On Coherent Variability Measures and Conditioning

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Abstract

Coherent upper and lower previsions are becoming more and more popular as a mathematical model for robust valuations under uncertainty. Likewise, the mathematically equivalent class of coherent risk measures is attracting a lot attention in mathematical finance. In this paper, we show that a misinterpretation of upper previsions demands a closer examination of the basis of the theory of imprecise previsions. As a consequence, we obtain a new interpretation of coherent lower previsions as fair prices, a class of coherent variability measures, and a new type of conditioning for coherent lower previsions.

Keywords. Coherent previsions, coherent risk measures, variability measures, fair price, conditioning.

1 Introduction

The theory of imprecise previsions deals with the problem of consistently valuating gambles, i.e. bounded, real valued functions. A minimal consistency condition is "avoiding sure loss", a more sophisticated one is "coherence". A lower prevision represents the supremum of prices one is willing to pay for obtaining a gamble. Analogously, an upper prevision represents an infimum selling price.

After providing the preliminaries in section 2, we show in section 3 that upper previsions cannot be interpreted as selling prices as suggested by Walley (cf. Walley 1991, [11]). They are short selling prices which formally can be interpreted as a price for buying the negative gamble. We show that for a consistent theory of pricing gambles one has to take into account the gambles one holds – ones "portfolio gamble". Moreover, coherent lower previsions turn out not only to represent a supremum buying price but also to be a fair price of a gamble. Non-linearity is then a result of taking the risk of a gamble into account when evaluating the fair price. Therefore, we discuss in section 4 a decomposition of coherent lower previsions into a fair risk-neutral price (represented by a linear prevision, e.g. an expected value) and a price for the "riskiness" (variability) of the gamble (represented by the newly introduced coherent variability measure). Building up on the discussion of section 3, we define coherent lower previsions conditioned on a gamble in section 5 and point out some elementary properties and relations with coherent variability measures.

Although not explicitely mentioned in the following, all results on coherent lower previsions can easily be expressed in terms of coherent risk measures, too. This is due to the fact that a coherent risk measure is the negative of a coherent lower prevision (cf. Maaß 2002, [6, page 86]).

2 Preliminaries

Throughout this paper, let Ω be a non-empty set and $\mathcal{K}$ denote a non-empty subset of the linear space of bounded, real-valued functions on Ω. A functional $\underline{P} : \mathcal{K} \to \mathbb{R}$ is called a **lower prevision avoiding sure loss** if it satisfies one of the following equivalent conditions (cf. Walley 1991, [11]):

(a) For all $n \geq 1$ and $X_1, \ldots, X_n \in \mathcal{K}$,

$$\sup \sum_{i=1}^{n} X_i \geq \sum_{i=1}^{n} \underline{P}(X_i).$$

(b) There exists a linear prevision P, i.e. the restriction of a monotone, normalized, linear functional to $\mathcal{K}$, such that

$$\underline{P}(X) \leq P(X)$$

for all $X \in \mathcal{K}$.

Linear previsions can also be interpreted as the restriction of an expected value w.r.t. a finitely additive probability measure (also known as probability content).

$\underline{P}$ is called a **coherent lower prevision** if it satisfies one of the following equivalent conditions (cf. Walley 1991, [11]):

(a) If $\mathcal{K}$ is a linear space containg constant functions then $\underline{P}$ is a monotone, superlinear (i.e. positively homogenuous and superadditive) functional that is additive w.r.t. constants, i.e.

(i) $\underline{P}(X) \geq \inf X$

(ii) $\underline{P}(\lambda X) = \lambda \underline{P}(X)$ for any $\lambda \geq 0$

(iii) $\underline{P}(X+Y) \geq \underline{P}(X) + \underline{P}(Y)$.

(b) There exists a non-empty set $\mathcal{M}(\underline{P})$ of linear previsions such that

$$\underline{P}(X) = \min_{P \in \mathcal{M}(\underline{P})} P(X)$$

for all $X \in \mathcal{K}$.

(c) There exists a set $\mathcal{D}$ of bounded, real valued functions on Ω with

(i) $\sup X < 0 \;\Rightarrow\; X \notin \mathcal{D}$

(ii) $\inf X > 0 \;\Rightarrow\; X \in \mathcal{D}$

(iii) $X \in \mathcal{D} \;\Rightarrow\; \lambda X \in \mathcal{D}$ for any $\lambda > 0$

(iv) $X, Y \in \mathcal{D} \;\Rightarrow\; X + Y \in \mathcal{D}$

such that

$$\underline{P}(X) = \sup\{\mu \in \mathbb{R} \mid X - \mu \in \mathcal{D}\}.$$

$\mathcal{D}$ can be obtained by $\mathcal{D} := \{X \mid \underline{P}(X) \geq 0\}$ when $\underline{P}$ is given.

Although the definitions of "avoiding sure loss" and "coherence" do not presuppose any structure on $\mathcal{K}$, it is often much more convenient to presuppose that $\mathcal{K}$ is a linear space containing constant functions. For coherent lower previsions it is almost always no restriction to presuppose this structure as the exists a natural extension of coherent lower previsions to the whole space of bounded, real-valued functions (cf. Walley 1991, [11, section 3.1]). Hence, we implicitly assume $\mathcal{K}$ to be an appropriate linear space when necessary.

For any lower prevision avoiding sure loss $\underline{P}$, the functional $\overline{P} : \mathcal{K} \to \mathbb{R}$, $\overline{P}(X) := -\underline{P}(-X)$, is called an **upper prevision avoiding sure loss**. For any coherent lower prevision $\underline{P}$, the functional $\overline{P} : \mathcal{K} \to \mathbb{R}$, $\overline{P}(X) := -\underline{P}(-X)$, is called an **coherent upper prevision**.

This setting is usually interpreted as follows. Any function $X \in \mathcal{K}$ is considered as a **gamble**. The values of X are then uncertain rewards. A lower prevision is interpreted as the the supremum of prices μ one is willing to pay for X or just the **supremum buying price** for the gamble X. The price $\underline{P}(X)$ is denoted to be **almost desirable**. If a lower prevision $\underline{P}$ avoids sure loss then it is not possible to build up a portfolio of gambles that surely results in a loss. Coherence is a more sophisticated condition that implies avoiding sure loss. The set $\mathcal{D}$ in characterization (c) is called the **class of (almost) desirable gambles**. Using this characterization, the value $\underline{P}(X)$ is the supremum of all prices μ you would pay for X such that the resulting gamble $X - \mu$ is desirable.

3 Interpreting Coherent Previsions

From characterization (c) of coherent lower previsions, one gets

$$\begin{aligned}\overline{P}(X) &= -\sup\{\mu \in \mathbb{R} \mid -X - \mu \in \mathcal{D}\} \\ &= \inf\{\mu \in \mathbb{R} \mid \mu - X \in \mathcal{D}\},\end{aligned}$$

what caused Walley to interpret $\overline{P}(X)$ as an infimum selling price for X since one marginally desires to sell X for μ (cf. Walley 1991, [11, page 65]). In fact, this equation does not justify this interpretation what will now be elaborated on.

To be consistent with the interpretation of coherent lower prevision, the above equation just states that one desires to get μ for obtaining the gamble $-X$ for any $\mu > \overline{P}(X)$. Buying $-X$ equals selling X if and only if one already holds X. Otherwise, buying $-X$ is tantamount to short selling X. In the first case one ends up with holding a risky gamble $-X$ and in the second case one ends up with the constant function in the value of the selling price. Since we have not assumed to hold a gamble $-X$ when interpreting $\underline{P}(X)$, we also should not assume to hold X when interpreting $\overline{P}(X)$. Hence, we have to interpret $\overline{P}(X)$ as an almost desirable short selling price for X. It should be mentioned that this interpretation of coherent upper previsions can also be found in Walley's book (Walley 1991, [11, page 95]) where he stated that coherent upper previsions can be used to model betting rates that are set by bookmakers but this is definitely not the prevalent interpretation. The difference between selling and short selling would disappear when one would not consider the current portfolio of the gambler, i.e. the gamble X representing the sum of his currently owned gambles. It will turn out shortly that this supposition is not reasonable.

The interpretation of coherent previsions is directly linked to the interpretation of the set $\mathcal{D}$, as $\underline{P}$ is uniquely determined by $\mathcal{D}$ and vice versa. Walley does not deliberate on the question whether $\mathcal{D}$ is the class of gambles one has a disposition to *accept when offered* or to *have as a portfolio*. In the first case, the acceptability of an offered gamble is independent of the portfolio hold whereas in the second case, the valuation of an offered gamble relies on the valuation of the portfolio with and without the offered gamble. The following example shows that the first interpretation of $\mathcal{D}$ is not reasonable.

Example 1 *Let $\mathcal{K} := \{0,1\}^{\mathbb{R}}$ and the coherent lower prevision $\underline{P}$ be defined by $\underline{P} := \inf$. Suppose, that a gambler already posseses the gamble X defined by $X(0) := -1$ and $X(1) := 1$.*
If the class $\mathcal{D}$, $\mathcal{D} = \{Z \mid \underline{P}(Z) \geq 0\}$, is interpreted as the class of gambles the gambler desires to get *when offered to him then the gambler would have the disposition to reject the offered gamble Y, defined by $Y(0) := 1+\varepsilon$ and $Y(1) := -1+\varepsilon$ with $0 < \varepsilon < 1$ as it is not contained in the class $\mathcal{D}$ and this means that he strictly prefers X to ε. This decision of the gambler is exceedingly questionable as the gambler would strictly prefer the constant positive gamble ε to X when starting with another gamble as initial portfolio, e.g. 0.*

The previous example has shown that interpreting $\mathcal{D}$ as the class of gambles desirable to *get* makes it impossible to determine the class of gambles desirable to *have*. It is then also impossible to introduce a preference relation on the class of gambles as the preference relation would not be asymmetric. This situation is completely unsatisfactory. Hence, we have to interpret $\mathcal{D}$ as the class of gambles desirable to have. We will be shown in section 5 that with this interpretation it is also possible to determine the class of gambles desirable to get and that reasonable results can be obtained.

Now, having clarified the meaning of $\mathcal{D}$, the same has to be done for coherent previsions. Since $\underline{P}(X) = \sup\{\mu \in \mathbb{R} \mid X - \mu \in \mathcal{D}\}$, the resulting portfolio after buying X for some μ has to be $X - \mu$ as $\mathcal{D}$ only contains desirable *resulting* portfolios. Hence, the gambler starts with initial portfolio 0 as he buys X for μ and ends up with $X-\mu$. In this special case, it is the same if $\mathcal{D}$ is considered as the class of desirable gambles to buy or to finally hold. This might be the reason why Walley has not reflected on the precise meaning of desirability and the two different possible choices stated above.

As a consequence of fixing 0 as initial gamble for coheren lower previsions, the interpretation of $\overline{P}(X)$ as the infimum selling price for X becomes unsustainable. In order to be able to interpret $\overline{P}(X)$ as a selling price for X, one had to presuppose the gambler to hold X before (otherwise it would be short-selling X and not selling X). As $\overline{P}$ can also be defined via $\mathcal{D}$, $\overline{P}(X) = \inf\{\mu \in \mathbb{R} \mid \mu - X \in \mathcal{D}\}$, the same reasoning as for coherent lower previsions applies to coherent upper previsions such that the initial gamble has to be 0, too. Since $\overline{P}(X) = -\underline{P}(-X)$, $\overline{P}(X)$ has to be interpreted as supremum short-selling price (i.e. supremum buying price of $-X$). The difference between selling and short-selling disappears in the case one exclusively consideres the trades and forgets about the resulting portfolios. But this case has been shown in Example 1 to lead to some extremely undesirable implications. A second consequence of fixing 0 as initial gamble for coherent previsions is the fact that the whole theory of imprecise previsions fails to be applicable after the first buying or short sale. We will overcome this deficiency in section 5 by the introduction of coherent lower previsions conditioned on a gamble.

Now, we show that coherent lower previsions are in fact fair prices and not only supremum buying prices. By definition, any price $\mu < \underline{P}(X)$ is a desirable buying price for X and any price $\mu > \underline{P}(X)$ is undesirable. But $\underline{P}(X)$ is also the infimum selling price. To show this, suppose one already holds X. Since selling X for some μ equals buying $-X$ for $-\mu$, one should accept any selling price μ such that the resulting gamble μ is more desirable than X. The only reasonable definition of the relation "is more desirable than" is: X_1 is more desirable than X_2 if one is willing to pay more for getting X_1 than for X_2, i.e. if $\underline{P}(X_1) \geq \underline{P}(X_2)$[1]. Hence, one should accept any selling price μ such that $\mu = \underline{P}(\mu) = \underline{P}(X + (-X + \mu)) > \underline{P}(X)$, and the infimum selling price is then $\underline{P}(X)$. In this context, non-linearity of the fair price functional $\underline{P}$ can be interpreted as taking also the risk (variability) of a gamble into account when calculating its fair price. For instance, diversification reduces risk such that the fair price of the sum of gambles should be higher than the single prices. In the next section, we tackle the problem of decomposing a coherent lower prevision into a fair, risk-neutral valuation (linear previson, e.g. expected value) and a variability measure and characterize the class of variability measures that can be a part of a coherent lower prevision.

Having identified $\underline{P}(X)$ as the fair price of a gamble X, the interval $]\underline{P}(X), \overline{P}(X)[$, up to now interpreted as the set of prices being undesirable both to buy and to sell X, can now be interpreted to be the set of prices being too high to buy but yet too low to

[1]The preference relation $\geqq$ on gambles introduced by Walley (cf. Walley 1991, [11, page 154]) can not count as a reasonable choice for representing desirability as the cancellation axiom implies $X + Y \geqq X$ if and only if $Y \geqq 0$, which bars one from buying an undesirable gamble Y even if $\underline{P}(X+Y) \geq \underline{P}(X)$ (cf. also section 5)

short sell X. In the next section, we show that this interval can naturally be obtained when interpreting coherent lower previsions as the difference of a risk-neutral price and price of the risk.

4 Coherent Variability Measures

In this section we will characterize those functionals on $\mathcal{K}$ that can serve as a coherent variability measure for a given risk-neutral valuation functional (linear prevision) P.

Definition 1 *A functional $V : \mathcal{K} \to \mathbb{R}_+$ is called a* ***coherent variability measure*** *if there exists a linear prevision P such that $P - V$ is a coherent lower prevision.*

The class of coherent variability measures is non-empty since the zero functional, $V = 0$, is contained in this class.

Of course, any coherent lower prevision $\underline{P}$ can be decomposed into a linear prevision and a coherent variability measure V: Using characterization (b) of coherent lower previsions, one trivially obtains $\underline{P} = P - V$ for any $P \in \mathcal{M}(\underline{P})$ and $V := P - \underline{P}$. This shows that there does not exist a unique decomposition of a coherent lower prevision unless it is linear.

For a given linear prevision P, we observe, again using characterization (b) of coherent lower previsions, that a variability measure V is coherent if and only if there is a set $\mathcal{M}$ of linear previsions including P such that $V(X) := P(X) - \min_{P' \in \mathcal{M}} P'(X)$. Another characterization which is easier to check refers to characterization (a) of coherent lower previsions.

Lemma 1 *A functional $V : \mathcal{K} \to \mathbb{R}_+$ is a coherent variability measure if and only if there exists a linear prevision P such that*

(a) $P(X) - V(X) \geq \inf X$

(b) $V(\lambda X) = \lambda V(X)$ *for all* $\lambda \geq 0$

(c) $V(X + Y) \leq V(X) + V(Y)$

As a consequence of properties (a) – (c), we obtain $V(X + c) = V(X)$ for every real constant c.

There are a lot of variability measures known and widely used but most of them are not coherent. In Table 1, it is listed whether the above properties are satisfied (+) or not (−) for the variability measures range ($\rho(X) := \sup X - \inf X$), variance ($\sigma^2(X) := E(X - EX)^2$), standard deviation ($\sigma(X) := E^{1/2}(X - EX)^2$) and average absolute deviation from median ($\tau(X) := E(|X - MX|)$).

Variability Measure	ρ	σ^2	σ	τ
Property				
$E(X) - V(X) \geq \inf X$	−	−	−	+
$V(\lambda X) = \lambda V(X)$	+	−	+	+
$V(X + Y) \leq V(X) + V(Y)$	+	−	+	+

Table 1: Properties of some variability measures.

We sketch the proof of τ being a coherent variability measure and provide an example for $E(X) - V(X) < \inf X$ for $V \in \{\rho, \sigma^2, \sigma\}$.

First, we show that $E - \tau$ is a coherent lower prevision. To do this, we use some results from Denneberg (Denneberg 1997, [4, Example 5.4]) who suggested to use $E + \tau$ as a premium principle in insurance mathematics (cf. Denneberg 1990, [3]). The functional $E + \tau$ can be represented as a Choquet integral w.r.t. a concave set function μ, $\int X\, d\mu = E(X) + \tau(X)$. Since for the Choquet integral w.r.t. the conjugate, convex set function $\overline{\mu}$ we have

$$\int X\, d\overline{\mu} = -\int -X\, d\mu = E(X) - \tau(-X)$$
$$= E(X) - \tau(X),$$

$\int \cdot\, d\overline{\mu}$ is a coherent lower prevision (cf. Maaß 2001, [7, Proposition 3.4]) and τ therefore is a coherent variability measure.

Now, we show $E(X) - V(X) < \inf X$ for every $V \in \{\rho, \sigma^2, \sigma\}$. Let $\Omega := \{0, 1\}$, $\mu : 2^\Omega \to [0, 1]$ be a probability measure defined by $\mu(\{0\}) = .8$, and $X : \Omega \to \mathbb{R}$ be $X(0) := -4$ and $X(1) = 1$. Then $E(X) = -3$, $\rho(X) = 5$, $\sigma^2(X) = 4$, $\sigma(X) = 2$ and $E(X) - V(X) < \inf X$ for every $V \in \{\rho, \sigma^2, \sigma\}$.

Finally, we provide a characterization of the class of coherent variability measures without starting with a coherent lower prevision or a linear prevision.

Proposition 1 *A functional $V : \mathcal{K} \to \mathbb{R}_+$ is a coherent variability measure if and only if*

(a) *V restricted to $\mathcal{K}' := \{X \in \mathcal{K} \mid \inf X = 0\}$ is a lower prevision avoiding sure loss, i.e.* $\sup \sum_{i=1}^n X_i \geq \sum_{i=1}^n V(X_i)$ *whenever* $n \geq 1$ *and* $X_1, \ldots, X_n$ *are in* $\mathcal{K}'$

(b) $V(\lambda X) = \lambda V(X)$ *for all* $\lambda \geq 0$

(c) $V(X + Y) \leq V(X) + V(Y)$

(d) $V(X + c) = V(X)$ *for all real constants* c.

Proof. We only have to show that part (a) is equivalent to part (a) of Lemma 1 provided that the remaining properties hold. Using $V(X) = V(X - \inf X)$,

$$\begin{aligned} &P(X) - V(X) \geq \inf X \\ &\Leftrightarrow P(X - \inf X) \geq V(X - \inf X). \end{aligned}$$

Thus, property (a) of Lemma 1 is equivalent to the existence of a linear prevision dominating V on $\mathcal{K}'$. This is again equivalent to V, restricted to $\mathcal{K}'$, being a lower prevision avoiding sure loss by characterization (b) of lower previsions avoiding sure loss. □

Coherent variability measures are therefore translation invariant, sublinear lower previsions which avoid sure loss on $\mathcal{K}'$.

It should be noticed that though we use the denotation "lower prevision avoiding sure loss" in the preceding paragraph, this does not mean that V should be interpreted as a lower prevision. $V(X)$ is the "fair variability price" of a gamble X with fair price $\underline{P}(X)$. The linear prevision $\underline{P} + V$ is interpreted as the fair price when being risk-neutral.

As mentioned before, $]\underline{P}(X), \overline{P}(X)[$ is that interval of prices for X that are considered to be too high to be desirable but not high enough to be willing to bet against the gamble X. The interpretation for the existence of such an interval was that there is a "risk of X" that causes one to demand for a discount of a risk-neutral valuation when buying or short selling X. Now, after having coherent variability measures introduced, this interpretation can be justified mathematically. Since

$$\begin{aligned} &]\underline{P}(X), \overline{P}(X)[\\ &=]P(X) - V(X), -P(-X) + V(-X)[\\ &= P(X) +] - V(X), V(-X)[, \end{aligned}$$

this interval is uniquely determined by the variability of the gamble X.

I'd like to thank one anonymous reviewer for pointing out that there is an online available research report (cf. Rockafellar, Uryasev, and Zabarankin, 2004 [10]) dealing with generalized deviation measures obtained by some axiomitization. In terms of the report, coherent variability measures are lower range dominated deviation measures associated with a coherent lower prevision (which can be identified with coherent risk measures). The scope of this section goes beyond the research report as two characterizations of this special type of deviation measures is given, where the second, i.e. Proposition 1, even does not need to refer to a given linear prevision (or probability measure).

5 Conditioning on Gambles

In this section, we extend the definition of coherent lower prevision to that effect that it can depend on a gamble that one already holds. We have seen in section 3 that $\mathcal{D}$ is the class of gambles being desirable to have as the resulting portfolio of gambles and that one naturally should accept all gambles Y that increase the fair price of the current portfolio X, i.e. $\underline{P}(X+Y) \geq \underline{P}(X)$ as $X+Y$ is more desirable (i.e. one is willing to pay more for $X+Y$) than X. Thus, the following definitions of the class $\mathcal{D}_X$ of desirable gambles *to get* when owning X and the corresponding coherent lower prevision given X are quite natural.

Definition 2 *Let $\underline{P}$ be a coherent lower prevision.*

(a) The set $\mathcal{D}_X$ of almost desirable gambles given X is defined by

$$\mathcal{D}_X := \{Y \mid \underline{P}(X+Y) - \underline{P}(X) \geq 0\}.$$

(b) The coherent lower prevision $\underline{P}(\,\cdot\,\|X)$ given a the gamble X is defined by

$$\underline{P}(Y\|X) := \sup\{\mu \mid Y - \mu \in \mathcal{D}_X\}.$$

For a given gamble X, the value $\underline{P}(Y\|X)$ can be interpreted as the supremum buying price, or, more precisely, as the fair price of Y given X, one is willing to pay for Y.

While it was not necessary to impose conditions on the domain of a coherent lower prevision, the same does not apply to coherent lower previsions conditioned on a gamble as in this case the class of gambles to buy does not coincide anymore with the class of gambles to finally have. Hence, for Y to be contained in the domain of $\underline{P}(\,\cdot\,\|X)$ one has to ensure that X and $X+Y$ are contained in the domain of $\underline{P}(\,\cdot\,)$. But, as mentioned in section 2, by the existence of the natural extension for coherent lower previsions, possible problems concerning the domain are not substantial.

We now collect some elementary properties of coherent lower previsions conditioned on a gamble.

Proposition 2

(a) $\mathcal{D}_X = \{Y \mid \underline{P}(Y\|X) \geq 0\}$, i.e., together with Definition 2 (b), $\mathcal{D}_X$ and $\underline{P}(\,\cdot\,\|X)$ determine each other.

(b) $\mathcal{D}_X \supset \mathcal{D}$ for every $X \in \mathcal{K}$, i.e. every gamble Y being desirable to have is also desirable to get independently of ones portfolio X.

(c) $\mathcal{D}_X$ is convex.

(d) $\underline{P}(Y\|X) = \underline{P}(X+Y) - \underline{P}(X)$, i.e. $\underline{P}(Y\|X)$ is the increment of the fair price of ones portfolio X when getting Y.

(e) $\inf Y \leq \underline{P}(Y) \leq \underline{P}(Y\|X) \leq \overline{P}(Y) \leq \sup Y$*, as by superadditivity of* $\underline{P}$*, adding* Y *to any gamble* X *increases the fair price of the result to a greater extent than* $\underline{P}(Y)$*.*

(f) $\underline{P}(Y\|c) = \underline{P}(Y)$*, i.e. the fair price of a gamble* Y *to get is the same as the fair price of* Y *to have when starting with a constant gamble,*

(g) $\underline{P}(\lambda X\|X) = \lambda\underline{P}(X)$*, i.e. the fair price of a gamble* λX *to get is the same as the fair price of this gamble to have when starting with* X*.*

(h) $\underline{P}(\,\cdot\,\|X)$ *is generally not positively homogenuous, i.e. it is not coherent.*

(i) $\underline{P}(\,\cdot\,\|X)$ *is a convex lower prevision, i.e. it satisfies*

(i) $Y \leq Z$ *implies* $\underline{P}(Y\|X) \leq \underline{P}(Z\|X)$

(ii) $\underline{P}(Y + c\|X) = \underline{P}(Y\|X) + c$

(iii) $\underline{P}(\lambda Y + (1-\lambda)Z\|X) \geq \lambda\underline{P}(Y\|X) + (1-\lambda)\underline{P}(Z\|X)$.

$\underline{P}(\,\cdot\,\|X)$ *is even centered convex for every* X *(cf. e.g. Pelessoni and Vicig 2003, [9]), i.e.* $\underline{P}(0\|X) = 0$ *which is equivalent to the property that* $\underline{P}(\,\cdot\,\|X)$ *avoids sure loss. As a convex lower prevision,* $\underline{P}(\,\cdot\,\|X)$ *is representable by*

$$\underline{P}(Y\|X) = \inf_{P \in \mathcal{M}(\underline{P})} \{P(Y) + \alpha(P)\}$$

and the property of being centered convex additionally implies $\inf_{P\in\mathcal{M}(\underline{P})} \alpha(P) = 0$*. A natural choice for the function* α *in this situation is* $\alpha(P) := P(X) - \underline{P}(X)$*.*

(j) Let $\underline{P} = P - V$ *with a linear prevision* P *satisfying* $P(X) = \underline{P}(X)$ *and a coherent variability measure* V*. Then* $\underline{P}(Y\|X) = P(Y) - V(X+Y)$*, i.e. the fair price of a gamble* Y *to get is its risk-neutral fair price reduced by the price for the variability (in terms of* V*) of the resulting portfolio.*

The proof consists of the application of Definition 2 and some simple calculations.

Remark 1 *The reason why one should choose in Proposition 2 (j) a linear prevision* P *satisfying* $P(X) = \underline{P}(X)$ *can be justified by using the interpretation of* $\underline{P}$ *to represent for every* X *the worst of all possible outcomes* $P(X)$*,* $P \in \mathcal{M}(\underline{P})$ *(cf. characterization (b) of coherent lower previsions). Using such a prudent interpretation of coherent lower previsions, it is quite natural that once one holds the gamble* X*, one only needs to consider those possible linear prevision* $P \in \mathcal{M}(\underline{P})$ *yielding the worst possible outcome* $\underline{P}(X)$*.*

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