Envelope Theorems and Dilation with Convex Conditional Previsions

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Abstract

This paper focuses on establishing envelope theorems for convex conditional lower previsions, a recently investigated class of imprecise previsions larger than coherent imprecise conditional previsions. It is in particular discussed how the various theorems can be employed in assessing convex previsions. We also consider the problem of dilation for these kinds of imprecise previsions, and point out the role of convex previsions in measuring conditional risks.

Keywords. Convex Prevision, Envelope Theorem, Dilation, Convex Risk Measure.

1 Introduction

Generally speaking, envelope theorems relate a function in a certain set $\mathcal{F}$ to a set $\mathcal{P}$ of other functions with well specified features. These theorems either ensure that by performing the (pointwise) infimum or supremum on the elements of $\mathcal{P}$ we get a function $f \in \mathcal{F}$, or else guarantee that every $f \in \mathcal{F}$ may be expressed as an infimum or supremum over some set $\mathcal{P}$, or both (thus characterising the functions in $\mathcal{F}$). Envelope theorems are found in many different research areas, like for instance cooperative games [10] or convex analysis [9].

In the theory of imprecise probabilities, a fundamental envelope theorem [13] states that a real function $\underline{P}$ is a coherent lower prevision over a set $\mathcal{D}$ of (unconditional) random variables (or gambles) if and only if $\underline{P}(X) = \inf_{P \in \mathcal{P}}\{P(X)\}, \forall X \in \mathcal{D}$, where all $P \in \mathcal{P}$ are coherent precise previsions. The theorem on one hand points out a way of assessing coherent lower previsions, on the other hand relates the behavioural approach to imprecise previsions with the indirect approach, which defines imprecise previsions or probabilities in terms of sets of other uncertainty measures (precise previsions, or probabilities).

In the language of risk measures, a version of this theorem characterises coherent risk measures [1] and the precise previsions are called 'scenarios' (see also [6] and, for a unifying approach, [4]).

Envelope theorems were introduced also for other kinds of imprecise previsions, including coherent lower previsions for unbounded random variables [12], convex previsions [7] and conditional coherent lower previsions [16]. The conditional framework is intrinsically more complex, because the set $\mathcal{P}$ is generally not convex and because conditioning events may be allowed to have zero probability.

This paper is concerned with establishing some envelope theorems for conditional convex previsions. Convex and centered convex previsions were introduced in [7] in a framework close to Walley's approach to imprecise previsions. Centered convex previsions are a special subset of previsions that avoid sure loss, are close to coherent lower previsions, but do not require positive homogeneity. In particular, convex risk measures [2] are a special case of convex (not necessarily centered) previsions. Conditional convex previsions and their basic properties were studied in [8].

The notions about convex and conditional convex previsions needed in the sequel are included in Sections 2.1 and 2.2, following the approach in [7, 8], where proofs of the results may be found. Some alternative approaches, like that in [5], are also discussed in [8]. Section 2.3 contains some preliminary material on conditional precise probabilities.

Envelope theorems are stated and discussed in Section 3. In particular, Theorems 5 and 6 point out ways of assessing conditional convex previsions. It is assumed in Theorem 5 that conditioning events have non-zero probability under every prevision in $\mathcal{P}$, while this assumption is dropped in Theorem 6. Theorem 7 characterises implicitly conditional convex previsions, while Theorem 8 gives an explicit characterisation. These results are then compared and their role in assessing or extending convex previsions is discussed.

Section 4.1 deals with the important problem of dilation, considered in [11] for coherent imprecise probabilities; some results in [11] are generalised to convex previsions. Section 4.2 contains a discussion of how the preceding notions can be applied for measuring conditional risk. Section 5 concludes the paper.

2 Preliminaries

In the sequel, $\mathcal{D}$ is an *arbitrary* (non-empty) set of conditional or unconditional *bounded* random variables (or gambles).

2.1 Convex Previsions

Convex lower previsions were defined in [7] as follows:

Definition 1 $\underline{P} : \mathcal{D} \to \mathbb{R}$ *is a* convex lower prevision *on $\mathcal{D}$ iff, for all $n \in \mathbb{N}^+$, $\forall\ X_0, X_1, \ldots, X_n \in \mathcal{D}$, $\forall\ s_1, \ldots, s_n \geq 0$ such that* $\sum_{i=1}^{n} s_i = 1$ (convexity condition), *defining* $\underline{G} = \sum_{i=1}^{n} s_i(X_i - \underline{P}(X_i)) - (X_0 - \underline{P}(X_0))$, $\sup \underline{G} \geq 0$.

The definition can be equivalently stated requiring that $\sum_{i=1}^{n} s_i = s_0 > 0$, where s_0 is multiplied for $(X_0 - \underline{P}(X_0))$ in Definition 1. When dropping the convexity condition $\sum_{i=1}^{n} s_i = s_0$, but asking for $s_0 \geq 0$, the definition reduces to that of coherent lower prevision [13]. If further the sign of the real numbers $s_0, \ldots, s_n$ is unconstrained, we get the definition of coherent *precise* prevision. However, the gap between the properties of convex and coherent imprecise previsions may be wide: for instance, a convex prevision does not necessarily avoid sure loss, nor does it require that $\underline{P}(0) = 0$, when $0 \in \mathcal{D}$. In spite of this, convex previsions deserve some attention because they can explain some uncertainty models, and they allow (under mild conditions) a convex natural extension, a generalisation of the natural extension in [13], with similar properties.

Further interesting properties hold for the subset of convex lower previsions such that ($0 \in \mathcal{D}$ and) $\underline{P}(0) = 0$: these are called *centered* convex previsions or *C-convex* previsions.

In particular, C-convex lower previsions always avoid sure loss and are such that $\underline{P}(\lambda X) \leq \lambda \underline{P}(X)$, $\forall \lambda \in]-\infty, 0[\cup]1, +\infty[$ and $\underline{P}(\lambda X) \geq \lambda \underline{P}(X)$ $\forall \lambda \in [0, 1]$. These properties are useful to incorporate some forms of risk aversion or, in a financial environment, liquidity risks. Convex previsions, centered or not, are characterised by an envelope theorem, as follows.

Theorem 1 (Envelope theorem) *$\underline{P}$ is convex on $\mathcal{D}$ iff there exist a set $\mathcal{P}$ of coherent precise previsions on $\mathcal{D}$ and a function $\alpha : \mathcal{P} \to \mathbb{R}$ such that:*

(a) $\underline{P}(X) = \inf_{P \in \mathcal{P}} \{P(X) + \alpha(P)\}$, $\forall X \in \mathcal{D}$ (inf *is attained).*

Moreover, $\underline{P}$ is C-convex iff ($0 \in \mathcal{D}$ and) both (a) and the following (b) hold:

(b) $\inf_{P \in \mathcal{P}} \alpha(P) = 0$ (inf *is attained).*

The customary envelope theorem for coherent lower previsions is a special case of Theorem 1, with $\alpha \equiv 0$.

2.2 Convex Conditional Previsions

In this section and later on $\mathcal{D}$ is a set of (bounded) conditional random variables $X|B$, where B is a non-impossible event. If the numbers $s_1, \ldots, s_n$ are associated to, respectively, $X_1|B_1, \ldots, X_n|B_n \in \mathcal{D}$, we call *support* of $\underline{s} = (s_1, \ldots, s_n)$ the event $S(\underline{s}) = \bigvee\{B_i : s_i \neq 0, i = 1, \ldots, n\}$. We shall use the same symbol B to denote either an event or its indicator function (de Finetti's convention).

Definition 2 $\underline{P} : \mathcal{D} \to \mathbb{R}$ *is a* convex conditional lower prevision *on $\mathcal{D}$ iff, $\forall n \in \mathbb{N}^+$, $\forall X_0|B_0, \ldots, X_n|B_n \in \mathcal{D}$, $\forall s_1, \ldots, s_n \geq 0 : \sum_{i=1}^{n} s_i = 1$, defining* $\underline{G} = \sum_{i=1}^{n} s_i B_i(X_i - \underline{P}(X_i|B_i)) - B_0(X_0 - \underline{P}(X_0|B_0))$, $\sup\{\underline{G}|S(\underline{s}) \vee B_0\} \geq 0$.

Clearly, this definition is a generalisation of Definition 1. Terming s_0 the number associated to $X_0|B_0$, the convexity condition is equivalently written as $\sum_{i=1}^{n} s_i = s_0 > 0$. From this form we obtain:

- Williams' definition of coherence for lower previsions [16] (in an equivalent form), when dropping the condition $\sum_{i=1}^{n} s_i = s_0$ (requiring $s_0 \geq 0$) and replacing $S(\underline{s}) \vee B_0$ with $S^*(\underline{s}) = \vee\{B_i : s_i \neq 0,\ i = 0, 1, \ldots, n\}$;
- the notion of coherence for *precise* conditional previsions, when further allowing $s_0, \ldots, s_n$ to be any real numbers.

We recall that if P is a coherent precise conditional prevision on $\mathcal{D}$, it has at least one coherent extension on any $\mathcal{D}' \supset \mathcal{D}$; in particular, $\forall X|B, Y|B \in \mathcal{D}, \forall h, k \in \mathbb{R}$,

$$P(kX + hY|B) = kP(X|B) + hP(Y|B). \qquad (1)$$

Convex conditional previsions may be characterised through a set of axioms when $\mathcal{D}$ has a special structure:

Theorem 2 *Let $\mathcal{X}$ be a linear space of bounded random variables, $\mathcal{E} \subset \mathcal{X}$ the set of all indicator functions* *of events in $\mathcal{X}$. Let also $1 \in \mathcal{E}$ and $BX \in \mathcal{X}$, $\forall B \in \mathcal{E}$, $\forall X \in \mathcal{X}$.*[1] *Define $\mathcal{E}^{\varnothing} = \mathcal{E} - \{\varnothing\}$, $\mathcal{D}_{LIN} = \{X|B : X \in \mathcal{X}, B \in \mathcal{E}^{\varnothing}\}$.*

$\underline{P} : \mathcal{D}_{LIN} \to \mathbb{R}$ *is a convex conditional lower prevision if and only if:*

(D1) $\underline{P}(X|B) - \underline{P}(Y|B) \leq \sup\{X - Y|B\}, \forall X, Y \in \mathcal{X}, \forall B \in \mathcal{E}^{\varnothing}$

(D2) $\underline{P}(\lambda X + (1-\lambda)Y|B) \geq \lambda \underline{P}(X|B) + (1-\lambda)\underline{P}(Y|B), \forall X, Y \in \mathcal{X}, \forall B \in \mathcal{E}^{\varnothing}, \forall \lambda \in]0,1[$

(D3) $\underline{P}(A(X - \underline{P}(X|A \wedge B))|B) = 0, \forall X \in \mathcal{X}, \forall A, B \in \mathcal{E}^{\varnothing}: A \wedge B \neq \varnothing$.

We term (D3) *Generalised Bayes Rule* (GBR), after the name it was given in [13] in the special case $B = \Omega$. When $\underline{P}$ is a coherent precise prevision P, (D3) reduces to the familiar Bayes rule

$$P(AX|B) = P(A|B)P(X|A \wedge B). \tag{2}$$

Definition 3 *A convex conditional lower prevision $\underline{P}$ is* centered *if $0|B \in \mathcal{D}$ and $\underline{P}(0|B) = 0, \forall X|B \in \mathcal{D}$.*

Centered previsions play a prominent role within the class of convex conditional previsions, much like or even more than the unconditional case. In fact, C-convex conditional previsions are guaranteed to avoid uniform loss, and the following notion of (conditional) convex natural extension may be developed for them:

Definition 4 *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a conditional lower prevision, $Z|B$ an arbitrary bounded conditional random variable. Define $g_i = s_i B_i(X_i - \underline{P}(X_i|B_i))$, $L(Z|B) = \{\alpha : \sup\{\sum_{i=1}^{n} g_i - B(Z - \alpha)|S(\underline{s}) \vee B\} < 0$, for some $n \geq 1, X_i|B_i \in \mathcal{D}, s_i \geq 0$, with $\sum_{i=1}^{n} s_i = 1\}$. The* convex natural extension *of $\underline{P}$ to $Z|B$ is $\underline{E}_c(Z|B) = \sup L(Z|B)$.*

It can be shown that a sufficient condition for the convex natural extension not to be vacuous is that the conditional lower prevision is centered and $0|B \in \mathcal{D}$, while there is no analogous need in the unconditional case (cf. [8] for details).

We recall that the notion of avoiding uniform loss may be defined as follows (cf. also [13, 14])

Definition 5 *$\underline{P} : \mathcal{D} \to \mathbb{R}$ is a conditional lower prevision* avoiding uniform loss *on $\mathcal{D}$ iff, for all $n \in \mathbb{N}^+$, $\forall X_1|B_1, \ldots, X_n|B_n \in \mathcal{D}$, $\forall s_1, \ldots, s_n \geq 0$, defining $\underline{G} = \sum_{i=1}^{n} s_i B_i(X_i - \underline{P}(X_i|B_i))$, $\sup\{\underline{G}|S(\underline{s})\} \geq 0$*

[1] The assumptions imply that the set of events whose indicator functions are in $\mathcal{E}$ is an algebra. We shall use the same symbol $\mathcal{E}$ to denote also this set.

and that lower previsions that avoid uniform loss may be characterised by the following result [16]:

Theorem 3 *$\underline{P}$ avoids uniform loss on $\mathcal{D}$ iff there is a coherent precise prevision P such that $P(X|B) \geq \underline{P}(X|B)$, $\forall X|B \in \mathcal{D}$.*

2.3 Conditional Probabilities

If $\mathcal{A}$ is an algebra of events and $\mathcal{A}^{\varnothing} = \mathcal{A} - \{\varnothing\}$, a coherent precise conditional prevision on $\mathcal{A}|\mathcal{A}^{\varnothing} = \{A|B : A \in \mathcal{A}, B \in \mathcal{A}^{\varnothing}\}$ specialises to a (coherent or) finitely additive conditional probability P. As well known [3], P induces some relations that let us rank zero probability events. These results are recalled here *partly*, in a form suitable for their later usage in Section 3. First, define relation $\preceq_P$ in $\mathcal{A}^{\varnothing}$: $A \preceq_P B$, or B is of order (of probability) not lower than A, iff $P(B|A \vee B) > 0$. Relation $\preceq_P$ is a weak order, to which the strict weak order $\prec_P$ is canonically associated: $A \prec_P B$, or B is of higher order (of probability) than A, iff $P(A|A \vee B) = 0$. Further, say that $A \approx_P B$, or A and B have the same order of probability, iff $A \preceq_P B$ and $B \preceq_P A$, that is iff $P(A|A \vee B)P(B|A \vee B) > 0$. Relation $\approx_P$ is an equivalence in $\mathcal{A}^{\varnothing}$ and $\preceq_P$ induces a simple order in the set $\mathcal{C} = \{K_i : i \in I\}$ of the corresponding equivalence classes: if $A \in K_i$ and $B \in K_j$, then $K_i \preceq_P^* K_j$ iff $A \preceq_P B$. A minimal class may or may not exist, whilst there is always a maximal class which is formed by those and only those events having (unconditional) positive probability.

Theorem 4 *Let P be a finitely additive conditional probability on $\mathcal{A}|\mathcal{A}^{\varnothing}$. Let also $\mathcal{C}$ be the set of equivalence classes determined by $\approx_P$. Then, for each $K \in \mathcal{C}$ there exists a positive function $\pi_P : K \to \mathbb{R}$ (determined up to a positive constant factor and called* weight function*) such that, $\forall A, B \in K : \varnothing \neq A \wedge B \in K$,*

$$\pi_P(A \wedge B) = P(A|B)\pi_P(B). \tag{3}$$

3 Envelope Theorems

The first result we present generalises to convex conditional lower previsions a statement already established for coherent [13] or convex unconditional [7] lower previsions. The proof is similar to those in [7, 13] and is omitted.

Proposition 1 *Let $\mathcal{P}$ be a set of convex conditional lower previsions defined on $\mathcal{D}$. If $\underline{P}(X|B) = \inf_{\underline{Q} \in \mathcal{P}} \{\underline{Q}(X|B)\}$ is finite $\forall X|B \in \mathcal{D}$, $\underline{P}$ is convex on $\mathcal{D}$.*

Theorem 1 tells us that unconditional convex previsions can be characterised in terms of functions $P(X)+\alpha(P)$. The point now is whether some generalisations of these functions may play a similar role in the conditional case. The next results allow answering this question.

Notation Given $\mathcal{D}$, let $\mathcal{B}=\{B:\exists X|B\in\mathcal{D}\}$, $\mathcal{H}=\{B|B\vee C:B,C\in\mathcal{B}\}$. □

Theorem 5 (Envelope Theorem) *Let $\mathcal{P}$ be a set of coherent precise previsions on $\mathcal{D}\cup\mathcal{B}$ such that $\forall P\in\mathcal{P}$, $P(B)>0\ \forall B\in\mathcal{B}$, and let $\alpha:\mathcal{P}\to\mathbb{R}$ be a real function. Then*

$$\underline{P}(X|B)=\inf_{P\in\mathcal{P}}\{P(X|B)+\frac{\alpha(P)}{P(B)}\}\ \forall X|B\in\mathcal{D} \quad (4)$$

is a convex conditional lower prevision on $\mathcal{D}$, whenever the infimum in (4) is finite. Further, $\underline{P}$ is centered iff $\inf_{P\in\mathcal{P}}\{\frac{\alpha(P)}{P(B)}\}=0$, $\forall B\in\mathcal{B}$.

Proof. We prove that $\forall P\in\mathcal{P}$, $\forall\alpha\in\mathbb{R}$, $\underline{P}_c=P(X|B)+\frac{\alpha}{P(B)}$ is convex. The main thesis of the theorem then follows from Proposition 1.

To prove that $\underline{P}_c$ is a convex conditional lower prevision, we show that a generic $\underline{G}$ in Definition 2 may be referred to P, after substituting $\underline{P}_c(X|B)$ with $P(X|B)+\frac{\alpha}{P(B)}$, and hence its supremum is non-negative because P is coherent. In fact, let $X_0|B_0,\ldots,X_n|B_n\in\mathcal{D}$, $s_1,\ldots,s_n\geq 0$ such that $\sum_{i=1}^n s_i=1$. Then $\underline{G}$ can be written as $\underline{G}=\sum_{i=1}^n s_iB_i(X_i-P(X_i|B_i)-\alpha/P(B_i))-B_0(X_0-P(X_0|B_0)-\alpha/P(B_0))=\sum_{i=1}^n s_iB_i(X_i-P(X_i|B_i))+\sum_{i=1}^n s_i(B_i\vee B_0)(Z_i-P(Z_i|B_i\vee B_0))-B_0(X_0-P(X_0|B_0))$, where

$$Z_i=\alpha(B_0/P(B_0)-B_i/P(B_i))$$

and $P(Z_i|B_i\vee B_0))=\alpha(P(B_0|B_i\vee B_0)/P(B_0)-P(B_i|B_i\vee B_0)/P(B_i))=\alpha(1/P(B_i\vee B_0)-1/P(B_i\vee B_0))=0$ is, by (1), the only coherent extension of P on $Z_i|B_i\vee B_0$, $i=1,\ldots,n$. Since $S'(\underline{s})=\bigvee\{B_i: s_i\neq 0, i=1,\ldots,n\}\vee\bigvee\{B_i\vee B_0: s_i\neq 0, i=1,\ldots,n\}\vee B_0=S(\underline{s})\vee B_0$, it follows $\sup\underline{G}|S(\underline{s})\vee B_0=\sup\underline{G}|S'(\underline{s})\geq 0$ by coherence of P.

The proof of the second part of the proposition follows at once from noting that when $X|B=0|B$ (4) reduces to $\underline{P}(0|B)=\inf_{P\in\mathcal{P}}\{\frac{\alpha(P)}{P(B)}\}$. ■

We note that condition $\inf_{P\in\mathcal{P}}\{\frac{\alpha(P)}{P(B)}\}=0$ implies that $\inf_{P\in\mathcal{P}}\{\alpha(P)\}=0$, but this simpler condition is equivalent to the former one only in some special cases, for instance when $\mathcal{P}$ is finite.

Theorem 5 lets us assess a convex conditional prevision $\underline{P}$ using (4). We shall further comment on this in Section 3.1. Let us now turn to another point: how can a convex prevision be assessed, when $P(B)=0$ for some $P\in\mathcal{P}$ and some $B\in\mathcal{B}$?

Taking (4) as a starting point, we investigate convexity of function $\underline{P}(X|B)=P(X|B)+\phi(B)$. The next result and its proof are helpful for this.

Proposition 2 *Let $\mathcal{X}$, $\mathcal{E}^{\varnothing}$ and $\mathcal{D}_{LIN}$ be defined as in Theorem 2. Let also P be a coherent precise prevision on $\mathcal{D}_{LIN}$, $\phi:\mathcal{E}^{\varnothing}\to\mathbb{R}$ and suppose $P(E)>0\ \forall E\in\mathcal{E}^{\varnothing}$. Then $\underline{P}(X|B)=P(X|B)+\phi(B)$ is a convex lower prevision on $\mathcal{D}_{LIN}$ iff $\phi(B)=\alpha/P(B)$, $\alpha\in\mathbb{R}$.*

Proof. Clearly, $\underline{P}$ is convex iff it satisfies properties (D1)÷(D3) in Theorem 2. It is easy to check that (D1) and (D2) are true for any $\phi(B)$, using coherence of P and (1) ((D2) holds with equality). As for (D3), we have $\underline{P}(A(X-\underline{P}(X|A\wedge B))|B)=P(A(X-P(X|A\wedge B)-\phi(A\wedge B))|B)+\phi(B)=P(A(X-P(X|A\wedge B))|B)-P(A\cdot\phi(A\wedge B)|B)+\phi(B)=-\phi(A\wedge B)P(A|B)+\phi(B)$, by applying (1), and then (2) at the last equality. Therefore, (D3) is satisfied for $\underline{P}$ iff $\forall A,B\in\mathcal{E}^{\varnothing}:A\wedge B\neq\varnothing$

$$\phi(B)=\phi(A\wedge B)P(A|B). \quad (5)$$

Using (2), (5) holds when $\phi(\cdot)=\alpha/P(\cdot)$ with $\alpha\in\mathbb{R}$. Conversely, putting $B=\Omega$ in (5), we must have $\phi(A)=\phi(\Omega)/P(A)$. ■

Obviously, Proposition 2 does not apply when $P(B)=0$ for some B, but (5) holds no matter whether zero probabilities are involved.

We exploit this important fact to show that *it is necessary for $P(X|B)+\phi(B)$ to be convex that $\phi(B)=0$, for any $B\in\mathcal{E}^{\varnothing}$ which does not belong to the minimal class $K_m=K_m(P)$* (when existing) in the ordering induced by the restriction of P on $\mathcal{E}|\mathcal{E}^{\varnothing}$ (recall that $\mathcal{E}$ is an algebra). In fact, if $B\notin K_m$, there is $A\in\mathcal{E}^{\varnothing}$ such that $A\prec_P B\approx_P A\vee B$. Therefore, if $P(X|B)+\phi(B)$ is convex, by (5) $\phi(A\vee B)=\phi(A)P(A|A\vee B)=0$ and also $\phi(A\vee B)=\phi(B)P(B|A\vee B)$, which implies $\phi(B)=0$.

Hence a convex prevision obtained by the rule $\underline{P}(X|B)=P(X|B)+\phi(B)$ can possibly differ from the precise prevision P only at events in the minimal class, if existing. In particular $\phi(B)\equiv 0$ if no minimal class exists.

The preceding considerations are useful in stating the next Theorem 6.

Notation Given a coherent precise prevision P on $\mathcal{D}_{LIN}$, $\alpha(P)\in\mathbb{R}$, define a function ϕ_P such that

$$\phi_P(B)=\begin{cases}0 & \text{if } B\notin K_m\\ \frac{\alpha(P)}{\pi_P(B)} & \text{if } B\in K_m\end{cases} \quad (6)$$

where π_P is the weight function defined on the minimal class K_m (cf. Section 2.3). □

Theorem 6 (Envelope Theorem) *Let $\mathcal{P}$ be a set of coherent precise previsions on $\mathcal{D}_{LIN}$, and for every $P \in \mathcal{P}$ define ϕ_P as above. Then, defining $\forall X|B \in \mathcal{D}_{LIN}$*

$$\underline{P}(X|B) = \inf_{P\in\mathcal{P}} \{P(X|B) + \phi_P(B)\}, \qquad (7)$$

$\underline{P}$ (when finite) is a convex conditional lower prevision on $\mathcal{D}_{LIN}$. $\underline{P}$ is centered iff $\inf_{P\in\mathcal{P}} \phi_P(B) = 0\ \forall B \in \mathcal{E}^{\varnothing}$.

Proof. It is sufficient to prove convexity of $\underline{P}_c(X|B) = P(X|B) + \phi_P(B)$, since then convexity of $\underline{P}$ follows from Proposition 1. Recalling the proof of Proposition 2, we can prove that $\underline{P}_c$ satisfies properties (D1)÷(D3) in Theorem 2. Again, (D1) and (D2) are not difficult, while (D3) is equivalent to (5). Hence there remains to show that (5) holds if ϕ_P has the form (6).

Suppose that ϕ_P is given by (6), and that $K_m \neq \emptyset$ (if not, (5) holds trivially, from what noted after Proposition 2). Given $A, B \in \mathcal{E}^{\varnothing} : A \wedge B \neq \varnothing$, we distinguish two cases:

i) *If* $B \notin K_m$, (5) holds because $\phi_P(B) = \phi_P(A \wedge B) = 0$ when $A \wedge B \notin K_m$, and because $\phi_P(B) = 0$ and $P(A|B) = P(A\wedge B|(A\wedge B)\vee B) = 0$, when $A \wedge B \in K_m$, since then $A \wedge B \prec_P B$.

ii) *If* $B \in K_m$, then necessarily $A \wedge B \in K_m$ since $B \succeq_P A\wedge B$. We can then replace ϕ_P (as defined in (6)) in (5), and use (3) to check that equality holds in (5).

Finally, put $X|B = 0|B$ in (7) to check that $\underline{P}$, given by (7), is centered iff $\inf_{P\in\mathcal{P}} \phi_P(B) = 0\ \forall B \in \mathcal{E}^{\varnothing}$. ■

The simplest way to use Theorem 6 is to put $\alpha \equiv 0$, getting $\underline{P}^*(X|B) = \inf_{P\in\mathcal{P}} P(X|B)$. This gives a coherent lower prevision $\underline{P}^*$, as well known [13, 16]. Even when $\alpha \not\equiv 0$, the convex prevision obtained using Theorem 6 will be coherent at least on $\mathcal{D}_c \subset \mathcal{D}_{LIN}$, where $\mathcal{D}_c = \{X|B : \forall P \in \mathcal{P}, B \notin K_m(P)\}$.

When considering also the reverse problem of how a convex prevision could be characterised, more general forms of envelope theorems for convex previsions can be obtained. The following lemma is a preliminary result.

Lemma 1 *Given $\underline{P}$ on domain $\mathcal{D}$, define, for all $X_0|B_0 \in \mathcal{D}$, $\mathcal{D}_{X_0|B_0} = \{B(X - \underline{P}(X|B)) - B_0(X_0 - \underline{P}(X_0|B_0))|B \vee B_0 : X|B \in \mathcal{D}\}$ and $\underline{P}_{X_0|B_0} = 0$ on $\mathcal{D}_{X_0|B_0}$. $\underline{P}$ is convex iff every $\underline{P}_{X_0|B_0}$ avoids uniform loss on its domain $\mathcal{D}_{X_0|B_0}$.*

Proof. Defining $Y_i = B_i(X_i - \underline{P}(X_i|B_i)) - B_0(X_0 - \underline{P}(X_0|B_0))$ and recalling that $\sum_{i=1}^n s_i = 1$, $\underline{G}$ in Definition 2 may be written as $\underline{G} = \sum_{i=1}^n s_i(B_i \vee B_0)(Y_i - \underline{P}_{X_0|B_0}(Y_i|B_i \vee B_0))$. Putting $S'(\underline{s}) = \bigvee\{B_i \vee B_0 : s_i \neq 0, i = 1,\dots,n\} = S(\underline{s}) \vee B_0$, $\sup\{\underline{G}|S(\underline{s}) \vee B_0\} = \sup \underline{G}|S'(\underline{s}) \geq 0$ iff $\underline{P}_{X_0|B_0}$ avoids uniform loss on $\mathcal{D}_{X_0|B_0}$, by Definition 5. The thesis follows. ■

Theorem 7 (Implicit Envelope Theorem) *Let $\mathcal{D}' = \mathcal{D}\cup\mathcal{H}$. $\underline{P} : \mathcal{D} \to \mathbb{R}$ is convex iff there exists a set $\mathcal{P}$ of coherent precise previsions on $\mathcal{D}'$ such that $\forall X_0|B_0 \in \mathcal{D}, \exists\ P_{X_0|B_0} \in \mathcal{P}$: $\forall X|B \in \mathcal{D}$*

$$\begin{aligned}&P_{X_0|B_0}(B|B \vee B_0)(P_{X_0|B_0}(X|B) - \underline{P}(X|B)) \geq\\ &P_{X_0|B_0}(B_0|B \vee B_0)(P_{X_0|B_0}(X_0|B_0) - \underline{P}(X_0|B_0)).\end{aligned} \qquad (8)$$

Proof. We begin by proving that convexity of $\underline{P}$ implies the existence of $\mathcal{P}$. From Lemma 1 (with the same notation) and Theorem 3, $\underline{P}$ is convex iff for every $X_0|B_0 \in \mathcal{D}$ there exists a coherent prevision $P_{X_0|B_0}$ on $\mathcal{D}_{X_0|B_0}$ such that $P_{X_0|B_0}(Y|C) \geq \underline{P}_{X_0|B_0}(Y|C) = 0, \forall Y|C \in \mathcal{D}_{X_0|B_0}$, that is, extending $P_{X_0|B_0}$ where necessary and exploiting (1), $P_{X_0|B_0}(B(X - \underline{P}(X|B))|B \vee B_0) - P_{X_0|B_0}(B_0(X_0 - \underline{P}(X_0|B_0))|B \vee B_0) \geq 0\ \forall X|B \in \mathcal{D}$. Since by (2) $P_{X_0|B_0}(B(X - \underline{P}(X|B))|B \vee C) = P_{X_0|B_0}(B|B \vee C)P_{X_0|B_0}(X - \underline{P}(X|B)|B) = P_{X_0|B_0}(B|B \vee C)(P_{X_0|B_0}(X|B) - \underline{P}(X|B))\ \forall X|B \in \mathcal{D}, \forall B|B \vee C \in \mathcal{H}$, the thesis follows defining $\mathcal{P} = \{P_{X_0|B_0} : X_0|B_0 \in \mathcal{D}\}$. Sufficiency can be obtained by reversing the preceding argument. ■

We now characterise C-convex previsions.

Proposition 3 *Let $\underline{P}$ be a convex lower prevision on $\mathcal{D}$ and let $0|B \in \mathcal{D}\ \forall X|B \in \mathcal{D}$. Denote also with $P_{X_0|B_0}$ a coherent prevision associated to $X_0|B_0$ in Theorem 7, $\forall X_0|B_0 \in \mathcal{D}$. Then $\underline{P}$ is centered iff*

a) $P_{0|B}(X|B) \geq \underline{P}(X|B)\ \forall X|B \in \mathcal{D}$

b) $P_{X_0|B_0}(X_0|B_0) \leq \underline{P}(X_0|B_0)\ \forall X_0|B_0 \in \mathcal{D}$

Proof. If a) and b) hold, putting there $X|B = 0|B$ and $X_0|B_0 = 0|B$, we get $0 = P_{0|B}(0|B) \geq \underline{P}(0|B) \geq P_{0|B}(0|B) = 0$, hence $\underline{P}(0|B) = 0\ \forall 0|B \in \mathcal{D}$. Conversely, suppose $\underline{P}$ is centered convex. Then a) can be obtained putting $X_0|B_0 = 0|B$ in (8), whilst b) follows letting $X|B = 0|B_0$ in (8). ■

Theorem 8 (Explicit Envelope Theorem) *Let $\underline{P}$ be a lower prevision on $\mathcal{D}_{LIN}$. Then $\underline{P}$ is convex if and only if there exist a set of coherent precise previsions on $\mathcal{D}_{LIN}$, $\mathcal{P} = \{P_{Y|C} : Y|C \in \mathcal{D}_{LIN}\}$, and $\alpha : \mathcal{P} \to \mathbb{R}$ such that, denoting for each $X|B \in \mathcal{D}_{LIN}$ by $\pi_{P_{Y|C}}(B)$ the weight of B determined by $P_{Y|C}$ and* *defining* $\forall X|B, Y|C \in \mathcal{D}_{LIN}$,

$$\mathcal{P}_{X|B} = \{P_{Y|C} \in \mathcal{P} : B \succeq_{P_{Y|C}} C\}, \qquad (9)$$

$$\phi_{P_{Y|C}}(B) = \begin{cases} 0 & \text{if } B \succ_{P_{Y|C}} C \\ \frac{\alpha(P_{Y|C})}{\pi_{P_{Y|C}}(B)} & \text{if } B \approx_{P_{Y|C}} C, \end{cases} \qquad (10)$$

the following statements hold:

a) $\underline{P}(X|B) = \min_{P \in \mathcal{P}_{X|B}}\{P(X|B) + \phi_P(B)\}$ $\forall X|B \in \mathcal{D}_{LIN}$;

b) if $X|B$, $X_0|B_0 \in \mathcal{D}_{LIN}$ *and* $B_0 \succ_{P_{X_0|B_0}} B$, $\alpha(P_{X_0|B_0}) \geq 0$.

Proof. We show first that convexity of $\underline{P}$ implies the existence of a set $\mathcal{P}$ such that a) holds. By Theorem 7 there exists a set $\mathcal{P} = \{P_{Y|C} : Y|C \in \mathcal{D}_{LIN}\}$ of coherent precise previsions on $\mathcal{D}_{LIN}$ such that $\forall X_0|B_0 \in \mathcal{D}_{LIN}, \exists\, P_{X_0|B_0} \in \mathcal{P}$: $\forall X|B \in \mathcal{D}_{LIN}$, (8) holds. Let now $X|B \in \mathcal{D}_{LIN}$ and $P_{X_0|B_0} \in \mathcal{P}_{X|B}$ (i.e. $B \succeq_{P_{X_0|B_0}} B_0$). We distinguish two cases:

i) If $B \succ_{P_{X_0|B_0}} B_0$, $P_{X_0|B_0}(B_0|B \vee B_0) = 0$ and $P_{X_0|B_0}(B|B \vee B_0) = 1$. Therefore, it ensues from (8) in Theorem 7 that $\underline{P}(X|B) \leq P_{X_0|B_0}(X|B) = P_{X_0|B_0}(X|B) + \phi_{P_{X_0|B_0}}(B)$.

ii) If $B \approx_{P_{X_0|B_0}} B_0$, by Theorem 4 we have

$$\begin{aligned} P_{X_0|B_0}(B|B \vee B_0) &= \frac{\pi_{P_{X_0|B_0}}(B)}{\pi_{P_{X_0|B_0}}(B \vee B_0)} \\ P_{X_0|B_0}(B_0|B \vee B_0) &= \frac{\pi_{P_{X_0|B_0}}(B_0)}{\pi_{P_{X_0|B_0}}(B \vee B_0)}. \end{aligned} \qquad (11)$$

Substituting (11) into (8) and putting

$$\begin{aligned} &\alpha(P_{X_0|B_0}) = \\ &\pi_{P_{X_0|B_0}}(B_0)(\underline{P}(X_0|B_0) - P_{X_0|B_0}(X_0|B_0)), \end{aligned} \qquad (12)$$

we get

$$\begin{aligned} &\underline{P}(X|B) \leq P_{X_0|B_0}(X|B) + \frac{\alpha(P_{X_0|B_0})}{\pi_{P_{X_0|B_0}}(B)} = \\ &P_{X_0|B_0}(X|B) + \phi_{P_{X_0|B_0}}(B). \end{aligned} \qquad (13)$$

Since (13) becomes an equality when $X_0|B_0 = X|B$ (therefore $P_{X_0|B_0} = P_{X|B} \in \mathcal{P}_{X|B}$), a) holds. If further $B_0 \succ_{P_{X_0|B_0}} B$ for $X_0|B_0, X|B \in \mathcal{D}_{LIN}$, then $P_{X_0|B_0}(B|B \vee B_0) = 0$ and $P_{X_0|B_0}(B_0|B \vee B_0) = 1$, so that (8) becomes $P_{X_0|B_0}(X_0|B_0) \leq \underline{P}(X_0|B_0)$, i. e. $\alpha(P_{X_0|B_0}) \geq 0$.

To prove sufficiency note that by a), given $X_0|B_0 \in \mathcal{D}_{LIN}$, there exists $P_{\overline{X}|\overline{B}} \in \mathcal{P}_{X_0|B_0}$ such that $B_0 \succeq_{P_{\overline{X}|\overline{B}}} \overline{B}$ and

$$\underline{P}(X_0|B_0) = P_{\overline{X}|\overline{B}}(X_0|B_0) + \phi_{P_{\overline{X}|\overline{B}}}(B_0). \qquad (14)$$

Further, if $X|B$ is such that $B \succeq_{P_{\overline{X}|\overline{B}}} \overline{B}$ (i.e. $P_{\overline{X}|\overline{B}} \in \mathcal{P}_{X|B}$), by a) again,

$$\underline{P}(X|B) \leq P_{\overline{X}|\overline{B}}(X|B) + \phi_{P_{\overline{X}|\overline{B}}}(B). \qquad (15)$$

Fix now $X_0|B_0 \in \mathcal{D}_{LIN}$ and let $X|B$ be any other random variable in $\mathcal{D}_{LIN}$. The proof then consists in showing that (8) holds no matter which are the positions of events B_0, B and $\overline{B}$ in the ordering given by $\succeq_{P_{\overline{X}|\overline{B}}}$. This requires the lengthy work of distinguishing several cases. Since most cases are similar, we examine here only two of them. In the following, we denote $P_{\overline{X}|\overline{B}}$ also with the notation $Q_{X_0|B_0}$ to emphasise that $P_{\overline{X}|\overline{B}}$ is associated to $X_0|B_0$, correspondingly to the notation in (8).

Suppose $B \approx_{P_{\overline{X}|\overline{B}}} B_0 \approx_{P_{\overline{X}|\overline{B}}} \overline{B}$. Recalling the definition of $\phi_{P_{\overline{X}|\overline{B}}}$, from (14) and (15) we obtain $\pi_{Q_{X_0|B_0}}(B)(\underline{P}(X|B) - Q_{X_0|B_0}(X|B)) \leq \alpha(Q_{X_0|B_0}) = \pi_{Q_{X_0|B_0}}(B_0)(\underline{P}(X_0|B_0) - Q_{X_0|B_0}(X_0|B_0))$. Dividing each term of this inequality by $\pi_{Q_{X_0|B_0}}(B \vee B_0)$ and recalling the analogous of (11) for $Q_{X_0|B_0}$ we get (8).

Let now $\overline{B} \approx_{P_{\overline{X}|\overline{B}}} B_0 \succ_{P_{\overline{X}|\overline{B}}} B$. Since in this case $Q_{X_0|B_0}(B|B \vee B_0) = 0$, $Q_{X_0|B_0}(B_0|B \vee B_0) = 1$ and $\phi_{P_{\overline{X}|\overline{B}}}(B_0) = \frac{\alpha(Q_{X_0|B_0})}{\pi_{Q_{X_0|B_0}}(B_0)}$, (8) is equivalent to the inequality $\frac{\alpha(Q_{X_0|B_0})}{\pi_{Q_{X_0|B_0}}(B_0)} \geq 0$, which is true since $\overline{B} \succ_{P_{\overline{X}|\overline{B}}} B$ implies $\alpha(P_{\overline{X}|\overline{B}}) = \alpha(Q_{X_0|B_0}) \geq 0$ by b).

The remaining cases can be proved analogously. ∎

Theorem 8 gives an explicit characterisation of convexity for conditional previsions. It is however more involved than other envelope theorems. In particular note that the set $\mathcal{P}_{X|B}$ in a) on which the minimum is performed *depends on* $X|B$. Another point is that the argument of the minimum in a) is generally *not* a convex prevision (apply the considerations following Proposition 2 to (10)); two notable exceptions are when $\forall Y|C \in \mathcal{D}_{LIN}$ the equivalence $\approx_{P_{Y|C}}$ identifies one equivalence class and when $B \approx_{P_{Y|C}} C$ in (10) implies $B \in K_m(P_{Y|C})$, $\forall X|B \in \mathcal{D}_{LIN}$. In fact, the 'if' statement of Theorem 8 implies, under one or the other of these two conditions, Theorem 5 and Theorem 6, respectively.

3.1 Assessing and Extending Convex Previsions with Envelope Theorems

An important feature of the envelope theorem for convex unconditional previsions (Theorem 1) is that it allows assessing convex or C-convex previsions indirectly: by taking the infimum on a given set $\mathcal{P}$ of precise previsions, each possibly modified by adding a term $\alpha(P)$ to it, a convex or C-convex lower prevision is *uniquely* determined. A notable special case arises when each precise prevision P is given by an expert: $\alpha(P)$ can then be assigned *freely* by the final assessor, also according to his/her degree of confidence towards the expert.

With convex conditional previsions, there are some differences. Let us discuss how the envelope theorems introduced before can be employed. Theorem 5 lets us obtain a unique convex prevision analogously to Theorem 1. However, the correction term is now $\alpha(P)/P(B)$. When $\underline{P}$ is C-convex, then necessarily $\alpha(\cdot) \geq 0$. This means that among those previsions such that $\alpha(\cdot) > 0$, prevision $P \in \mathcal{P}$ will tend to contribute little or nothing to forming $\underline{P}(X|B)$ when $\alpha(P)$ is 'high' or also when its $P(B)$ is 'small' compared to those of the other previsions in $\mathcal{P}$. Therefore, apart from the contribution due to function α, C-convexity tends to rule out previsions giving a comparatively small probability to the conditioning event B when determining $\underline{P}(X|B)$. On the contrary, when $\underline{P}$ is convex but not C-convex, $P \in \mathcal{P}$ may be relevant in forming $\underline{P}(X|B)$ even when $P(B)$ is comparatively very small, if $\alpha(P) < 0$.

The correction term ϕ_P in Theorem 6, when non-zero, works in the same way as $\frac{\alpha(P)}{P(B)}$ in Theorem 5 but operates, for a given P, only on those $X|B$, if any, such that $B \in K_m(P)$. This implies that if $P(B) > 0$, then the correction term can modify $P(X|B)$ only if $\approx_P$ forms just one equivalence class, i.e. if P assigns positive probability to all events in $\mathcal{E}^{\varnothing}$. In practical situations, there will often be a proper subset $\mathcal{D} \subset \mathcal{D}_{LIN}$ such that $\forall X|B \in \mathcal{D}$, $\forall P \in \mathcal{P}$, $P(B) > 0$, and having the chance of correcting evaluations on $\mathcal{D}$ will be more important than modifying those on random variables $Y|C \notin \mathcal{D}$, where $P(C) = 0$ for some $P \in \mathcal{P}$ might suggest that $Y|C$ could be very unlikely. If such is the case, Theorem 5 can be used alternatively to Theorem 6, on the restriction $\mathcal{D}$. This shows also that Theorem 6 is not properly a generalisation of Theorem 5.

In principle, we may also resort to Theorems 7 or 8. The point here is that, given a set $\mathcal{P}$ of precise previsions and given $X_0|B_0$, Theorem 7 does not tell us which $P \in \mathcal{P}$ should play the role of $P_{X_0|B_0}$ in (8). Correspondingly, there is no unquestionably preferable way of getting each $\mathcal{P}_{X|B}$ from $\mathcal{P}$ in Theorem 8. In both cases, different selection criteria can lead to different results, as is easy to verify.

Therefore Theorems 5 and 6, although formally special cases of Theorem 7, seem preferable when assessing convex previsions.

Theorem 5 may be used for extending a previously assessed convex prevision $\underline{P}$ as long as its hypotheses are preserved. In fact, the functions $P(X|B) + \frac{\alpha(P)}{P(B)}$ in (4) of Theorem 5 are convex previsions and can be extended on any $\mathcal{D}'$ by extending P. If $P(C) > 0\ \forall Y|C \in \mathcal{D}'$, the extensions $P(Y|C) + \frac{\alpha(P)}{P(C)}$ are still convex. We can then apply Theorem 5 again on $\mathcal{D}'$, thus extending $\underline{P}$ there.[2]

We recall that envelope theorems let us extend convex previsions *indirectly*; alternatively, one might extend them *directly* by computing the convex natural extension. When $\mathcal{D}$ is finite and each $X|B \in \mathcal{D}$ is *simple*, i.e. has finitely many distinct values, as we will assume in the remaining part of this section, this can be done in ways similar to those developed for the natural extension of coherent conditional lower probabilities [14]. This is intuitively sound, as the basic concepts for convex and coherent previsions formally differ only by the linear constraint $\sum_{i=1}^n s_i = 1$. To give just an idea of the matter, we note that the convex natural extension $\underline{E}_c(Z|B)$ may be also equivalently defined as follows:

Definition 6 *Let $\mathcal{D} = \{X_1|B_1, \ldots, X_n|B_n\}$ and $\underline{P} : \mathcal{D} \to \mathbb{R}$ a conditional lower prevision, $Z|B$ an arbitrary simple conditional random variable. The* convex natural extension *of $\underline{P}$ to $Z|B$ is $\underline{E}_c(Z|B) = \sup\{\alpha : \exists\ \varepsilon > 0, s_i \geq 0\ (i = 1, \ldots, n)$, with $\sum_{i=1}^n s_i = 1$, such that $\sum_{i=1}^n s_i B_i(X_i - \underline{P}(X_i|B_i) + \varepsilon) \leq B(Z - \alpha)\}$.*

The proof that Definition 6 and Definition 4 are equivalent follows closely the proof of Lemma 3 in [14]. Similarly to what developed in [14], Definition 6 suggests the following parametric linear programming problem to evaluate $\underline{E}_c(X|B)$:

$$\textit{maximise } \alpha \textit{ subject to}$$

$$\sum_{i=1}^n s_i B_i(X_i - \underline{P}(X_i|B_i) + \varepsilon) + \alpha B \leq BZ$$

$$\varepsilon > 0, s_i \geq 0\ (i = 1, \ldots, n), \sum_{i=1}^n s_i = 1.$$

The above problem does not always have solutions, unlike the corresponding one in [14], but is feasible when $\underline{P}$ is centered convex, and further $0|B \in \mathcal{D}$, $\underline{P}(0|B) = 0$. The latter condition is not actually restrictive (see [8], Proposition 7 (b)). Solving the problem with a very 'small' fixed value for $\varepsilon > 0$ gives an approximate evaluation for $\underline{E}_c(Z|B)$.

[2] In general (without additional assumptions) the extension will be not unique, nor will it coincide with the convex natural extension.

4 Conditioning with Centered Convex Previsions

4.1 Dilation

Conditional random variables are often introduced by conditioning X on the (non-impossible) events of a given partition $\mathbb{P}$. Roughly speaking, dilation then occurs when the uncertainty evaluation on $X|B$ is vaguer then the evaluation on X, whatever is $B \in \mathbb{P}^{\varnothing} = \mathbb{P} - \{\varnothing\}$. The case when both lower ($\underline{P}$) and upper ($\overline{P}$) previsions are assessed is particularly meaningful, since then there is *strict dilation* [11] when

$$\underline{P}(X|B) < \underline{P}(X) \leq \overline{P}(X) < \overline{P}(X|B),\ \forall B \in \mathbb{P} \quad (16)$$

and we say that $\mathbb{P}$ *dilates strictly* X, while $\mathbb{P}$ *dilates* X when one of the strict inequalities in (16) is replaced by a weak inequality. However assuming, as usual,

$$\overline{P}(X|B) = -\underline{P}(-X|B), \quad (17)$$

which specialises to $\overline{P}(X) = -\underline{P}(-X)$ when $B = \Omega$, strict dilation can be discussed also referring to lower or alternatively upper previsions only. A situation where dilation is conveniently presented referring to upper previsions is sketched out in Section 4.2. Dilation is a somewhat baffling phenomenon, but was shown [11] to be not unusual with coherent imprecise probabilities. We shall now discuss some aspects of dilation with C-convex previsions, extending some results in [11] to this case.

To comply with the situation described above, let us suppose that the set $\mathcal{D}$ is formed by unconditional random variables only, and that:

- a partition $\mathbb{P}$ is given and $\mathbb{P} \subset \mathcal{D}$;
- $X \in \mathcal{D} \Rightarrow BX \in \mathcal{D}$, $\forall B \in \mathbb{P}$;
- $X \in \mathcal{D} \Rightarrow -X \in \mathcal{D}, X + c \in \mathcal{D}, \forall c \in \mathbb{R}$.

Let now a C-convex lower prevision $\underline{P}$ be assigned on $\mathcal{D}$. By Theorem 1, $\underline{P}(X) = \min_{P \in \mathcal{P}}\{P(X) + \alpha(P)\}$, where $\mathcal{P}$ is a set of coherent precise previsions on $\mathcal{D}$ and $\inf_{P\in\mathcal{P}} \alpha(P) = 0$. Note that we do not assume at this stage that $P(B) > 0$, $\forall P \in \mathcal{P}$, $B \in \mathbb{P}$.

Notation Define, $\forall X \in \mathcal{D}$, $M_*(X) = \{P \in \mathcal{P} : \underline{P}(X) = P(X) + \alpha(P)\}$, $M^*(X) = M_*(-X)$ (using (17) with $B = \Omega$, $M^*(X) = \{P \in \mathcal{P} : \overline{P}(X) = P(X) - \alpha(P)\}$).

Further, for $X \in \mathcal{D}$, $B \in \mathbb{P}^{\varnothing}$, define

$\Sigma^-_\alpha(X, B) = \{P \in \mathcal{P} : P(X)P(B) > P(BX) + \alpha(P)\}$,
$\Sigma^+_\alpha(X, B) = \Sigma^-_\alpha(-X, B) =$
$\{P \in \mathcal{P} : P(X)P(B) < P(BX) - \alpha(P)\}$.

In particular $\Sigma^-_0(X, B) = \{P \in \mathcal{P} : P(X)P(B) > P(BX)\}$. Note that when $P(B) > 0$, P belongs to $\Sigma^-_0(X, B)$ iff $P(X|B) < P(X)$, i.e. iff B is *negatively relevant* for X.[3] Similar considerations apply to $\Sigma^+_0(X, B)$. □

Let now $X \in \mathcal{D}$ and suppose that $\underline{P}$ is extended, preserving C-convexity, on $\mathcal{D}' = \mathcal{D} \cup \{X|B, 0|B : B \in \mathbb{P}^{\varnothing}\}$. We explore strict dilation of X with respect to $\mathbb{P}$.

Lemma 2 *If $P \in \Sigma^-_\alpha(X, B)$, then $P(B) > 0$ and $P(X) > \underline{P}(X|B)$; if $P \in \Sigma^+_\alpha(X, B)$, $P(B) > 0$ and $P(X) < \overline{P}(X|B)$.*

Proof. Let $P \in \Sigma^-_\alpha(X, B)$. By Theorem 1 and the GBR, $P(B(X - \underline{P}(X|B))) + \alpha(P) \geq \underline{P}(B(X - \underline{P}(X|B))) = 0$, therefore $P(BX) + \alpha(P) \geq \underline{P}(X|B)P(B)$. Since $P \in \Sigma^-_\alpha(X, B)$, $P(X)P(B) > P(BX) + \alpha(P) \geq \underline{P}(X|B)P(B)$. Then $P(X)P(B) > \underline{P}(X|B)P(B)$, and hence $P(B) > 0$, $P(X) > \underline{P}(X|B)$. The second part follows from the first one when referring to $-X|B$. ■

Proposition 4 *If, $\forall B \in \mathbb{P}^{\varnothing}$, $M_*(X) \cap \Sigma^-_\alpha(X, B) \neq \emptyset$ and $M^*(X) \cap \Sigma^+_\alpha(X, B) \neq \emptyset$, then $\mathbb{P}$ dilates strictly X.*

Proof. Suppose $P \in M_*(X) \cap \Sigma^-_\alpha(X, B)$. Then we obtain, using also Theorem 1, and Lemma 2 at the last inequality (recall that $\alpha(P) \geq 0$),

$$P(X) + \alpha(P) = \underline{P}(X) \geq P(X) > \underline{P}(X|B),$$

which proves the first inequality in (16).

When $M^*(X) \cap \Sigma^+_\alpha(X, B) = M_*(-X) \cap \Sigma^-_\alpha(-X, B) \neq \emptyset$, we get from the first part $\underline{P}(-X|B) < \underline{P}(-X)$ and therefore, recalling (17), $\overline{P}(X|B) > \overline{P}(X)$. ■

We make now the further assumptions that $\forall P \in \mathcal{P}$, $\forall B \in \mathbb{P}^{\varnothing}$, $P(B) > 0$ and that the extension of $\underline{P}$ from $\mathcal{D}$ to $\mathcal{D}'$ is made by putting $\underline{P}(X|B) = \inf_{P\in\mathcal{P}}\{\frac{P(BX)}{P(B)} + \frac{\alpha(P)}{P(B)}\}$ (a similar kind of extension is considered in [11] for coherent imprecise probabilities). By Theorem 5, $\underline{P}$ is convex. Define then the following set:

$$M_*(X|B) = \{P \in \mathcal{P} : \underline{P}(X|B) = P(X|B) + \frac{\alpha(P)}{P(B)}\}.$$

Proposition 5 *$\mathbb{P}$ does not dilate strictly X if there exists $B \in \mathbb{P}^{\varnothing}$ such that either $M_*(X|B) \not\subset \Sigma^-_0(X, B)$ or $M_*(-X|B) \not\subset \Sigma^+_0(X, B)$.*

Proof. When $M_*(X|B) \not\subset \Sigma^-_0(X, B)$ there is $P \in \mathcal{P}$ such that $\underline{P}(X|B) = P(X|B) + \frac{\alpha(P)}{P(B)}$ and $P(BX) \geq$

[3] B is *irrelevant* for X when $P(X|B) = P(X)$.

$P(B)P(X)$. From this $\underline{P}(X|B) = \frac{P(BX)}{P(B)} + \frac{\alpha(P)}{P(B)} \geq P(X) + \alpha(P) \geq \underline{P}(X)$.

The proof that condition $M_*(-X|B) \not\subset \Sigma_0^+(X,B)$ is sufficient to guarantee that $\overline{P}(X|B) \leq \overline{P}(X)$, and hence to avoid strict dilation, follows easily recalling that $\Sigma_0^+(X,B) = \Sigma_0^-(-X,B)$ and applying (17). ■

When $\underline{P}$ is a coherent lower probability $\Sigma_\alpha^-(X,B) = \Sigma_0^-(X,B)$; Propositions 4 and 5 specialise to Theorems 2.3 and (in an equivalent statement) 2.2 in [11], respectively.

In general, Proposition 4 states a sufficient condition for strict dilation, while Proposition 5 gives two conditions, each one sufficient to avoid strict dilation.

Suppose now that $\mathcal{P}$ is formed by n previsions expressing the opinions of n experts. Proposition 5 implies that X is not affected by strict dilation if, for at least one $B \in I\!P^\varnothing$, the experts agree on B being positively relevant or irrelevant for X (alternatively: B is unanimously considered negatively relevant or irrelevant for X): in fact either $\Sigma_0^-(X,B)$ or $\Sigma_0^+(X,B)$ is then empty. General agreement on the relevance of B for X is a stronger requirement than those in Proposition 5, but may be fairly natural if there is a clear relationship between B and X.

4.2 Convex Risk Measures

Given $X|B$, a risk measure $\rho(X|B)$ is a real number which evaluates the 'riskiness' of $X|B$. In a financial setting $\rho(X|B)$, when positive, is often a benchmark to establish the amount of money which an investor holding the financial asset X should reserve to have an adequate protection against the riskiness of X, in the hypothesis that B is true.

Risk measures may be interpreted [6, 8] as upper previsions:

$$\rho(X|B) = \overline{P}(-X|B) \ (= -\underline{P}(X|B)). \tag{18}$$

This enables us to apply the theory of imprecise previsions to them. In the unconditional case, convex risk measures were discussed in [7], showing that convex previsions generalise a notion of convex risk measure introduced in [2].

The results of the preceding sections, when cast into their specular form for upper previsions, concern convex conditional risk measures: from (18) a convex risk measure for the (conditional) risk $X|B$ is a convex conditional upper prevision for $-X|B$.

Convex risk measures may be assessed using the envelope theorems introduced in Section 3 for upper previsions. For instance, Theorem 6 and (18) allow assessing a convex conditional risk measure $\rho(\cdot|\cdot)$ as

$$\rho(X|B) = \sup_{P \in \mathcal{P}} \{P(-X|B) - \phi_P(B)\}. \tag{19}$$

The previsions $P \in \mathcal{P}$ may be termed 'scenarios'; assessing risk measures via envelope theorem by (19) is a generalisation of the method of scenarios mentioned, for instance, in [1].

Using (17) and (18), the strict dilation condition (16) is written as follows for risk measures:

$$\rho(X|B) > \rho(X), \rho(-X|B) > \rho(-X), \forall B \in I\!P^\varnothing. \tag{20}$$

In words, strict dilation implies that the money an investor should reserve to cover risks from his/her holding either X or $-X$ must be increased when assuming that B will be true, no matter which $B \in I\!P$ is chosen. Since one $B \in I\!P^\varnothing$ is certainly true, the reserve money should be raised in all cases. A crucial point here is the choice of $I\!P$: if $I\!P$ is well-chosen, in the sense that the influence (or relevance) of at least one $B \in I\!P^\varnothing$ on X is relatively easy to state, dilation does not occur (this is the special case of Proposition 5 mentioned at the end of Section 4.1).

Example X is the random return at a fixed future time t_0 of an investment in country C. Country C is moderately unstable, but is trying to enter the European Union (EU) by time t_0. To evaluate $\rho(X)$, an investor may resort to a group of experts, each one supplying a 'scenario' (i.e. $P(X)$). To have a more detailed evaluation, each expert is subsequently asked to assess also $P(X|B)$, $P(X|B^c)$, where $B =$ 'Country C will enter the EU by time t_0'. The investor can then easily obtain also $\rho(X|B)$, $\rho(X|B^c)$. Given the well-established stability policy of the EU, there should intuitively be a large consensus on B being positively relevant for X, i.e. one might expect that $P(X|B) > P(X)$ for each expert. If so, this will prevent dilation; if not, dilation may or may not occur, but the investors should rather focus on the reasons why the experts' unanimity failed.

5 Conclusions

This paper complements the theory of convex conditional previsions studied in [8], solving in particular the question left open there of stating envelope theorems for this kind of imprecise previsions. This is an interesting matter both theoretically, as it shows how these theorems work when departing from coherence, and operationally, pointing out ways of assessing convex previsions. The important phenomenon of dilation, subsequently considered in the paper, may be related with envelope theorems too. In fact, a result given here (Proposition 5) relies on a specific extension of a given convex unconditional prevision; it would be interesting to examine how dilation operates under different extensions, and envelope theorems may be useful to construct them. Intuitively, the (convex) natural extension, being the vaguest possible (convex) extension, should tend to amplify dilation. This justifies the future work of investigating other kinds of extensions, in particular generalisations of those considered in [15], which, as discussed in [6] for the case of coherent (unconditional) previsions, are also meaningful in a risk measurement view, since they can be interpreted as 'prudential' extensions.

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References

[1] P. Artzner, F. Delbaen, S. Eber and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9:203-228, 1999.

[2] H. Föllmer and A. Schied. Convex measures of risk and trading constraints. *Finance and Stochastics*, 6:429-447, 2002.

[3] P. H. Krauss. Representation of conditional probability measures. *Acta Math. Acad. Sci. Hungar.*, 19:229-241, 1968.

[4] S. Maaß. Exact functionals and their core. *Statistical Papers*, 43:75–93, 2002.

[5] R. F. Nau. Indeterminate probabilities on finite sets. *The Annals of Statistics*, 20:1737–1767, 1992.

[6] R. Pelessoni and P. Vicig. Imprecise previsions for risk measurement. *Int. Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 11(4): 393–412, 2003.

[7] R. Pelessoni and P. Vicig. Convex imprecise previsions. *Reliable Computing*, 9(6):465–485, 2003.

[8] R. Pelessoni and P. Vicig. Uncertainty modelling and conditioning with convex imprecise previsions. *International Journal of Approximate Reasoning*, 39(2–3):297–319, 2005.

[9] R. T. Rockafellar. *Convex Analysis*, Princeton University Press, 1970.

[10] D. Schmeidler. Cores of exact games, I. *Journal of Mathematical Analysis and Applications*, 40: 214–225, 1972.

[11] T. Seidenfeld and L. Wasserman. Dilation for sets of probabilities. *The Annals of Statistics*, 21(3):1139-1154, 1993.

[12] M. C. M. Troffaes and G. de Cooman. Lower previsions for unbounded random variables. In *Soft Methods in Probability, Statistics and Data Analysis*, Advances in Soft Computing, P. Grzegorzewski, O. Hryniewicz, M. Á. Gil (eds.), 146–155, Physica-Verlag, New York, 2002.

[13] P. Walley. *Statistical Reasoning with Imprecise Probabilities.* Chapman and Hall, 1991.

[14] P. Walley, R. Pelessoni and P. Vicig. Direct algorithms for checking consistency and making inferences from conditional probability assessments. *Journal of Statistical Planning and Inference*, 126(1):119-151, 2004.

[15] K. Weichselberger. *Elementare Grundebegriffe einer allgemeineren Wahrscheinlinchkeitsrechnung I – Intervallwahrscheinlichkeit als umfassendes Konzept.* Physika, Heidelberg, 2001.

[16] P. M. Williams. Notes on conditional previsions. *Research Report*, School of Math. and Phys. Science, University of Sussex, 1975.