An overview of economic applications of David Schmeidler's models of decision making under uncertainty *

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July 28, 2003

Abstract

This paper surveys some economic applications of the decision theoretic framework pioneered by David Schmeidler to model effects of ambiguity. We have organized the discussion principally around three themes: financial markets, contractual arrangements and game theory. The first section discusses papers that have contributed to a better understanding of financial market outcomes based on ambiguity aversion. The second section focusses on contractual arrangements and is divided into two sub-sections. The first sub-section reports research on optimal risk sharing arrangements, while in the second sub-section, discusses research on incentive contracts. The third section concentrates on strategic interaction and reviews several papers that have extended different game theoretic solution concepts to settings with ambiguity averse players. A final section deals with several contributions which while not dealing with ambiguity *per se*, are linked at a formal level, in terms of the pure mathematical structures involved, with Schmeidler's models of decision making under ambiguity. These contributions involve issues such as, inequality measurement, intertemporal decision making and multi-attribute choice.

JEL classification number: D81

Keywords: Ellsberg Paradox, ambiguity aversion, uncertainty aversion

1 Introduction

What do ambiguity averse decision makers do when they are not picking balls out of urns–when they find themselves in contexts that are "more realistic" in terms of economic institutions involved? In this part, the reader is provided with a sample of economic applications of the decision theoretic framework pioneered by David Schmeidler. Indeed, decision theoretic models are designed, at least in part, to eventually be used to answer questions about behavior and outcomes in interesting economic environments. Does it make a difference for the outcome of a given game, or market interaction or contractual arrangement if we were to assume that decision makers are ambiguity averse rather than Bayesian? What kind of insights are gained by introducing ambiguity averse agents in our economic models? What are the phenomena that can be explained in the "ambiguity aversion paradigm" that did not have a (convincing) explanation in the expected utility framework? Do equilibrium conditions (e.g., rational expectations equilibrium or any sort of equilibrium in a game) that place constraints on agents' beliefs rule out certain types of beliefs and attitudes towards uncertainty? These are but a few of the questions that the contributions collected in this part of the volume have touched upon. In this introduction we discuss these contributions along with several other papers which, while not

*This paper is to appear in the forthcoming volume, "*Uncertainty in Economic Theory: A collection of essays in honor of David Schmeidler's 65th birthday*," edited by Itzhak Gilboa (to be published in June, 2004, by Routledge Publishers). We thank Thibault Gajdos and Itzakh Gilboa for helpful comments.

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included in the volume, make important related points and therefore play a significant role in the literature.

We have organized the discussion of economic applications principally around three themes: financial markets, contractual arrangements and game theory. In all these contexts, it is found that ambiguity aversion does make a difference in terms of qualitative predictions of the models and, furthermore, often provides an explanation of contextual phenomena that is, arguably, more straightforward and intuitive than that provided by the expected utility model. The first section discusses papers that have contributed to a better understanding of financial market outcomes based on ambiguity aversion. The second section focusses on contractual arrangements and is divided into two sub-sections. The first sub-section reports research on optimal risk sharing arrangements, while in the second sub-section, discusses research on incentive contracts. The third section concentrates on strategic interaction and reviews several papers that have extended different game theoretic solution concepts to settings with ambiguity averse players. A final section deals with several contributions which while not dealing with ambiguity per se, are linked at a formal level, in terms of the pure mathematical structures involved, with Schmeidler's models of decision making under ambiguity. These contributions involve issues such as, inequality measurement, intertemporal decision making and multi-attribute choice.

2 Financial market outcomes

In pioneering work, Dow and Werlang (1992) applied the Choquet expected utility model of Schmeidler (1989) to the portfolio choice and identified an important implication of Schmeidler's model. They showed, in a model with one risky and one riskless asset that there exists a non-degenerate price interval at which a CEU agent will strictly prefer to take a zero position in the risky asset (rather than to sell it short or to buy it). This constitutes a striking difference with an expected utility decision maker, for whom this price interval is reduced to a point (as known since Arrow (1965).

The intuition behind this finding may be grasped in the following example. Consider an asset that pays off 1 in state L and 3 in state H and assume that the DM is of the Choquet expected utility type with capacity $\nu(L) = 0.3$ and $\nu(H) = 0.4$ and linear utility function. The expected payoff (that is, the Choquet integral computed in a way explained in the introduction of this volume) of buying an unit of the risky asset (the act z_b) is given by $\mathbb{CE}_\nu(z_b) = 0.6 \times 1 + 0.4 \times 3 = 1.8$. On the other hand, the payoff from going short on an unit of the risky asset (the act z_s) is higher at L than at H. Hence, the relevant minimizing probability when evaluating $\mathbb{CE}_\nu(z_b)$ is that probability in the core of (ν) that puts most weight on H. Thus, $\mathbb{CE}_\nu(z_s) = 0.3 \times (-1) + 0.7 \times (-3) = -2.4$. Hence, if the price of the asset z were to lie in the open interval $(1.8, 2.4)$, then the investor would strictly prefer a zero position to either going short or buying. Unlike in the case of unambiguous beliefs there is no single price at which to switch from buying to selling. Taking a zero position on the risky asset has the unique advantage that its evaluation is not affected by ambiguity. The 'inertia' zone demonstrated by Dow and Werlang was a statement about optimal portfolio choice corresponding to *exogenously* determined prices, given an initially *riskless* position.[1] It leaves open the issue whether this could be an *equilibrium* outcome.

Epstein and Wang (1994) is the first paper that looked at an equilibrium model with multiple prior agents. The main contribution is twofold. First, they provide an extension of Gilboa and Schmeidler's multiple prior model to a dynamic (infinite horizon) setting. Gilboa and Schmeidler model is static and considers only one-shot choice. The extension to a dynamic setting poses the difficult problem of ensuring dynamic consistency of choices, together with the choice of a revision rule for beliefs that are not expressed probabilistically. The recursive approach developed in Epstein

[1] See Mukerji and Tallon (2003) for a derivation of this inertia property from a primitive notion of ambiguity without relying on a parametric preference model.

and Wang (1994) (and which was subsequently axiomatized by Epstein and Schneider (2003)) allows one to bypass these problems and ensures that intertemporal choices are dynamically consistent. The authors also provide techniques enabling one to find optimal solutions of such a problem; techniques that amount to generalizing Euler equalities and dealing with Euler *inequalities*.

Second, they develop a model of intertemporal asset pricing à la Lucas (1978). They thus construct a representative agent economy, in which the price of the assets are derived at equilibrium. One central result shows that asset prices can be indeterminate at equilibrium. Indeterminacy, for instance, would result when there are factors that influence the dividend process while leaving the endowment of the agent unchanged. In that case, we are back to the intuition developed in Dow and Werlang (1992): there exists a multiplicity of prices supporting the initial endowment. The economic important consequence of this finding is that large volatility of asset prices is consistent with equilibrium. Chen and Epstein (2002) develops the continuous-time counterpart of that in Epstein and Wang (1994). They show that excess returns for a security can be decomposed as a sum of a risk premium and an ambiguity premium. Epstein and Miao (2003) use this model to provide an explanation of the home-bias puzzle: when agents perceive domestic assets as non ambiguous and foreign asset as ambiguous, they will hold "too much" (compared to a model with probabilistic beliefs) of the latter.

The framework developed in Epstein and Wang (1994) has the feature that the equilibrium is Pareto optimal and, somewhat less importantly, the equilibrium allocation necessarily entails no-trade (given the representative agent structure). Mukerji and Tallon (2001) develops a static model with heterogeneous agents. They show that ambiguity aversion could be the cause of less than full risk sharing, and, consequently, of an imperfect functioning of financial markets. Actually, ambiguity aversion could lead to the absence of trade on financial markets. This could be perceived to be a direct generalization of Dow and Werlang (1992)'s no-trade price interval result. However, simply closing Dow and Werlang's model is not enough to obtain this result, as can be seen in an Edgeworth box. Hence, some other ingredient has to be inserted. Similar to the crucial ingredient leading to equilibrium price indeterminacy in Epstein and Wang (1994), what is needed here is the introduction of a component in asset payoffs that is independent of the endowments of the agents. Actually, one also needs to ensure that some component of an asset payoff is independent of both the endowments and the payoff of any other asset as well. Mukerji and Tallon (2001) prove that, when the assets available to trade risk among agents are affected by this kind of idiosyncratic risk, and if agents perceive this idiosyncratic component as being ambiguous and the ambiguity is high enough, then financial market equilibrium entails no trade at all and is suboptimal. This is to be contrasted with the situation in which agents do not perceive any ambiguity, in which standard replication and diversification arguments ensure that, eventually, full risk sharing is obtained and the equilibrium is Pareto optimal. Thus, ambiguity aversion is identified to cause market breakdown: assets are there to be traded, but agents, because of aversion towards ambiguity, prefer to hold on to their (sub-optimal) endowments, rather than bear the ambiguity associated to holding the assets. The absence of trade is of course an extreme result, which in particular is due to the fact that all the assets are internal assets. It would in particular be interesting to obtain results concerning the volume of trade, in particular if outside assets were present as well.

Building on a similar intuition, Mukerji and Tallon (2000) explain why unindexed debt is often preferred to indexed debt: the indexation potentially introduces some extra-risk in one's portfolio (essentially, the risk due to relative price variation of goods that appear in the indexation bundle but that the asset holder does not consume nor possess in his endowments). This provides further evidence that risk sharing and market functioning might be altered when ambiguity is perceived in financial markets.

Financial applications of the decision theoretic models developed by David Schmeidler are, of course, not limited to the ones reported above. There is by now a host of studies that address issues such as under-diversification (Uppal and Wang (2003), cross-sectional properties of asset prices in presence of uncertainty (Kogan and Wang (2002)), liquidity when the model of the economy is uncertain (Routledge and Zin (2001)). What is probably most needed now is an econometric evaluation of these ideas. Thus more work is needed to precisely assess how (non-probabilistic) uncertainty can be measured in the data. Some econometric techniques are being developed (see Henry (2001)) but applications are still rare.

In a series of contributions, L.P. Hansen, T .Sargent and coauthors[2] have developed an approach to understanding a decision maker's concern about "model uncertainty", which, although not formally building on Schmeidler's work, is based upon a similar intuition. The idea goes back to what the "rational expectations revolution" in macroeconomics wanted to achieve: that the econometrician modeler and the agents within the model be placed on equal footing concerning the knowledge they have of the model. This led to the construction of models in which agents have an exact knowledge of the model, in particular in which they knew the equilibrium price function. However, econometricians typically acknowledge that their models might be misspecified. Thus, Hansen and Sargent argue, these doubts should also be present in the agents' minds. They hence came to develop a model of robust decision making, wherein agents have a model in mind but also acknowledge the fact that this model might be wrong: they therefore want to take decisions that are robust against possible mis-specifications. Since a particular model implies a particular probability distribution over the evolution of the economic system, a consideration for robustness can be understood, in familiar terms of Schmeidler's decision theory, as a concern for the uncertainty about what probability distribution is a true description of the relevant environment. Wang (2003) examines the axiomatic foundation a related decision model and compares it with the multiple prior model.

The paper Anderson, Hansen, and Sargent (2003) included in the volume, belongs to this line of research and takes an important step in formulating the kind of model mis-specifications the decision maker may take into consideration. As the authors emphasize "a main theme of the present paper is to advocate a workable strategy for actually specifying those [Gilboa-Schmeidler] multiple priors in applied work". The analysis is based on the assumption that the agents, given that they have access to a limited amount of data, cannot discriminate among various models of the economy. This makes them value decision rules that perform well across a set of models. What is of particular interest is that this cautious behavior will show in the data generated by the model as an uncertainty premium incorporated in equilibrium security market prices, which goes towards an explanation of the equity premium puzzle.

3 Optimal contractual arrangements

3.1 Risk sharing

Optimal risk-sharing arrangements have been studied extensively, in contact theory, in general equilibrium analysis and so on. It is a priori not clear whether the risk-sharing arrangements that were optimal under risk remain so when one reconsiders their efficacy in the context of non-Bayesian uncertainty. Chateauneuf, Dana, and Tallon (2000) studies the way economy-wide risk-sharing is affected when agents behave according to the Choquet expected utility model. They show that the Pareto optimal allocations of an economy in which all agents are of the von-Neumann and Morgenstern type are still optimal in the economy in which agents behave according to Choquet expected utility provided all agents' beliefs are described by the same convex capacity. Things are however different when agents have different beliefs. To understand why, consider the particular case of betting: there is no aggregate uncertainty and agents have certain endowments. The only reason why they might be room for Pareto improving trade is if agents have different beliefs. This is the situation treated in Billot, Chateauneuf, Gilboa, and Tallon (2000). They show that in this case, Pareto optimal allocations are full insurance allocations (i.e., all agents have a constant across state consumption profile) if and only if the intersection of the cores of the capacities representing their beliefs is non-empty. This

[2] See for instance Hansen, Sargent, and Tallarini (1999), Hansen, Sargent, Turmuhambetova, and Williams (2001) and their forthcoming book Hansen and Sargent (2004)

is to be contrasted with the case in which agents have probabilistic beliefs: then, betting will take place as soon as agents have different beliefs, not matter how "small" this difference is. Thus, the fact that people do not bet against one another on many issues could be interpreted not as evidence that they have the same beliefs but rather that they have vague beliefs about these issues, and that vagueness is sufficiently large to ensure that agents have overlapping beliefs. Rigotti and Shannon (2002) look at similar issues when agents have "multiple priors with unanimity ranking" à la Bewley (1986) (see the introduction to this volume).

Mukerji and Tallon (2002) also considers a risk-sharing problem, but in the context of a wage contract. The paper studies the optimal degree of (price) indexation in a wage contract between an risk-averse employee and a risk neutral firm, given the presence of two types of price risk. The two types of risk are, an *aggregate* price risk, arising from an economy wide shock (possibly, monetary) that multiplies prices of all goods by the same factor, and *specific* risks, arising from demand/supply shocks to specific commodities that affects the price of a single good or a restricted class of goods. If contracting parties were SEU maximizers, an optimal wage contract will typically involve partial indexation (i.e., a certain fraction of wages, strictly greater than zero, will be index linked). However, this paper shows that if agents are ambiguity averse with CEU preferences and if they have an ambiguous belief about specific price risks involving goods that are neither in the employee's consumption basket nor in the firm's production set, then *zero* indexation coverage is optimal so long as the variability of inflation is anticipated to lie within a certain bound. What is crucial is the ambiguity of belief about specific price risks. The intuition for this result again rather simple: ambiguity averse workers will not want to bear the risk associated to changes in relative prices of the goods composing the indexation bundle if these changes are difficult to anticipate. Thus, even though indexation insures them against the risk of inflation, if this risk is well apprehended (which is the case in most countries where inflation is low and not very variable) workers prefer to bear this (known) risk to the ambiguous risk associated to relative price movements which are less predictable.

3.2 Incentive contracts

Typically, incentive contracts involve arrangements about contingent events. As such, the relevant trade-offs hinge crucially on the likelihoods of the relevant contingencies. Hence, it is a reasonable conjecture that the domain of contractual transactions is one area of economics that is significantly affected by agents' knowledge of the odds. Thus such contractual relations are a natural choice as a particular focus of the research on the principal economic effects of ambiguity aversion.

Why firms exist, what productive processes and activities are typically integrated within the boundaries of a firm, is largely explicated on the understanding that under certain conditions it is difficult or impossible to write supply and delivery contracts that are complete in relevant respects. A contract may be said to be incomplete if the contingent instructions included in the contract do not exhaust all possible contingencies; for some contingencies arrangements are left to be completed ex post. Incomplete contracts are, typically, inefficient. It is held, firms emerge to coordinate related productive activities through administrative hierarchies if such productive activities may only be inefficiently coordinated using incentives delivered through contracts, as would happen if conditions are such that the best possible contractual arrangements are incomplete. Mukerji (1998) shows that uncertainty and an ambiguity averse perception and attitude to this uncertainty is one instance of a set of conditions wherein the best possible contracts may be incomplete and inefficient. The formal analysis there basically involves a reconsideration of the canonical model of a vertical relationship (i.e., a relationship in which one firm's output is an input in the other firm's production activity) between two contracting firms under the assumption that the agents' *common* belief about the contingent events (which affect the gains from trade) is described by a convex capacity rather than a probability. A complete contract which is appropriate in the sense of being able to deliver enough incentives to the contracting parties to undertake efficient actions, will require that the payments from the contract be uneven across contingencies. For instance, the contract would reward a party in those contingencies which are more likely if the party takes the "right" action. However, the Choquet evaluation of such contract, for either party, may be low because the expected value of the contracted payoffs vary significantly across the different probabilities in the core of the convex capacity. Thus a null contract, an extreme example of an incomplete contract, may well be preferred to the contract which delivers "more appropriate" incentives. This would be so because the null contract would imply that the ex post surplus is divided by an uncontingent rule and, as such, deliver payoffs that are more even across contingencies thereby ensuring that the expected value is more robust to variation in probabilities. Hence, the best contractual agreement under ambiguity might not be a good one, in the sense of being unable deliver appropriate incentives, and therefore may be improved upon by vertical integration which delivers incentives by a hierarchical authority structure.

Why might an explanation like the one given above be of interest? A recurrent claim among business people is that they integrate vertically because of uncertainty in input supply, a point well supported in empirical studies (see discussion and references in Shelanski and Klein (1999)). The claim, however, has always caused difficulties for economists in the sense that it has been hard to rationalize on the basis of standard theory (see, for instance, remarks in Carlton (1979)). The analysis in the present paper explains how the idea of ambiguity aversion provides one precise understanding of the link between uncertainty and vertical integration.

In a related vein, Mukerji (2002) finds that ambiguity could provide a theoretical link between a very prevalent contractual practice in procurement contracting and uncertainty. The prevalent practice in question is the use of contracts which reimburse costs (wholly or partially) ex post and therefore provide very weak cost control incentives to the contractor. It is argued in that paper, while there is ample empirical evidence for this link, for instance in the case of research and development procurement by the U.S. Defense Department, the existing theoretical underpinnings for this link based on expected utility are less than plausible. It is worth pointing that the analyzes in both papers, Mukerji (1998) and Mukerji (2002), model firms as ambiguity averse entities. Economists have traditionally preferred to model firms as risk neutral, citing diversification opportunities. Based on formal laws of large numbers results proved in Marinacci (1999), and following upon the intuition in Mukerji and Tallon (2001), it may be conjectured that diversification opportunities, even in the limit, are not good enough to neutralize ambiguity, the way they can neutralize risk. The conjecture, to date, remains an interesting open question.

Intriguingly, the optimal contractual forms characterized in both Mukerji (1998) and Mukerji (2002) are instances where ambiguity aversion leads to contracts with low powered incentives. It is an interesting but open question as to how general this is. It has been widely observed that optimal contracts, say under moral hazard, as predicted on the basis of expected utility analysis are far more high powered than seen in the real world. It is an intriguing conjecture that high powered, fine tuned incentive contracts are not robust to the ambiguity about relevant odds and that optimal incentive schemes under, say, moral hazard would be far less complex when ambiguity considerations are taken into account than what they are in standard theory. While this question is an important one, finding an answer is not likely to be easy. Ghirardato (1994) investigated the principal agent problem under moral hazard with the player's preferences given by CEU. A significant finding there is that many of the standard techniques used in characterizing optimal incentive schemes in standard theory, do not seemingly work with CEU/MEU preferences. For instance, the Grossman-Hart "trick" of separating the principal's objective function into a revenue (from implementing a given action) component and a cost (of implementing a given action) component is not possible with CEU/MEU preferences. On a more positive note, the paper reports interesting findings about the comparative statics of the optimal incentive scheme with respect to changes in the agent's uncertainty aversion. One result shows that as uncertainty aversion decreases the agent will be willing too implement an arbitrary action for a *uniformly* lower incentive scheme. The point of interest is that this is in contrast with what happens for decreases in risk aversion. Typically, a decrease in risk aversion will have an asymmetric effect on contingent payments: it will make high payments higher and low payments lower.

4 Strategic interaction

In recent years non-cooperative game theory, the theory of strategic decision making, has come to be the basic building block of economic theory. Naturally, one of the first points of inspiration stimulated by Schmeidler's ideas was the question of incorporating these ideas into non-cooperative game theory. The general research question was, "What if players in a game were allowed to have beliefs and preferences as in the CEU/MEU model?" More particularly, there were at least three interrelated sets of questions: (1) a set of purely modeling/conceptual questions: e.g. how should solution concepts, such as strategic equilibrium, be defined given the new decision theoretic foundations; (2) questions about the general behavioral implications of the new solution concepts; (3) questions about insights such innovations might bring to applied contexts. The research so far has largely focussed on clarifying conceptual questions like defining the appropriate analogue of solution concepts like Nash equilibrium, and that too almost exclusively in the domain of strategic form games with complete information. Questions about the appropriate equilibrium concepts in incomplete information games and refinements of equilibrium in extensive form games remain largely unanswered. However, the progress on conceptual clarification has provided significant clues about behavioral implications and, in turn, has led to some important insights in some applied contexts.

One reason why the progress has been largely limited to complete information normal form games, is the host of supplementary questions that one has to face up to in order to tackle the question of defining equilibrium even in this simplest of strategic contexts. Defining a strategic equilibrium under ambiguity involves making several non-trivial modeling choices - namely, whether to use multiple priors or capacities to represent beliefs and if the latter what specific class of capacities, whether to allow for a strict preference for randomization, whether to fix actions explicitly in the description of the equilibrium, or whether, instead of explicitly describing actions, to simply describe the supports of the beliefs, and if the latter, what among the various possible notions of support to adopt (see Ryan (1999) for a perspective on this choice). Unsurprisingly, the definition of equilibrium varies across the literature, each definition involving a particular set of modeling choices.

Lo (1996) considers the question of an appropriate notion of strategic equilibrium, for normal form games, when players' preferences conform to the multiple priors MEU model. In Lo's conceptualization, equilibrium is a profile of beliefs (about other players' strategic choice) that satisfies certain conditions. To see the key ideas, consider a 2-player game. The component of the equilibrium profile that describes player i's belief, about strategic choice of player j, is a (convex) set of priors such that all j's strategies in the support of each prior are best responses for j given j's belief component in the equilibrium profile. (Lo also extends the concept to n-player games requiring players' beliefs to satisfy stochastic independence as defined in Gilboa and Schmeidler (1989).) This notion of equilibrium predicts that player i chooses some (possibly mixed) strategy that is in the set of priors describing player j's belief about i's choice. In terms of behavioral implications, the notion implies that an outsider who can only observe the actual strategy choices (and not beliefs), will not be able to distinguish uncertainty averse players from Bayesian players. Intuitively, the reason why uncertainty aversion has seemingly so limited "bite" in this construction is that players' belief set is severely restricted by equilibrium knowledge: every prior in i's belief set about j's strategic choice must be a best response mixed strategy. In other words, given equilibrium knowledge, there are too few possible priors, too little to be uncertain about, so to speak. Dow and Werlang (1994), Klibanoff (1996) and Marinacci (2000), all offer equilibrium concepts with uncertainty aversion that differ from Lo's in one key way. They do not restrict the equilibrium belief to only those priors which are best responses (as mixed strategies); other priors are also possible thus enriching the uncertainty, in a manner of speaking. One principal effect of this is that these notions of equilibria "rationalize" more strategy profiles compared to Lo's concept, indeed even strategy profiles that are not Nash equilibria.

Dow and Werlang (1994) defines equilibrium in two player normal form games where players' have CEU preferences Schmeidler (1989). Equilibrium is simply a pair of capacities, where each capacity gives a particular player's belief about the strategic choice made by the other player. Further, the support of each capacity is restricted to include only such strategies which are best responses with respect to the counterpart capacity. Significantly, the equilibrium notion only considers pure strategies. A pure strategy is deemed to be a best response if it maximizes the Choquet expectation over the set of pure strategies. Much depends on how the *support* of a capacity is defined. Indeed, the only restriction on equilibrium behavior is that only those strategies which appear in the set defined to be the support of the equilibrium beliefs may be played in an equilibrium. Dow and Werlang (1994) defines support A of a capacity μ_i to be a set (a subset of S_j, the set of strategies of player j, in the present context) such that $\mu_i(A^c) = 0$ and $\mu_i(B^c) > 0$ for all $B \subset A$. The nature of this definition may be more readily appreciated if we restrict attention to convex capacities and consider the set of priors in the core of such a capacity. Significantly, the set includes priors that put positive probability on pure strategies in A^c which *may not be best responses.* The convex capacity μ_i is the lower envelope of the priors in the core of μ_i; hence, as long as there is one prior in the core which puts zero probability weight on $s_j \in A^c$, $\mu_i(s_j) = 0$. Hence, a player i's evaluation of a pure strategy s_i will take into account the payoff $u_i(s_i, s_j)$, $s_j \in A^c$, even though s_j may not be a best response for player j, given j's equilibrium belief.

Klibanoff (1996) defines equilibrium in normal form games and like Lo, applies the multiple priors framework to model players' preferences. But there are important differences. Klibanoff defines equilibrium as a profile of tuples $(\{\sigma_i\}, \{m_i\})_{i=1}^n$ where σ_i is the actual mixed strategy used by player i and m_i is a set of priors of player i denoting his belief about opponents' strategy choices. The profile has to satisfy two "consistency" conditions. One, each σ_i has to be consistent with m_i in the sense that σ_i is a best response for i given his belief set m_i. Two, the strategy profile σ_{-i} chosen by other players should be considered possible by player i. The second condition is a consistency condition on the set m_i in the sense that it has to include the actual (possibly mixed) strategy chosen by the other players. However, it is permitted that m_i may contain priors that are not mixed strategies chosen by other players and indeed strategies that are not best responses. Hence, Klibanoff's equilibrium is different from Lo's in that it puts a weaker restriction on equilibrium beliefs, a restriction that is very similar to that implicit in Dow and Werlang's' definition. But Klibanoff's definition is different from Dow and Werlang's in that it explicitly allows players to choose a mixed strategy and allows for mixed strategies to be strictly preferred to pure strategies. Moreover, it is different from both Lo's definition and Dow and Werlang's in that it is specifies more than just equilibrium beliefs: as noted, it explicitly states which strategies will be played in equilibrium.

Marinacci (2000), defines equilibrium in 2 player normal form games and, like Dow and Werlang, applies the CEU framework to model players' preferences. He also defines an equilibrium in beliefs, again much like Dow and Werlang, where beliefs are modeled by convex capacities. However, he employs a slightly different notion of support for equilibrium capacities. His definition of support A of a capacity μ_i consists of all elements $s_i \in S_i$ such that $\mu_i(s_i) > 0$. This puts a weaker restriction on beliefs than Lo's definition, in very much the same spirit as Dow and Werlang's and Klibanoff's definitions. But the true distinctiveness of Marinacci's definition lies elsewhere. His definition includes an explicit, exogenous parametric restriction on the *ambiguity* incorporated in equilibrium beliefs. Given a capacity $\mu(\cdot)$, the ambiguity of belief about an event A, denoted $\Psi(A)$, is measured by $1 - \mu(A) - \mu(A^c)$. The measure is intuitive: $\Psi(A)$ is precisely the difference between the maximum likelihood put on A by any probability measure in the "core" of the $\mu(\cdot)$, and the minimum likelihood put on A by some other probability measure appearing in the core. Thus $\Psi(A)$ is indeed a measure of the fuzziness of belief about A. Marinacci views the ambiguity in the description of the strategic situation *as a primitive*, and characterizes a 2-player *ambiguous game* G by the tuple $\{S_i, u_i, \Psi_i : i = 1, 2\}$. The addition to the usual menu of strategies and utility functions is the ambiguity functional, $\Psi_i : S_j \to [0, 1]$, restricting the possible beliefs of the player i to have a given level of ambiguity: a player i's belief $\mu_i : 2^{S_j} \to [0, 1]$ must be such that $1 - \mu_i(A) - \mu_i(A^c) = \hat{\Psi}_i(A)$. In the models of Dow and Werlang, Lo, and Klibanoff, the equilibrium beliefs are freely equilibrating variables and as such, the level of ambiguity in the beliefs is endogenous. Given the endogeneity, it is not possible in these models, strictly speaking, to pose the comparative statics question, "What happens to the equilibrium if the ambiguity in the way players perceive information of the strategic environment changes?" In Marinacci's model, on the other hand, this question is well posed since beliefs, as equilibrating variables, are not free to the extent they are subject to the ambiguity level constraint imposed by the parameter Ψ. Hence, the answer to the question (a very natural one in an applied context) involves a well posed comparative static exercise showing how the equilibrium changes when Ψ changes.

Following is *one* way of understanding why notions of equilibrium in Dow and Werlang (1994), Klibanoff (1996) and Marinacci (2000) allow a "rationalization" of non-Nash strategy profiles. For instance, in a two player game, it is possible to have as equilibrium, (μ_1^*, μ_2^*), convex capacities denoting equilibrium beliefs of players 1 and 2, where $s_i^* = \sup p\left(\mu_j^*\right)$ is not a best response to $s_j^* = \sup p\left(\mu_i^*\right)$. The key to the understanding lies in the fact that the support of μ_i^*, $i = 1, 2$, is so defined that it may exclude strategies $\hat{s}_j \in \hat{S}_j \subset S_j$ such that $\mu_i^*\left(\hat{S}_j\right) > 0$, but $\mu_i^*\left(s_j\right) = 0$. Since the strategy $\hat{s}_j \in \hat{S}_j$ is not in the support of μ_i^* it is not required to be a best response to μ_j^*, but nevertheless the Choquet integral evaluation of s_i^*, with respect to the belief μ_i^*, may attach a positive weight to the payoff $u_i\left(s_i^*, s_j\right)$ given that $\mu_i^*\left(\hat{S}_j\right) > 0$. Hence, s_i^* can be an equilibrium best response even though it may not be a best response to a belief that puts probability 1 on s_j^*. It is as if the player i, when evaluating s_i^*, allows for the possibility that j may play a strategy that is not a best response.

The discussion in the preceding paragraph suggests that equilibrium, as defined Dow and Werlang (1994), Klibanoff (1996) and Marinacci (2000) incorporates a flavor of a (standard) Bayesian equilibrium involving "irrational types". This point has been further investigated in Mukerji and Shin (2002). That paper concerns the interpretation of equilibrium in non-additive beliefs in *two*-player normal form games. It is argued that such equilibria involve beliefs and actions which are consistent with a lack of common knowledge of the game. The argument rests on representation results which show that different notions of equilibrium in games with non-additive beliefs may be reinterpreted as standard notions of equilibrium in associated games of incomplete information with *additive* (Bayesian) beliefs where common knowledge of the (original) game does not apply. More precisely, it is shown that any pair of non-additive belief functions (and actions, to the extent these are explicit in the relevant notion of equilibrium) which constitute an equilibrium in the game with Knightian uncertainty/ambiguity may be replicated as beliefs and actions of a specific pair of *types*, one for each player, in an equilibrium of an orthodox Bayesian game, in which there is a common prior over the type space. The representation results provide one way of comparing and understanding the various notions of equilibrium for games with non-additive beliefs, such as those in Dow and Werlang (1994), Klibanoff (1996) and Marinacci (2000).

Greenberg (2000) analyzes an example of an equilibrium in a dynamic game wherein beliefs about strategic choice off the equilibrium path of play are modeled using ideas of Knightian uncertainty/ambiguity. The example illustrates both the appropriateness of this modeling innovation and its potential for generating singular insight in the context of extensive form games. The example is a game consisting of 3 players. Players 1 and 2 first play a "bargaining game" which can end in agreement or disagreement. Disagreement may arise due to the "intransigeance" of either player. However, player 3, who comes into play only in the instance of disagreement, does not have perfect information as to which of players 1 or 2 was responsible for the disagreement; at the point player 3 makes his choice the responsibility for disagreement is private information to players 1 and 2. Player 3 has two actions, one of which is disliked by player 1 while the other is disliked by player 2, and disliked more than disagreement. However, player 3 is indifferent between his two choices though he prefers the agreement outcome to either of them. The conditions of Nash equilibrium require that players have a common probabilistic belief about 3's choice. The details of payoffs are such that any common probability belief would make disagreement more attractive to at least one of players 1 and 2. Greenberg argues that agreement, even though not a Nash equilibrium, can be supported as the unique equilibrium outcome if the first two players's beliefs about what 3 will do in the event of disagreement (an off equilibrium choice) is ambiguous and players are known to be ambiguity averse. A common set of probabilities describes players 1 and 2's prediction about 3's choice. But given uncertainty aversion, say as in MEU, each of players 1 and 2 evaluates his options *as if* his belief is described by the probability that mirrors his most pessimistic prediction. Hence the two players, given their respective apprehensions, choose to agree, thereby behaving as if they had two different probability beliefs about player 3's choice. Greenberg further observes, player 3 may actually be able to facilitate this "good" equilibrium by not announcing or precommiting to the action he would choose if called upon to play following disagreement; the player would strictly prefer to exercise "the right to remain silent". The silence "creates" ambiguity of belief and given aversion to this ambiguity, in turn "brings about" the equilibrium. The question of appropriate modeling of beliefs about off-equilibrium path choices has been a source of vexation about as long as extensive form games have been around. It may be argued persuasively that on the equilibrium path of play, beliefs are pinned down by actual play . The argument is far less persuasive, if at all, for beliefs off the equilibrium path of play. Hence, the appropriateness of modeling such beliefs as ambiguous. But off equilibrium path beliefs may be crucial for the construction of equilibrium. As has been noted, the good equilibrium described in Greenberg's example would not obtain if players 1 and 2 were required to have a common probabilistic belief about 3's choice. Of course, this profile would not be ruled out by a solution concept that allows for "disparate" beliefs off the path of play, for instance, self-confirming equilibrium (Fudenberg and Levine (1993), subjective equilibrium (Kalai and Lehrer (1994)), extensive form rationalizability (Bernheim (1984), Pearce (1984)). What ambiguity aversion adds, compared to these solution concepts, is a positive theory as to why the players (1 and 2) would choose to behave *as if* they had particular differing probabilistic beliefs even though they are *commonly* informed. While Greenberg does not give a formal definition of equilibrium for extensive form games where players may be uncertainty averse, Lo (1999) does. However, Lo does not go far enough to consider the question of extensive form refinements. Hence, determining reasonable restrictions on beliefs about off equilibrium play, while allowing them to be ambiguous, remains an exciting open question, hopefully to be taken up in future research.

Analysis of behavior in auctions has been a prime area of application of game theory, especially in recent years. Traditional analysis of auctions assumes the seller's and each bidder's belief about rival bidders' valuations are represented by probability measures. Lo (1998) makes an interesting departure from this tradition by proceeding on the assumption that such beliefs are instead described by sets of multiple priors and players are uncertainty averse (in the sense of MEU). Lo's analyzes first and second price sealed bid auctions with independent private values. In a more significant finding, he shows, under "an interesting" parametric specification of beliefs, that the first price auction Pareto dominates the second price auction. A rough intuition for the result is as follows. Suppose the seller and the bidders are commonly informed about "others'" valuation, i.e., the information is described by the same set of probabilistic priors. When the seller is considering which of the two auction formats to adopt, the first or the second price sealed bid auction, he evaluates his options using that probabilistic prior (from the "common" set of priors) which reflects his worst fears, namely, that bidders have low valuations. Recall that bidders always bid their true valuation in the second price auction. Therefore, the usual Revenue Equivalence Theorem implies that the seller would be indifferent between the two auction formats, if bidders' strategy (in the first price auction) were based on the same probabilistic prior that the seller effectively applies for his own strategy evaluations. However, given uncertainty aversion, bidders will behave as if the probability relevant for their purposes is the one that reflects their apprehensions: the fear that their rivals have high valuation. Which means under uncertainty aversion the optimal bid will be higher. On the other hand, because of his apprehensions, the seller will choose a reserve price for first price auction that is strictly lower than the one he chooses for the second price auction. Hence, the first price auction format is Pareto preferred to the second price format. Ozdenoren (2002) generalizes Lo's results by relaxing the parametric restriction on beliefs significantly. These successful investigations of behavior in auctions point to the potential for further research to understand behavioral implications of uncertainty aversion in incomplete information environments, in general, and of implementation theory in particular. In general, strategic interaction in incomplete information environments would appear to be particularly appropriate a setting for investigation since the scope of play of ambiguity is far greater than what is possible under equilibrium restrictions in complete information settings. More particularly, Bayesian implementation theory has frequently been criticized for prescribing schemes which are "too finely tuned" to the principal's and agents' knowledge of the precise prior/posteriors. Perhaps introducing the effect of uncertainty aversion will lead to the rationalization of schemes which are more robust in this respect (and hopefully, an understanding of implementation schemes which are more implementable in the real world!).

Eichberger and Kelsey (2002) apply Dow and Werlang's notion of equilibrium to analyze the effect of ambiguity aversion on a public good contribution game. They show that it is possible to sustain as equilibrium (under ambiguity) a strategy profile which involves higher contributions than possible under standard beliefs. The rough idea is as follows. Recall our discussion about how the Dow and Werlang notion may allow a non-Nash profile to be the support of equilibrium beliefs. Working with the CEU model, Eichberger and Kelsey construct a equilibrium belief profile, wherein each player i behaves as if there is a chance that another player j plays a strategy lying outside the support of i's equilibrium belief, in particular, a strategy that is "bad" for i, i.e., make contribution lower than the equilibrium contribution. Essentially, it is this "fear" of lower contribution by others, given the strategic uncertainty, which drives up i's equilibrium contribution. The paper also extends the analysis from the public good provision games to consider more general games classified in terms of strategic substitutability and complementarity.

Ghirardato and Katz (2002) apply the MEU framework to the analysis of voting behavior to give an explanation of the phenomenon of selective abstention. Consider a multiple office election, i.e., an election in which the same ballot form asks for the voter's opinion on multiple electoral races. It is typically observed in such elections that voters choose to vote on the more high profile races (say, the state governor), but simultaneously abstain from expressing an opinion on other races (say, the county sheriff). Ghirardato and Katz's objective is to formalize a commonly held intuition that the reason the voters selectively abstain is because they believe that they are relatively poorly informed about the candidates involved in the low profile races. The paper contends that the key to the intuition is the issue of modeling the sensitivity of a decision maker's choice to what he perceives to be the quality of his information and that this issue cannot be adequately addressed in a standard Bayesian framework. On the other hand, they argue, it can be addressed in a decision theoretic framework which incorporates ideas of ambiguity; roughly put, information about an issue is comparatively more ambiguous and of lower quality if the information about the issue is represented by a more encompassing set of priors. This point would seem to be of wider interest and worth pursuing in future research.

5 Other applications

Finally, we survey some applications of the tools developed by David Schmeidler that do not, *per se*, involve decision making under uncertainty. The applications covered relate to the measurement of inequality, intertemporal choice and multi-attribute choice.

Inequality measurement

Decision theory under risk and the theory of inequality measurement essentially deal with the same mathematical objects (probability distributions). Therefore, these two fields are closely related, and their relationship has long been acknowledged.[3] However, surprisingly enough, almost all the

[3] For instance, it is easy to check that the well-known Gini index relies on a Choquet integral (with respect to a symmetric capacity). Indeed, as recalled in the introduction to this volume, the first axiomatization of a rank dependent model was provided in the framework of inequality measurement (Weymark (1981)). We will not expand on

literature on inequality measurement deals with *certain* incomes. This is probably due, in part, to a widely held opinion that the problem of measuring inequality of uncertain incomes can be reduced to a problem of individual choice under uncertainty (say, e.g., by first computing in each state a traditional welfare function *à la* Atkinson-Kolm-Sen, and then reducing the problem to a single decision maker's choice among prospects of welfare) or alternatively to a problem of inequality measurement over sure incomes (say, e.g., by evaluating each individual's welfare by his expected utility, and then considering the distribution of the certainty equivalent incomes). In a path-breaking paper, Ben Porath, Gilboa, and Schmeidler (1997) show that such is not the case, and that inequality and uncertainty should be analyzed jointly and not separately in two stages. They present the following example which serves to illustrate their point . Consider a society with two individuals, a and b, facing two equally likely possible states of the world, s and t, and assume that the planner has to choose among the three following social policies, P_1, P_2 and P_3:

P_1	a	b
s	0	0
t	1	1

P_2	a	b
s	1	0
t	0	1

P_3	a	b
s	1	0
t	1	0

Observe that in P_1, both individuals face the same income prospects as in P_2; but in P_1, there is no *ex post* inequality, whatever the state of the world. This could lead one to prefer P_1 over P_2. Similarly, P_2 and P_3 are *ex post* equivalent, since in both cases, whatever the state of the world, the final income distribution is identical; but P_3 gives 1 for sure to one individual, and 0 to the other, while P_2 provides both individuals with the same *ex ante* income prospects. On these grounds, it is reasonable to think that P_2 should be ranked above P_3. Thus, a natural ordering would be $P_1 \succ P_2 \succ P_3$.

Ben Porath et al., point out that there is no hope to obtain such an ordering by two-stage procedures. Indeed, the first two-stage procedure (mentioned earlier) would lead us to neglect *ex ante* considerations and to judge P_2 and P_3 as equivalent. In contrast, the second procedure would lead us to neglect *ex post* considerations and to see P_1 and P_2 as equivalent. In other words, these procedures would fail to simultaneously take into account the *ex ante* and the *ex post* income distributions.

They suggest solving this problem by considering a linear combination of the two procedures, i.e., a linear combination of the expected Gini index and the Gini index of expected income. This solution captures both *ex ante* and *ex post* inequalities. Furthermore, it is a natural generalization of the principles commonly used for evaluating inequality under certainty on the one hand, and for decision making under uncertainty on the other hand.

The procedure suggested in Ben Porath, Gilboa, and Schmeidler (1997) is not the only possible evaluation principle that takes into account both *ex ante* and *ex post* inequalities. Any functional that is increasing in both individuals' expected income and snapshot inequalities (say, measured by the Gini index) has the same nice property, provided that it takes its values between the expected Gini and the Gini of the expectation. Furthermore, it is unclear why we should restrict ourselves, as Ben Porath et. al did, to decision makers who behave in accordance with the multiple priors model. Finally, they do not provide an axiomatization for the specific functional forms they propose. This problem is partially dealt with in Gajdos and Maurin (2002), who provide an axiomatization for a broad class of functionals that can accommodate Ben Porath, Gilboa and Schmeidler's example.

Intertemporal choice and Multi-attribute choice

Schmeidler's models on decision under uncertainty have also been shown to hold new insights when applied to decision making contexts and questions that do not (necessarily) involve uncertainty. For instance, in the context of intertemporal decision making (under certainty). In terms of abstract construction, a intertemporal decision setting is essentially the same as that of decision making under uncertainty with time periods replacing states of nature. Gilboa (1989) transposes the Choquet expected utility model of decision under uncertainty to intertemporal choices. He shows that using

this literature, which has very close links with the RDEU model, here.

the axiomatization of Schmeidler (1989) and adding an axiom called “Variation preserving sure-thing principle" the decision rule is given by a weighted average of the utility in each period and the utility variation between each two consecutive periods. Aversion towards uncertainty is now replaced by aversion towards variation over time of the consumption profile. Wakai (2001) in a similar vein uses the idea that agents dislike time-variability of consumption and axiomatizes a notion of non-separability in the decision criterion. He then goes on to show how such a decision criterion modifies consumption smoothing and can help providing an explanation to the equity and the risk-free rate puzzle. Marinacci (1998) also uses a transposition of Gilboa and Schmeidler (1989) model to intertemporal choice and axiomatizes a complete patience criterion with a new choice criterion (the Polya index). De Waegenaere and Wakker (2001) generalizes the Choquet integral to signed Choquet integral, which captures both violations of separability and monotonicity. This tool can be used to model agents whose aversion towards volatility of their consumption is so high that they could prefer a uniformly smaller profile of consumption if it entails sufficiently less volatility. A work of related interest is Shalev (1997), which uses a mathematically similar technique to represent preferences incorporating a notion of loss aversion in an explicitly intertemporal setting, i.e., where objects of choice are intertemporal income/consumption streams and the decision maker is averse to consumption decreasing between successive periods.

In the context of decision making under uncertainty, the Choquet integral may be viewed as a way of aggregating utility across different states in order to arrive at an (ex ante) decision criterion. Multi-attribute choice concerns the question of aggregating over different attributes, or characteristics, of commodities in order to formulate an appropriate decision criterion for choosing among the multi-attributed objects (see for instance Grabisch (1996), Dubois, Grabisch, Modave, and Prade (2000).) The use of variants of the Choquet integral allows some flexibility in the way attributes are weighted and combined. In an interesting paper, Nehring and Puppe (2002) use the multi-attribute approach to modeling (bio-)diversity. In doing so, they develop tools to measure diversity, based on the notion of the (Möbius) inverse of a capacity. Interestingly, this is also related to another line of research developed by Gilboa and Schmeidler, namely Case-Based Decision Theory (Gilboa and Schmeidler (1995), Gilboa and Schmeidler (2001))

6 Concluding remarks

This was a review of a sample of the rich, varied, and very much "alive" literature that has been inspired by David Schmeidler's path breaking contributions to decision theory. We do want to emphasize that it is very much a sample; the list of papers discussed or cited is far from exhaustive. Nevertheless, we hope the following conclusions are justifiable even on the basis of the limited discussion. First, that thinking of decision making under uncertainty in the way Schmeidler's models prompt and allow us to do, incorporating issues such as sensitivity of decision makers to the ambiguity of information, does lead to new and important insights about economic phenomena. Second, that while it has long been suspected that issues like ambiguity could be of significance in economics, the great merit of Schmeidler's contribution has been to provide us with tools that have allowed us to develop tractable models to investigate these intuitions in formal exercises, per standard practice in economic theory. The opportunity we have had is quite unique. There are several other branches of decision theory that depart from the stand expected utility paradigm. But comparatively, these branches have seen far less applied work. One cannot help but speculate that the relative success of the ambiguity paradigm is in no small measure due to the tractability of Schmeidler's models. Tractability, is a hallmark of a classic model. Lastly, we hope the review has demonstrated that there are many important and exciting questions that remain unanswered. And indeed, enough progress has been made with modeling issues which gives us grounds to be optimistic that the answers to the open questions cannot be far away. Therefore, researchers should consider such questions definitely worth investigating.

While we have mentioned several issues worthy of future investigation in the course of our discussions, there are couple of issues that we have not touched upon. One, we hope there will be more directed empirical work, directed to testing predictions and also the basis of some of the assumptions, for instance in financial markets. Two, as would have been evident in our survey, most of the work has been of the "positive" variety, meant to help us "understand" puzzling phenomena. Perhaps, we also want to think about more normative questions: for instance, "What is a good way of deciding between alternatives, in a particular applied context, when information is ambiguous ?" The work of Hansen, Sargent and their coauthors is one notable exception, but more is needed, perhaps in fields like environmental policy making where, arguably, information is in many instances, ambiguous.

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