})$.

$$\begin{array}{ccccc}
(\langle S \rangle^{\omega_s^{(\cdot)}}_{A_{1..4,7}}, \otimes) & \subset & (\langle S \rangle^{\omega}_{A_{1..4,7}}, \otimes) & \subset & (\mathcal{R}, \otimes) \\
\wr & & \wr & & \\
(\langle S \rangle^{\Theta}_{A_{1..4,7}}, \otimes) & \subset & \begin{array}{c} (\langle S \rangle^{\Theta}_{A_{1..4,7}}, \otimes) \\ \times \\ (\langle S \rangle^{\Theta}_{A_{1..4,7}}, \otimes) \end{array} & & \cap \\
= & & & & \\
(\langle S \rangle^{\Theta}_{A_{1..4,7}}, \otimes) & \subset & (\mathcal{F}, \otimes) & \subset & \begin{array}{c} (\mathcal{F}, \otimes) \\ \times \\ (\mathcal{F}, \otimes) \end{array}
\end{array}$$

6 Lattice structure

It is well-known (see [5], page 456) that the internal operation of a monoid $\mathcal{M}$ induces an order relation among the elements of $\mathcal{M}$, namely $a|b \equiv \exists\, c\; s.t.\; b = a \cdot c$, In our case this becomes

$$\Theta_2 \geq \Theta_1 \equiv \exists\, \Theta_3\; s.t.\; \Theta_2 = \Theta_1 \otimes \Theta_3 \tag{8}$$

$\equiv \Theta_1 \otimes \Theta_2 = \Theta_2$, i.e. Θ_2 *is a refinement of* Θ_1.
Since both finite and general families of frames are monoids,

Proposition 7. $\langle S \rangle_{A_{1..4,7}}$ *and* $\mathcal{F}$ *are* partially ordered set *(poset) with respect to the order relation* (8).

Remark. It is interesting to note that Proposition (3) can be expressed in term of the order relation

$$\omega_2 \geq \omega_1 \Rightarrow \omega_1 \otimes \omega_2 = \omega_2.$$

Again, since $\omega_2 \geq \omega_1 \Leftrightarrow \omega_2 = \omega_1 \otimes \omega_3$ for some ω_3, it can rewritten as

$$\omega_1 \otimes (\omega_1 \otimes \omega_3) = (\omega_1 \otimes \omega_1) \otimes \omega_3 = \omega_1 \otimes \omega_3. \tag{9}$$

This means that $\omega \otimes \omega = \omega$ (*idempotence*) is a sufficient condition.

6.1 Minimal refinement as lower upper bound

In a poset the dual notions of *lower upper bound* and *greatest lower bound* of a pair of elements can be introduced.

Definition 11. *Given two elements* $x, y \in X$ *of a poset* X *their* least upper bound $\sup_X(x,y)$ *is the smallest element of* X *that is bigger than both* x *and* y, *i.e.* $\sup_X(x,y) \geq x, y$ *and*

$$if\; \exists z\; s.t.\; z \leq \sup_X(x,y),\; z \geq x,y \Rightarrow z = \sup_X(x,y).$$

Definition 12. *Given two elements* $x, y \in X$ *of a poset* X *their* greatest lower bound $\inf_X(x,y)$ *is the biggest element of* X *that is smaller than both* x *and* y, *i.e.* $\inf_X(x,y) \leq x, y$ *and*

$$if\; \exists z\; s.t.\; z \geq \inf_X(x,y),\; z \leq x,y \Rightarrow z = \inf_X(x,y).$$

The standard notation is $\inf(x,y) = x \wedge y$ and $\sup(x,y) = x \vee y$. By induction sup and inf can be defined for arbitrary finite collections too.
It must be pointed out that not any pair of elements of a poset admits inf and/or sup in general.

Definition 13. *A* lattice *is a poset in which any pair of elements admit both* inf *and* sup.

Definition 14. *An infinite lattice* L *is said* complete *if any arbitrary collection (even not finite) of points in* L *admits both* sup *and* inf.

In this case there exist $0 \equiv \wedge L$, $\quad 1 \equiv \vee L$ called respectively *initial* and *final* element of L.

Theorem 9. *In a family of frames* $\mathcal{F}$ *seen as a poset the sup of a finite collection of frames coincide with the minimal refinement,*

$$\sup_{\mathcal{F}}(\Theta_1, ..., \Theta_n) = \Theta_1 \otimes \cdots \otimes \Theta_n.$$

Proof. Of course $\Theta_1 \otimes \cdots \otimes \Theta_n \geq \Theta_i\; \forall i = 1, ..., n$ for there exists a refining between each Θ_i and the minimal refinement. Now, if exists another frame Ω greater than each Θ_i then Ω is a common refinement for $\Theta_1, ..., \Theta_n$, hence it is a refinement of the minimal refinement, i.e. $\Omega \geq \Theta_1 \otimes \cdots \otimes \Theta_n$ according to the order relation (8). $\square$

At a first glance is not clear what $\inf\{\Theta_1, ..., \Theta_n\}$ should represent.

6.2 Common coarsening

Let us introduce a new operation over finite collections of frames.

Definition 15. *A* common coarsening *of two frames* Θ_1, Θ_2 *is a set* Ω *such that* $\exists \omega_1 : 2^\Omega \rightarrow 2^{\Theta_1}$ *and* $\omega_2 : 2^\Omega \rightarrow 2^{\Theta_2}$ *refinings, i.e. is a coarsening of both the frames.*

Theorem 10. *If* $\Theta_1, \Theta_2 \in \mathcal{F}$ *are elements of a family of compatible frames then they have a common coarsening.*

Proof. From the proof of Theorem 7 they have at least **1** as a common coarsening. $\square$

6.3 Maximal coarsening

Theorem 11. *Given a collection* $\Theta_1, ..., \Theta_n$ *of elements of a family of compatible frames* $\mathcal{F}$ *there exists a unique element* $\Omega \in \mathcal{F}$ *such that:*

1. $\forall i$ *there exists a refining* $\omega_i : 2^\Omega \rightarrow 2^{\Theta_i}$;
2. $\forall \theta \in \Omega \quad \nexists\, A_1 \subseteq \omega_1(\{\theta\}), ..., A_n \subseteq \omega_n(\{\theta\})$ *s.t.* $\eta_1(A_1) = ... = \eta_n(A_n)$

where $\eta_i : 2^{\Theta_i} \rightarrow 2^{\Theta_1 \otimes \cdots \otimes \Theta_n}$.

We first need a simple intermediate result.

Lemma 2. *Suppose* $\Theta_1 \otimes \cdots \otimes \Theta_n$ *is the minimal refinement of* $\Theta_1, ..., \Theta_n$, *with refinings* $\eta_i : 2^{\Theta_i} \rightarrow$

$2^{\Theta_1 \otimes \cdots \otimes \Theta_n}$, *and there exist* $X_1 \subset \Theta_1, ..., X_n \subset \Theta_n$ *with* $\eta_1(X_1) = \cdots = \eta_n(X_n)$ *such that* $\not\exists\ A_1 \subseteq X_1, ..., A_n \subseteq X_n$ *s.t.* $\eta_1(A_1) = ... = \eta_n(A_n)$. *Then for every common coarsening* Θ *of* $\Theta_1, ..., \Theta_n$, *with refinings* $\omega_i : 2^\Theta \to 2^{\Theta_i}$, *there exists* $\theta \in \Theta$ *such that* $X_1 \subset \omega_i(\{\theta\})$.

Proof. Let us suppose there not exists such an element θ, but A_i is covered by a subset $\{\theta_1, ..., \theta_k\} \subset \Theta$. If we consider one of these elements $\bar{\theta}$ we have

$$\eta_i(\omega_i(\bar{\theta}) \cap X_i) = \eta(\omega_i(\bar{\theta})) \cap \eta_i(X_i)$$

but then $\eta_1(\omega_1(\bar{\theta})) = ... = \eta_n(\omega_n(\bar{\theta}))$ by definition of common coarsening, and $\eta_1(X_1) = ... = \eta_n(X_n)$ for hypothesis, so that

$$\eta_1(\omega_1(\bar{\theta}) \cap X_1) = ... = \eta_n(\omega_n(\bar{\theta}) \cap X_n)$$

with $A_i \doteq \omega_i(\bar{\theta}) \cap X_i \subseteq X_i\ \forall i = 1, ..., n$ that is a contradiction. □

Now we can afford the proof of Theorem 11.

Proof. Existence. The proof is constructive. Let us take an arbitrary coarsening $\mathcal{L}$ of $\Theta_1, ..., \Theta_n$ (existing for Theorem 10) and check for every $l \in \mathcal{L}$ whether there exists a collection of subsets $\{A_i \subset \omega_i(\{l\}),\ i = 1, ..., n\}$ such that $\eta_1(A_1) = ... = \eta_n(A_n)$. If the answer is negative we have the desired frame. Otherwise we can build a new common coarsening $\mathcal{L}'$ of $\Theta_1, ..., \Theta_n$ by simply splitting $\{l\}$ into a pair $\{l_1, l_2\}$ where

$$\omega_i'(\{l_1\}) = A_i\ \forall i, \quad \omega_i'(\{l_2\}) = B_i$$

having called $B_i = \omega_i(\{l\}) \setminus A_i$. This can be done for if $\omega_i(\{l\}) \setminus A_i \neq \emptyset$ for some i then it is not void $\forall i$.
This procedure can be iterated until there are not subsets satisfying condition 2. It terminates for the number of possible bijections of the images $\omega_i(\{l\})$ is finite. More precisely, the maximum number of steps is

$$\lceil \log_2 \max_{l \in \mathcal{L}} \min_{i=1,...,n} |\omega_i(\{l\})| \rceil.$$

Uniqueness. Suppose Ω' is another common coarsening satisfying condition 2, with refinings $\omega_i' : 2^{\Omega'} \to 2^{\Theta_i}$, distinct from Ω. If we define $X_i \doteq \omega_i(\{\theta\})$, $\theta \in \Omega$, for Lemma 2 there exists $\theta' \in \Omega'$ such that $\omega_i(\{\theta\}) \subset \omega_i'(\{\theta'\})$. But then condition 2 implies $\omega_i(\{\theta\}) = \omega_i'(\{\theta'\})$ for every pair θ, θ' so that $\Omega = \Omega'$. □

Definition 16. *This unique frame is denoted* $\Theta_1 \oplus ... \oplus \Theta_n$ *and is called* maximal coarsening *of* $\Theta_1, ..., \Theta_n$.

6.4 Maximal coarsening as greatest lower bound

Theorem 12. *If* Ω *is a common coarsening of a finite set of compatible frames* $\Theta_1, ..., \Theta_n$ *then* Ω *is coarsening of their maximal coarsening too, i.e. there exists a refining* $\omega : 2^\Omega \to 2^{\Theta_1 \oplus \cdots \oplus \Theta_n}$.

Proof. Let us consider another common coarsening Ω' of $\Theta_1, ..., \Theta_n$, with $\alpha_i : 2^{\Omega'} \to 2^{\Theta_i}$. If it satisfies condition 2 of Theorem 11 then for the uniqueness of $\Theta_1 \oplus \cdots \oplus \Theta_n$ it coincides with the maximal coarsening. Otherwise the procedure of the above proof can be used to produce such a frame. Again, uniqueness guarantees this is the maximal coarsening and by construction it is a refinement of Ω'. □

In other words,

Corollary 2. *The maximal coarsening* $\Theta_1 \oplus \cdots \oplus \Theta_n$ *of a collection of frames* $\Theta_1, ..., \Theta_n$ *coincides with the greatest lower bound of the collection of frames as elements of the poset* $(\mathcal{F}, \geq)$.

7 Semimodularity

Proposition 1 and Theorems 9, 11 and 12 have a straightforward consequence on the algebraic structure of $\mathcal{F}$.

Corollary 3. *The collection of sets* $\mathcal{F}$ *of a family compatible frames of discernment is a lattice, with respect to the internal operations of minimal refinement and maximal coarsening.*

It must be pointed out that $\mathcal{F}$ lacks the attribute of completeness: the axioms do not guarantee the existence of a minimal refinement for an infinite (even enumerable) collection of sets. The basic notions about lattices and other algebraic structures are exposed in [10] and [11].

Definition 17. *A lattice* L *is modular iff if* $a \geq b$, $a \wedge c = b \wedge c$, $a \vee c = b \vee c$ *then* $a = b$.

Definition 18. *A lattice* L *is* semimodular *(following* [5]*) when if* $a \vee b \succ a, b \Rightarrow a, b \succ a \wedge b$, *where* $x \succ y$ *(*x *"covers"* y*) means that* $x > y$ *and* $\nexists z : x > z > y$.

A poset is said to have *finite length* if the length of its chains (collections of consecutive elements) is limited.

Theorem 13. *A finite family of compatible frames* $(\langle S \rangle_{A_1,..,4,7}, \geq)$ *is a* semimodular complete lattice of finite length.

Proof. Proof. $(\langle S \rangle_{A_1,..,4,7}, \geq)$ is *complete.* Actually each finite lattice is complete for it does not contain any infinite collection of elements.

$(\langle S\rangle_{A_1,..,4,7}, \geq)$ is *semimodular*. Let us suppose Θ_1 and Θ_2 having their maximal coarsening covering them. Necessarily $\Theta_1 \oplus \Theta_2$ must have rank

$$|\Theta_1 \oplus \Theta_2| = |\Theta_1| + 1 = |\Theta_2| + 1,$$

so the refinings $\omega_1 : 2^{\Theta_1 \oplus \Theta_2 \to 2^{\Theta_1}}$ and $\omega_2 : 2^{\Theta_1 \oplus \Theta_2 \to 2^{\Theta_2}}$ leave inalterate each point but one, replaced by two new elements.
Now, on the other side Θ_1 and Θ_2 represent partitions of their minimal refinement; by construction these partitions coincide but the elements obtained by refining these two points, as showed in Figure 1. Analytically

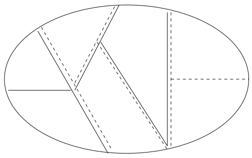

Figure 1: Example of partitions like Π_1 (solid line) and Π_2 (dashed line) in Theorem 13.

$$\Pi_1 = \{p'_1, p'_2, p_3, ..., p_n\},\ \Pi_2 = \{p_1, ..., p_{n-2}, p''_{n-1}, p''_n\}$$

having denoted the minimal refinement as $\{p_1, ..., p_n\}$, whose rank is then equal to $|\Theta_1 \otimes \Theta_2| = |\Theta_1| - 1 = |\Theta_2| - 1$. It is not difficult to see that it is covered by both the frames for there cannot exist any frame with intermediate size among Θ_1 or Θ_2 and $\Theta_1 \otimes \Theta_2$. □

For an alternative proof see [11]. It is interesting

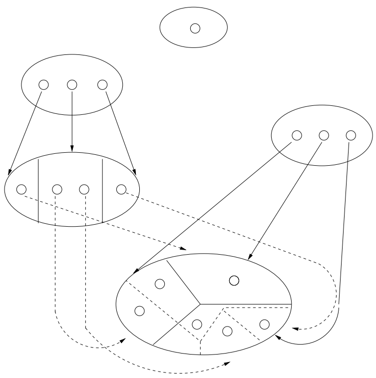

Figure 2: Non-modularity of finite families of frames: counterexample.

to note that finite lattices of frames *are not modular*: Figure 2 shows a simple counterexample, where the two frames on the left, linked by a refinings, have both the same minimal refinement and maximal coarsening (the unit frame). Instead, looking at the proof of Theorem 13 we observe that it states the semimodularity of *general* families of frames, too. In fact semimodularity is a *local* property inheriting the sublattice $[\Theta_1 \oplus \cdots \oplus \Theta_n, \Theta_1 \otimes \cdots \otimes \Theta_n]$.

Corollary 4. *The collection of sets $\mathcal{F}$ of a family of compatible frames is a* locally Birkhoff lattice *bounded below, i.e. is a semimodular lattice of locally finite length with initial element.*

Proof. It remains to point out that for Theorem 7 every arbitrary collection of frames in $\mathcal{F}$ admit a common coarsening **1**, that plays the role of initial element of the lattice. □

8 External and internal independence

Birkhoff lattices are exactly the minimum algebraic structure where the concept of *independence* ([11]) can be introduced.

Definition 19. *Consider the collection $\mathcal{F}(M)$ of all the finite subsets of a given set M and define in the product $M \times \mathcal{F}(M)$ a relation Λ. Λ is said to be a* linear dependence *on the set M if it satisfies the following conditions:*

1. *$p_j\ \Lambda\ \{p_1, ..., p_m\}$, $j = 1, ..., n$;*
2. *if $p\ \Lambda\ \{p_1, ..., p_m\}$ and $\forall j\ p_j\ \Lambda\ \{q_1, ..., q_n\}$, then $p\ \Lambda\ \{q_1, ..., q_n\}$;*
3. *if $p\ \Lambda\ \{p_1, ..., p_m, q\}$ but $p\ \Lambda'\ \{p_1, ..., p_m\}$, then $q\ \Lambda\ \{p_1, ..., p_m, p\}$.*

Definition 20. *If L is a finite length semimodular lattice bounded below, the* rank *$h(a)$ of an element a is the length of the interval $I[0, a]$ (i.e. the sublattice $\{x \in L : 0 \leq x \leq a\}$).*

It can be proved that (see [1], Chapter 6)

Theorem 14. *If L is a Birkhoff lattice bounded below, the equivalence relation $\mathcal{LD} \subset L \times \mathcal{F}(L)$ defined as*

$$p\ \mathcal{LD}\ \{p_1, ..., p_n\} \equiv$$

$$\equiv h(p \vee p_1 \vee \cdots \vee p_n) < h(p_1 \vee \cdots \vee p_n) + h(p) \tag{10}$$

where $p, p_1, ..., p_n \in L$ and h is the rank of the elements of the lattice is a linear dependence *relation.*

Theorem 15. *$\Theta_1, ..., \Theta_n$ are independent as compatible frames iff they are linear independent as elements of a Birkhoff lattice.*

Proof. Condition 4 of Theorem 1 is equivalent to $h(\Theta_1 \otimes \cdots \otimes \Theta_n) = h(\Theta_1) + ... + h(\Theta_n)$ where $h = \lg(|\Theta|)$ is the rank of Θ as element of the lattice $\mathcal{F}$. When we note that

$$p\ \overline{\mathcal{LD}}\ \{p_1, ..., p_n\} \equiv h(p \vee p_1 \vee \cdots \vee p_n) =$$

$$= h(p_1 \vee \cdots \vee p_n) + h(p) = h(p_1) + ... + h(p_n) + h(p)$$

and for the lattice $\mathcal{F}$, $\Theta \vee \Omega = \Theta \otimes \Omega$, the thesis immediately follows. □

9 Pseudo Gram-Schmidt algorithm

We are now able to treat conflicting measurements from an algebraic point of view. We know that:

- vector subspaces of a finite-dimensional vector space V and compatible families of frames share the structure of *complemented semimodular lattice* ($L(V)$ in particular is a modular lattice);
- the elements of these lattices (subspaces and frames) are *commutative monoids.*

A vector space, indeed, is a group with respect to the operation "+" and a frame is a Boolean algebra, i.e. a complete distributive lattice, hence they are both commutative monoids ([5]). This remark suggests a possible method to solve the conflict among measurement belief functions without resorting to the conflict graph mentioned in Section 3.
The well known Gram-Schmidt orthonormalization procedure is able to select a collection of independent vectors from an arbitrary one, resting on the *independence* condition and the *projection* of vectors onto other subspaces. Analogously, a pseudo Gram-Schmidt method, starting from a set of belief functions defined over a finite collection of discrete feature spaces, would detect another collection of *independent* frames of the same family

$$\Theta_1, ..., \Theta_n \in \mathcal{F} \longrightarrow \Theta'_1, ..., \Theta'_m \in \mathcal{F}$$

$$s_1, ..., s_n \mapsto s'_1, ..., s'_m$$

with $m \neq n$ in general. Once projected the n b.f.s $s_1, ..., s_n$ onto the new set of frames we would achieve a set of *surely combinable* feature data $s'_1, ..., s'_m$, *equivalent* to the previous one.
Although we think the proposed method is the most elegant and rigorous approach to cope with groups of conflicting belief functions, a lot of details needs to be investigated prior to a practical application to the information integration problems like the object tracking task we discussed in Section 3.
For instance, a precise definition of the *projection operator* in the context of semimodular lattices of commutative monoids has to be formulated. Furthermore power sets are not groups, so we cannot simply subtract the projected functions like in the original G-S algorithm.
Other interesting questions concern the meaning of the obtained collection of independent frames (in the object tracking context, we could ask whether they represent actual features) and the order in which the input data is "orthogonalized".

10 Conclusions and future work

In this work we have seen how a rigorous, algebraic description of the families of frames could open the possibility of solving the problem of conflicting evidence by means of algebraic tools. In particular, it seems useful to investigate the feasibility of a pseudo Gram-Schmidt algorithm, able to rearrange a set of belief functions into a new set of surely compatible b.f.s over independent frames.
Of course, the implications of this approach are far more extended, since many concepts of the theory of evidence are inherently connected to the structure of the underlying domains. For example, the notion of *support function* depends on the idea of refining and could quite likely be reformulated using the algebraic language. Its analysis in the light of the lattice structure of $\mathcal{F}$ can lead eventually to an alternative solution of the *canonical decomposition* problem.
Other properties of the frame lattices still need to be investigated: the presence and nature of complements and semi-complements, intervals and the relations of filters and ideal with finite and general families of frames.

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