Powerful algorithms for decision making under partial prior information and general ambiguity attitudes

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Decision making under incomplete data using the imprecise Dirichlet model

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1. The basic decision problem

- Comprehensive framework
  - **Actions** $a_i \in \mathbb{A}$ (treatment; investment)
  - **states of nature** $\vartheta_j \in \Theta$ (disease; development of economy)
  - **utility** $u(a_i, \vartheta_j) \implies$ random variable $u(a)$

- Find **optimal** action(s)!

- When everything is finite: utility table

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<tr>
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<th>$\vartheta_1$</th>
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<th>$\vartheta_j$</th>
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2. Classical Decision criteria

- Randomized actions: \( \lambda(a_i) \) probability to take action \( a_i \)

**Two classical criteria:**

- **Bayes optimality**
  * perfect probabilistic knowledge: prior \( \pi(\cdot) \) on \( \Theta \)
  * maximize expected utility \( \mathbb{E}_\pi u(a) \rightarrow \max_a \)

- **Maximin (Wald) optimality**
  * complete ignorance \( \implies \) focus on the worst state:
    \[
    \min_j u(a, \vartheta_j) \rightarrow \max_a
    \]

What to do in the case of **partial** prior knowledge?
3. Decision criteria under partial knowledge

- $\mathcal{M}$ convex polyhedron of classical probabilities (e.g. structure of F-probability); $\mathcal{E}(\mathcal{M})$ set of vertices

- $\mathcal{M} = \{ \pi(\cdot) | b_l \leq \mathbb{E}_\pi f_l \leq \bar{b}_l \} \quad l = 1, \ldots, r$

- interval-valued expected utility:
  $\mathbb{E}_\mathcal{M} u(a) := [\mathbb{E}_\mathcal{M} u(a), \bar{\mathbb{E}}_\mathcal{M} u(a)]$

  $:= [\inf_{\pi \in \mathcal{M}} \mathbb{E}_\pi u(a), \sup_{\pi \in \mathcal{M}} \mathbb{E}_\pi u(a)]$

- axiomatic justifications!
Some Criteria
(Survey: Troffaes (SIPTA-NL, Dec 2004))

- linear ordering
  a) $\mathbb{E}_M u(a) \rightarrow \max_a$
  Γ-Maximin
  Choquet-Integral
  $\mathbb{E}_M u(a) \rightarrow \max_a$
  Caution parameter $\eta$

- partial ordering
  b) $\eta \cdot \mathbb{E}_M u(a) + (1 - \eta)\mathbb{E}_M u(a) \rightarrow \max_a$
  E-admissibility (Levi)
  Maximalility (Walley)

Caution parameter $\eta$
4. Calculation of optimal actions

• Far from being straightforward; lack of feasible algorithms
• has hindered large scale applications
• Formulation in terms of linear programming problems also provides theoretical insight.
• Two different situations considered here
  * direct assessment of $\mathcal{M}$ (e.g. by an expert)
  * construction of partial knowledge based on previous observations on $\Theta$ (repeated decision problems)
4. a) Pessimistic decision making: Gamma-Maximin

- Bayes and minimax optimality as border cases
- Gamma-Minimax criterion (e.g., Berger (1984¹², Springer), Vidakovic (2000, Insua, D.R., and Ruggeri, F. (eds.))
- Maxmin expected utility model (Gilboa, Schmeidler (1989, Journal of Mathematical Economics)
- MaxEMin (Kofler, Menges (1976)) (cf. also Kofler (1989, Campus) and the references therein)
- maximinity (Walley (1991, Chapman Hall))
- In the case of two-monotonicity: Choquet expected utility (e.g., Chateauneuf, Cohen, Meilijson (1991, Finance))
\[ \mathbb{E}_M u(\lambda) \rightarrow \max \]

\[ \iff \min_{\pi \in \mathcal{M}} \sum_{j=1}^{m} \left( \sum_{i=1}^{n} u(a_i, \vartheta_j) \lambda(a_i) \right) \pi(\{\vartheta_j\}) \rightarrow \max_{\lambda} \]

subject to \[ \sum_{i=1}^{n} \lambda(a_i) = 1; \quad \lambda(a_i) \geq 0 \]

\[ \iff \]

\[ G \rightarrow \max \]

subject to \[ \sum_{i=1}^{n} \lambda(a_i) = 1; \quad \lambda(a_i) \geq 0 \]

\[ \sum_{j=1}^{m} \left( \sum_{i=1}^{n} u(a_i, \vartheta_j) \lambda(a_i) \right) \pi(\{\vartheta_j\}) \geq G, \quad \forall \pi \in \mathcal{M}. \]
\[ G \rightarrow \max \]
\[ \text{subject to } \sum_{i=1}^{n} \lambda(a_i) = 1; \quad \lambda(a_i) \geq 0 \quad \text{and} \]
\[ \sum_{j=1}^{m} \left( \sum_{i=1}^{n} u(a_i, \vartheta_j) \lambda(a_i) \right) \pi(\{\vartheta_j\}) \geq G, \quad \forall \pi \in \mathcal{E}(\mathcal{M}). \]

• needs, however, all vertices to be determined in advance

• In case of F-probability: \(|\mathcal{E}(\mathcal{M})|\) may be as large as \(m!\) (Wallner (2005, ISIPTA))

• considerable simplification in the case of two-monotonicity
Alternative: partial dualization

\[ \min_{\pi \in M} \sum_{j=1}^{m} \left( \sum_{i=1}^{n} u(a_i, \theta_j) \lambda(a_i) \right) \pi(\{\theta_j\}) \rightarrow \max \]

subject to \( \lambda \cdot 1 = 1. \)

\[ \text{• Fix } \lambda, \text{ and consider the dual problem of} \]

\[ \sum_{j=1}^{m} \left( \sum_{i=1}^{n} u(a_i, \theta_j) \lambda(a_i) \right) \pi(\{\theta_j\}) \rightarrow \min_{\pi \in M} \]

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With $\mathbf{C} = (c_1, \ldots, c_r)^T$, $\mathbf{D} = (d_1, \ldots, d_r)^T$:

$$\max_{c, \mathbf{C}, \mathbf{D}} \left\{ c + \mathbf{B}\mathbf{C} - \mathbf{B}\mathbf{D} \right\}$$

subject to $c \in \mathbb{R}$, $\mathbf{C}, \mathbf{D} \in \mathbb{R}^r$, and

$$c + \mathbf{F}_j (\mathbf{C} - \mathbf{D}) \leq \sum_{i=1}^{n} u(a_i, \vartheta_i) \lambda(a_i), \quad j = 1, \ldots, m. \quad (1)$$

---

Here $c, \mathbf{C}, \mathbf{D}$ are optimization variables such that the variable $c$ corresponds to the constraint $\sum_{j=1}^{m} \pi_j = 1$ in the primal form, $c_i$ corresponds to the constraints $b_i \leq E_{\pi} f_i$ and $d_i$ corresponds to the constraints $E_{\pi} f_i \leq \bar{b}_i$. 

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• By the general theory, the values at the optima coincide
\[
\min_{\pi \in \mathcal{M}} \sum_{j=1}^{m} \left( \sum_{i=1}^{n} u(a_i, \vartheta_j) \lambda(a_i) \right) \pi(\{\vartheta_j\}) = \max_{c,C,D} \left\{ c + BC - BD \right\},
\]

• Then the additional maximization over $\lambda$ gives the optimal action:
\[
\max_{c,C,D,\lambda} \left\{ c + BC - BD \right\}
\]
subject to $c \in \mathbb{R}$, $C, D \in \mathbb{R}^+_r$, $\lambda \cdot 1 = 1$ and
\[
c + F_j(C - D) \leq \sum_{i=1}^{n} u(a_i; \vartheta_j) \lambda(a_i), \ j = 1, \ldots, m.
\]

• Note: single linear programming problem, the vertices are not needed
4 b) Caution parameter $\eta$

- More sophisticated representations of interval-valued expected utility to avoid overpessimism
- take additionally into consideration the decision maker’s attitude towards ambiguity, e.g.:
  - Ellsberg (1961, QJE)
  - Jaffray (1989, OR Letters)
  - Schubert (1995, IJAR)
  - Weichselberger (2001, Physika, Chapter 2.6)
  - Weichselberger and Augustin (1998, Galata and Küchenhoff (eds.))
• Criterion:

\[ \eta \overline{E}_M u(\lambda) + (1 - \eta) \overline{E}_M u(\lambda) \rightarrow \max_\lambda \]

• Same tricks can not be applied again: unbounded solutions

• Ensure that in the previous systems some inequalities are equalities \( \Rightarrow \) several optimization problems to be solved

• Alternatively, in the approach based on the vertices, consider for every \( \tilde{\pi} \in \mathcal{E}(\mathcal{M}) \) the objective function

\[ \eta \cdot G + (1 - \eta) \sum_{j=1}^{m} \left( \sum_{i=1}^{n} u(a_i, \vartheta_j) \lambda(a_i) \right) \tilde{\pi}(\vartheta_j) \rightarrow \max \]

and maximize over all elements of \( \mathcal{E}(\mathcal{M}) \)
4 c) **E-admissibility (and maximality)**

- E-admissibility (e.g., Levi (1974, J Phil), Schervish et al. (2003, ISIPTA)):

- Consider all actions that are not everywhere suboptimal:

  \[ \exists \pi_{a^*} \in \mathcal{M} \text{ such that } a^* \text{ is Bayes with respect to } \pi_{a^*}: \]

  \[ \sum_{j=1}^{m} u(a^*, \vartheta_j) \pi_{a^*}(\vartheta_j) \geq \sum_{j=1}^{m} u(a, \vartheta_j) \pi_{a^*}(\vartheta_j), \forall a \in \mathcal{A} \]
Lemma 1  (Characterization of Bayes actions in classical decision theory) Fix \( \pi(\cdot) \) and let \( A_\pi^* \) be the set of all pure Bayes actions with respect to \( \pi(\cdot) \), and \( \Lambda_\pi^* \) the set of all randomized Bayes actions with respect to \( \pi(\cdot) \). Then

i) \( A_\pi^* \neq \emptyset \)

ii) \( \Lambda_\pi^* = \text{conv}(A_\pi^*) \).

Proof: The task of finding a Bayes action with respect to \( \pi(\cdot) \) can be written as a linear programming problem

\[
\sum_{j=1}^{m} \left( \sum_{i=1}^{n} u(a_i, \vartheta_j) \lambda(a_i) \right) \pi(\vartheta_j) \rightarrow \max_{\lambda}
\]

subject to \( \sum_{i=1}^{n} \lambda(a_i) = 1 \), and \( \lambda(a_i) \geq 0 \), for all \( i \).

i) One optimal solution must be attained at a vertex.

ii) Convexity of the set of optimal solutions.

\(^2\) Here every pure action \( a_i \in IA \) is identified with the randomized action \( \lambda(a) = 1 \) if \( a = a_i \) and \( \lambda(a) = 0 \) else, and with the corresponding \((n \times 1)\) vector.
A general algorithm for E-admissibility

• Turn the problem around!
  Now fix the actions!

• For every $a_i$ look at

$$\Pi_i := \{\pi(\cdot) \in \mathcal{M} | a_i \text{ is Bayes action with respect to } \pi(\cdot)\}$$

According to Lemma 1:

$$\Pi_i = \left\{ \pi(\cdot) \in \mathcal{M} \middle| \sum_{j=1}^{m} u(a_i, \vartheta_j) \pi(\vartheta_j) \geq \sum_{j=1}^{m} u(a_l, \vartheta_j) \pi(\vartheta_j), \quad \forall l = 1, \ldots, n \right\}$$
\[ \Pi_i = \text{conv} \left( \tilde{\pi}(\cdot) \in \mathcal{E}(\mathcal{M}) \left| \sum_{j=1}^{m} u(a_i, \vartheta_j) \tilde{\pi}(\vartheta_j) \geq \sum_{j=1}^{m} u(a_l, \vartheta_j) \tilde{\pi}(\vartheta_j), \forall l = 1, \ldots, n \right. \right) . \]
• Alternatively, without using $\mathcal{E}(\mathcal{M})$:

$$z \quad \longrightarrow \quad \max_{(\pi^T, z)^T}$$

$$\sum_{j=1}^{m} u(a_i, \theta_j) \pi(\theta_j) \quad \geq \quad \sum_{j=1}^{m} u(a_l, \theta_j) \pi(\theta_j), \quad \forall l = 1, \ldots, n$$

$$\sum_{j=1}^{m} \pi(\theta_j) = z, \quad z \leq 1, \quad \pi(\theta_j) \geq 0, \quad j = 1, \ldots, m,$$

$$b_l \leq \sum f_i(\theta_j) \pi(\theta_j) \leq \bar{b}_l, \quad l = 1, \ldots, r.$$  

• Iff $z = 1$ then $\Pi_i \neq 0$ and $a_i$ is $E$-admissible

• To determine all $E$-admissible pure actions: $|A|$ linear optimization problems have to be solved
By Lemma 1 ii) adaption possible to calculate all E-admissible actions:
For all \( I \subseteq \{1, \ldots, m\} \) check whether there is a prior \( \pi \) under which all \( a_i, i \in I \), are simultaneously optimal, i.e. replace (23) by

\[
\Pi_I := \left\{ \pi(\cdot) \middle| \sum_{j=1}^{m} u(a_i, \vartheta_j) \pi(\vartheta_j) \geq \sum_{j=1}^{m} u(a_l, \vartheta_j) \pi(\vartheta_j), \right. \\
\left. \forall i \in I, l = 1, \ldots, n. \right\}
\]

If \( \Pi_I \) is not empty, then all the elements of \( \text{conv}(a_i| i \in I) \) are E-admissible actions.

If \( \Pi_I = \emptyset \) for some \( I \) then all index sets \( J \supset I \) need not be considered anymore.
maximality

• If $\Pi_i$ contains $\pi$ with $\pi(\cdot) > 0$, then $a_i$ is admissible in the classical sense and therefore maximal.

• But if $\mathbb{A}$ is not convex, not all maximal actions are found in that way.

• uniform optimality of $a_{i^*}$:

If $\Pi_i = \mathcal{M}$ then $\mathbb{E}_\pi u(a_{i^*}) \geq \mathbb{E}_\pi u(a), \ \forall \pi \in \mathcal{M}, \ a \in \mathbb{A}$. (cp. Weichselberger (2001, Chapter 2.6): structure dominance)
Now (second paper) data on $\theta_1, \ldots \theta_n$

- $n_j$ observations of $\theta_j$, $j = 1, \ldots n$.
- more general: set-valued observations $\subseteq \Theta$
- calculate expected utility based on estimates $\hat{\pi}(\theta_j)$ resulting from the data

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To calculate optimal actions

• use previous techniques or

• considerable simplications due to the use of the IDM and belief functions: With Möbius inverse \( m(\cdot) \)

\[
\mathbb{E} u(a) = \left[ \sum_{A \subseteq \Theta} m(A) \cdot \min_{\theta \in A} u(a, \theta); \sum_{A \subseteq \Theta} m(A) \cdot \max_{\theta \in A} u(a, \theta) \right]
\]

Chateauneuf and Jaffray (1989, Math. Social Sc.; Cor.4), Strat (1991, IJAR)

• Leads to a frequency-based Hodges-Lehman criterion

• Be careful when specifying \( \Theta! \) The embedding principle is not valid in decision theory based on the IDM.
The (Imprecise) Dirichlet Model in decision making

- $N$ multinomial observations on space $\Omega$, Dirichlet prior with parameter $S$, $t = (t_1, \ldots t_m)$

- For every $A \subseteq \Omega$ predictive probability
  \[
P(A|n, t, s) = \frac{\sum_{\omega_j \in A} n_j + s \cdot \sum_{\omega_j \in A} t_j}{N + s}
  \]

- Walley (1996, JRSSB): Consider \textit{all} Dirichlet priors, i.e. vary $t \in S(1, m)$

  \[
P(A|n, t, s) = \left[ \frac{\sum_{\omega_j \in A} n_j}{N + s}, \frac{s + \sum_{\omega_j \in A} n_j}{N + s} \right]
  \]
• In decision making based on certain value of $t$

$$E_t u(\lambda) = \int S(1, m) \sum_{i=1}^{m} (u(\lambda, \omega_i) \cdot \pi_i) p(\pi) d\pi$$

$$= \sum_{i=1}^{m} u(\lambda, \omega_i) \cdot \int S(1, m) \pi_i \cdot p(\pi) d\pi = \sum_{i=1}^{m} u(\lambda, \omega_i) \cdot \mathbb{E}_p \pi_i,$$

where

$$\mathbb{E}_p \pi_i = \frac{n_i + st_i}{N + s}, \quad (1)$$

finally resulting in $E_t u(\lambda) = \sum_{i=1}^{m} u(\lambda, \omega_i) \frac{n_i + st_i}{N + s}. \quad (2)$

• For the IDM

$$\bar{E}u(\lambda) := [\underline{E}u(\lambda), \overline{E}u(\lambda)] := [\inf_{t \in S(1,m)} E_t u(\lambda), \sup_{t \in S(1,m)} E_t u(\lambda)].$$
Optimal actions in the case of pessimistic decision making

\[ \mathbb{E}u(\lambda) \rightarrow \max_{\lambda} \]

• use previous approaches or:

• for randomized actions solve

\[ G \rightarrow \max_{\lambda} \]

subject to \( G \in \mathbb{R}, \lambda \cdot 1 = 1, \) and for \( j = 1, \ldots, m, \)

\[ G \leq \frac{1}{N} + s \sum_{r=1}^{n} \lambda(a_r) \left( s \cdot u(a_r, \vartheta_j) + \sum_{j=1}^{m} u(a_r, \vartheta_j) \cdot n_j \right). \]
• for pure actions

\[
\left( \sum_{j=1}^{m} u(a_r, \vartheta_j) \cdot n_j + s \cdot \min_{j=1,...,m} u(a_r, \theta_j) \right) \rightarrow \max_r
\]

\[
\iff \frac{N}{N + s} \cdot \text{(MEU based on } \frac{n_i}{N}) + \frac{S}{N + s} \cdot \text{(Wald criterion)}
\]

\[
N \rightarrow \infty \quad \text{maximum expected utility (MEU)}
\]

\[
N = 0 \quad \text{Wald}
\]
Incomplete data

- **coarse** data, set-valued observations make no additional assumptions (like CAR (Heitjan and Rubin (1991, Ann. Stat.), Blumenthal (1968, JASA))) $\implies$ extended empirical belief functions (Utkin (2005, FSS))

- $c_i$ observations of $A_i \subseteq \Omega$, $i = 1, \ldots, M$ such that $\sum_{i=1}^{M} c_i = N$; $c := (c_1, \ldots, c_M)$

- leads to several IDM’s with observations $n^{(k)} = (n_1^{(k)}, \ldots, n_m^{(k)})$, $k = 1, \ldots, K$.
  (cp. also de Cooman and Zaffalon (2004, AI), Zaffalon (2002, JSPI))
• for fixed $t$

$$P(A|c, s) = \frac{\min_k \sum_{\omega_j \in A} n_j^{(k)} + s \cdot \sum_{\omega_j \in A} t_j}{N + s}$$

$$\overline{P}(A|c, s) = \frac{\max_k \sum_{\omega_j \in A} n_j^{(k)} + s \cdot \sum_{\omega_j \in A} t_j}{N + s}$$

• vary $t \in S(1, m)$

$$P(A|c, s) = \frac{\sum_{i: A_i \subseteq A} c_i}{N + s}, \quad \overline{P}(A|c, s) = \frac{\sum_{i: A_i \cap \overline{A} \neq \emptyset} c_i + s}{N + s}.$$
Relation to empirical belief functions/random sets

- Empirical belief functions: set \( m(A_i) = \frac{c_i}{N} \).

- Naive approach does not reflect the sample size,

- leads to \( Bel_{emp}(\cdot) \) and \( Pl_{emp}(\cdot) \)

- Extended empirical belief functions can be written as

\[
P(A|c, s) = \frac{N \cdot Bel_{emp}(A)}{N + s}, \quad \overline{P}(A|c, s) = \frac{N \cdot Pl_{emp}(A) + s}{N + s}
\]

with Möbius inverse

\[
m^*(A_i) = \frac{c_i}{N + s}; \quad m^*(A_\infty) = \frac{s}{N + s}.
\]
Optimal randomized actions (with $J_i := \{j|\omega_j \in A_i\}$)

$$\frac{1}{N + s} \left( s \cdot V_0 + \sum_{k=1}^{M} c_k \cdot V_k \right) \rightarrow \max_{\lambda},$$

subject to $V_0, V_i \in \mathbb{R}, \lambda \cdot 1 = 1$.

$$V_i \leq \sum_{r=1}^{n} u(a_r, \omega_j) \cdot \lambda(a_r), \ i = 1, \ldots M, \ j \in J_i$$

$$V_0 \leq \sum_{r=1}^{n} u(a_r, \omega_j) \cdot \lambda(a_r), \ i = 1, \ldots m.$$  

Optimal pure actions

$$\frac{1}{N + s} \left( s \cdot \min_j u(a_r, \omega_j) + \sum_{k=1}^{M} c_k \cdot \min_{\omega_j \in A_k} u(a_r, \omega_j) \right) \rightarrow \max_{r=1, \ldots n}$$
Concluding remarks

- Other optimality criteria

- Alternative approach:

- Alternative models to learn from multinomial data:
  inference within the frame of Weichselberger’s (e.g. 2005, ISIPTA) theory of symmetric probability or circular–$A(n)$–based inference: (Coolen and Augustin (2005, ISIPTA)).