Some theoretical properties of interval-valued conditional probability assessments

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Abstract

In this paper we consider interval-valued conditional probability assessments on finite families of conditional events. Based on the coherence principle of de Finetti, we give some preliminary results on precise and imprecise probability assessments, by recalling the properties of avoiding uniform loss (AUL), which coincides with the notion of g-coherence, and of coherence introduced by Walley. Among other results, we generalize to interval-valued assessments a connection property, obtained in a previous paper, for the set Π_n of precise coherent assessments on a family $\mathcal{F}_n$ of n conditional events. More specifically, we prove that, with any pair of AUL interval-valued assessments X'_n, X''_n on $\mathcal{F}_n$, we can associate an infinite class $\mathcal{X}$ of AUL interval-valued imprecise assessments which are convex combination between X'_n and X''_n and connect them. Then, we examine the extension of g-coherent imprecise assessments. We also give a result on totally coherent imprecise assessments, by examining its relationship with a necessary and sufficient condition of total coherence for interval-valued assessments.

Keywords: conditional events, g-coherence, avoiding uniform loss, coherence, interval-valued probability assessments, connection property, total coherence.

1 Introduction

The probabilistic treatment of uncertainty plays a relevant role in many applications of Artificial Intelligence, e.g. uncertain reasoning. In such applications typically the set of uncertain quantities at hand has no particular algebraic structure; moreover, the experts may have a vague and partial information. Then, a flexible approach can be obtained by using imprecise probabilities, based on a suitable generalization of the coherence principle of de Finetti, or on similar principles like that ones adopted for lower and upper probabilities ([1], [3], [4], [5], [7], [10], [11], [12], [13]).

In this paper we examine interval-valued probability assessments on finite families of conditional events. We use a notion of generalized coherence which coincides with the property of avoiding uniform loss (AUL) introduced by Walley ([12]). We also recall how we can determine the coherent (in the sense of Walley) interval-valued assessment associated with a given AUL assessment (then, the theoretical results obtained for g-coherent assessments can be suitably adapted to coherent ones; in the paper we explicitly consider only the case of AUL, i.e. g-coherent, interval-valued assessments).

We recall some recent results on precise probability assessments ([2]). Then, we generalize such results to interval-valued assessments; in particular, we consider a connection property of the set Π_n of precise coherent assessments on a family $\mathcal{F}_n$ of n conditional events and we generalize this property to interval-valued assessments. More specifically, we prove that, with any pair of AUL interval-valued assessments X'_n, X''_n on $\mathcal{F}_n$, we can associate an infinite class $\mathcal{X}$ of AUL interval-valued imprecise assessments which connects X'_n and X''_n. Then, based on such result, we examine the extension of g-coherent imprecise assessments. We also give a result on totally coherent set-valued probability assessments on $\mathcal{F}_n$ and we examine its relationship with a necessary and sufficient condition of total coherence for interval-valued assessments.

The paper is organized as follows. In Section 2 we recall some preliminary notions and results. In particular, in sub-section 2.1 we consider the case of precise probability assessments; in sub-section 2.2 we consider the case of interval-valued probability assessments. In Section 3 we make some remarks on avoiding uniform loss and coherent interval-valued probability assessments. In Section 4 we give some theoretical results on interval-valued assessments; we also construct some classes of AUL interval-valued probability assessments. In Section 5 we give a result on totally coherent imprecise probability assessments, by examining its relationship with a necessary and sufficient condition of total coherence which holds for interval-valued assessments. Finally, in Section 6 we give some conclusions.

2 Preliminaries notions and results

We recall some notions and results on coherence and generalized coherence of precise and imprecise conditional probability assessments. For each integer n, we set $J_n = \{1, 2, \dots, n\}$. We denote by A^c the negation of A and by $A \vee B$ (resp., AB) the disjunction (resp., the conjunction) of A and B. We use the same symbol to denote an event and its indicator.

2.1 Precise probability assessments

Given a real function P defined on an arbitrary family of conditional events $\mathcal{K}$, let $\mathcal{F}_n = \{E_i|H_i,\, i \in J_n\}$ be a finite subfamily of $\mathcal{K}$ and $\mathcal{P}_n$ the vector $(p_i,\, i \in J_n)$, where $p_i = P(E_i|H_i)$. Then, let us consider the disjunctive normal form obtained by expanding the expression

$$(E_1H_1 \vee E_1^cH_1 \vee H_1^c) \wedge \cdots \wedge (E_nH_n \vee E_n^cH_n \vee H_n^c)\,.$$

In such disjunctive normal form we eliminate all the conjunctive terms which, due to the logical relationships among the events $E_1, \dots, E_n, H_1, \dots, H_n$, coincide with the impossible event. The remaining conjunctive terms are the constituents generated by $\mathcal{F}_n$. We denote by $C_1, \dots, C_m$ the constituents contained in $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$; moreover, we set $C_0 = H_1^c \cdots H_n^c$ (of course, it may be $C_0 = \emptyset$). Notice that $m \leq 3^n - 1$. Then, with the pair $(\mathcal{F}_n, \mathcal{P}_n)$ we associate the random gain $G_n = \sum_{i \in J_n} s_i H_i (E_i - p_i)$, where $s_1, \dots, s_n$ are arbitrary real numbers and E_i, H_i denote the indicators of the corresponding events. We denote by g_h the value of G_n corresponding to C_h and by $G_n|\mathcal{H}_n$ the restriction of G_n to $\mathcal{H}_n$. Of course, $G_n|\mathcal{H}_n \in \{g_1, \dots, g_m\}$. Then, using the *betting scheme* of de Finetti, we recall the following

Definition 1. The function P is said coherent if and only if

$$\max G_n|\mathcal{H}_n \geq 0\,,\ \forall\, n \geq 1,\ \forall\, \mathcal{F}_n \subseteq \mathcal{K},\ \forall\, s_1, \dots, s_n \in \mathbb{R}.$$

Given any vector $(\lambda_r,\, r \in J_m)$, we denote by $\sum_{H_j} \lambda_r$ (resp., $\sum_{E_jH_j} \lambda_r$) the sum of the λ_r's such that $C_r \subseteq H_j$ (resp., $C_r \subseteq E_jH_j$). Then, given a probability assessment $\mathcal{P}_n = (p_j, j \in J_n)$ on $\mathcal{F}_n$, let $\mathcal{S}$ be the following system, with vector of (nonnegative) unknowns $\Lambda = (\lambda_r,\, r \in J_m)$,

$$\begin{cases} \sum_{E_jH_j} \lambda_r = p_j \sum_{H_j} \lambda_r\,, & j \in J_n, \\ \sum_{r \in J_m} \lambda_r = 1,\ \lambda_r \geq 0, & r \in J_m. \end{cases} \tag{1}$$

We set

$$\begin{array}{c} \Phi_j(\Lambda) = \sum_{H_j} \lambda_r\,,\ \ j \in J_n\,; \\ I_0 = \{j \in J_n : max_{\Lambda \in S} \Phi_j(\Lambda) = 0\}. \end{array} \tag{2}$$

Then, denoting by $\mathcal{P}_0$ the sub-assessment associated with I_0, we have ([7])

Theorem 1. The probability assessment $\mathcal{P}_n$ on $\mathcal{F}_n$ is coherent if and only if the following conditions are satisfied:
1. the system (1) is solvable;
2. if $I_0 \neq \emptyset$, then $\mathcal{P}_0$ is coherent.

We recall below some results obtained in ([2]). We set $\Gamma_0 = J_n \setminus I_0$; then, we have

Theorem 2. Given a probability assessment $\mathcal{P}_n = (p_i,\ i \in J_n)$ on $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$, assume that system (1) is solvable. Then, there exists a solution Λ of system (1) such that $\Phi_j(\Lambda) > 0\,,\ \forall\, j \in \Gamma_0$.

Theorem 3. Given a probability assessment $\mathcal{P}_n = (p_i,\ i \in J_n)$ on $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$, assume that the system (1) is solvable. Then, for every $\Gamma \subseteq \Gamma_0$, the sub-vector $\mathcal{P}_\Gamma = (p_i,\ i \in \Gamma)$ is a coherent probability assessment on the sub-family $\mathcal{F}_\Gamma$.

We denote by Π_n the set of coherent probability assessments on $\mathcal{F}_n$. Of course, Π_n is a suitable subset of the unitary hypercube of $\mathbb{R}^n$ and, in geometrical terms, a conditional probability assessment $\mathcal{P} = (p_i,\, i \in J_n)$ on $\mathcal{F}_n$ is coherent if and only if $\mathcal{P}$ is a "point" of the set Π_n. Given two points

$$\mathcal{P}' = (p'_i,\, i \in J_n) \in \Pi_n\,, \quad \mathcal{P}'' = (p''_i,\, i \in J_n) \in \Pi_n\,,$$

we set

$$\begin{array}{l} p_i^m = \min\{p'_i, p''_i\}\,,\ \ p_i^M = \max\{p'_i, p''_i\}\,, \\ \mathcal{P}^m = \mathcal{P}' \wedge \mathcal{P}'' = (p_i^m,\ i \in J_n)\,, \\ \mathcal{P}^M = \mathcal{P}' \vee \mathcal{P}'' = (p_i^M,\ i \in J_n)\,. \end{array} \tag{3}$$

Moreover, given any pair of points

$$\mathbf{x} = (x_i,\, i \in J_n), \quad \mathbf{y} = (y_i,\, i \in J_n)\,,$$

we set $\mathbf{x} \leq \mathbf{y}$ if and only if $x_i \leq y_i,\ \forall\, i \in J_n$.
Then, $\mathcal{P}^m \leq \mathcal{P}^M$, for every $\mathcal{P}', \mathcal{P}''$.
Based on the ordinary topology of the space $\mathbb{R}^n$, we have

Theorem 4. Let $\mathcal{P}' = (p'_i,\ i \in J_n)$, $\mathcal{P}'' = (p''_i,\ i \in J_n)$ be two coherent probability assessments defined on $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$. There exists a continuous curve $\mathcal{C}$ with extreme points $\mathcal{P}', \mathcal{P}''$ such that:

$$(i)\ \ \mathcal{P}^m \leq \mathcal{P} \leq \mathcal{P}^M\,,\ \forall\, \mathcal{P} \in \mathcal{C}\,; \qquad (ii)\ \ \mathcal{C} \subseteq \Pi_n\,.$$

Theorem 4 assures that, for every pair of coherent assessments $\mathcal{P}', \mathcal{P}''$ on $\mathcal{F}_n$, we can construct (at least) a continuous curve $\mathcal{C} \subseteq \Pi_n$ (from $\mathcal{P}'$ to $\mathcal{P}''$) whose points are intermediate coherent assessments between $\mathcal{P}'$ and $\mathcal{P}''$. We remark that in general the number of such curves is infinite. By Theorem 4, we obtain

Corollary 1. Given any quantities

$$p_1 \,, \, \dots \,, \, p_{i-1} \,, \, l_i \, \leq \, u_i \,, \, p_{i+1} \,, \, \dots \,, \, p_n \,,$$

let us define

$$\begin{array}{l} \mathcal{P}' = (p_1, \dots, p_{i-1}, l_i, p_{i+1}, \dots, p_n)\,, \\ \mathcal{P}'' = (p_1, \dots, p_{i-1}, u_i, p_{i+1}, \dots, p_n)\,. \end{array}$$

Moreover, let $\mathcal{I} = \mathcal{P}'\mathcal{P}''$ be the segment $\{(p_1, \dots, p_i, \dots, p_n) : l_i \leq p_i \leq u_i\}$, with set of vertices $\mathcal{V} = \{\mathcal{P}', \mathcal{P}''\}$. Then: $\mathcal{I} \subseteq \Pi_n \iff \mathcal{V} \subset \Pi_n$.

2.2 Imprecise probability assessments

Given any interval-valued probability assessment $X_n = ([l_i, u_i],\ i \in J_n)$ on a family $\mathcal{F}_n = \{E_i|H_i,\ i \in J_n\}$, we use the following definition of generalized coherence (g-coherence) ([1]).

Definition 2. An interval-valued probability assessment $X_n = ([l_i, u_i], i \in J_n)$, defined on a family of n conditional events $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$, is g-coherent if there exists a coherent precise probability assessment $\mathcal{P}_n = (p_i, i \in J_n)$ on $\mathcal{F}_n$, with $p_i = P(E_i|H_i)$, which is consistent with X_n, that is such that $l_i \leq p_i \leq u_i$ for each $i \in J_n$.

Generalizing the system (1) to the case of interval-valued assessments, we obtain the following system

$$\left\{ \begin{array}{l} \sum_{E_j H_j} \lambda_r \;\geq\; l_j \sum_{H_j} \lambda_r\,, \quad j \in J_n, \\ \sum_{E_j H_j} \lambda_r \;\leq\; u_j \sum_{H_j} \lambda_r\,, \quad j \in J_n, \\ \sum_{r \in J_m} \lambda_r = 1, \quad \lambda_r \geq 0, \quad r \in J_m. \end{array} \right. \tag{4}$$

We can suitably adapt to interval-valued assessments the definition of the set I_0 and of the functions $\Phi_j(\Lambda), j \in J_n$. We remark that, for each solution Λ of system (4) it is

$$\sum_{j \in J_n} \Phi_j(\Lambda) = \sum_{j \in J_n} \sum_{H_j} \lambda_r \;\geq\; \sum_{r \in J_m} \lambda_r = 1\,,$$

hence $\Phi_j(\Lambda) > 0$ for at least a subscript j; therefore I_0 is a strict subset of J_n. Then, denoting by X_0 the sub-assessment associated with I_0, in the next result we generalize Theorem 1 to interval-valued assessments.

Theorem 5. The assessment X_n on $\mathcal{F}_n$ is g-coherent if and only if the following conditions are verified:
1. The system (4) is solvable;
2. if $I_0 \neq \emptyset$, then X_0 is g-coherent.

3 Some remarks on avoiding uniform loss and coherent interval-valued probability assessments

We recall that a lower probability $\underline{P}$ on a family of conditional events $\mathcal{K}$ avoids uniform loss (AUL) if and only if, for every $\mathcal{F}_n = \{E_i|H_i,\ i \in J_n\} \subseteq \mathcal{K}$ and for every $\sigma_1 \geq 0, \dots, \sigma_n \geq 0$, denoting by $L_n = (l_i, i \in J_n)$ the restriction of $\underline{P}$ to $\mathcal{F}_n$, the random gain

$$\underline{G}_n = \sum_{i \in J_n} \sigma_i H_i (E_i - l_i)\,,$$

associated with the pair $(\mathcal{F}_n, L_n)$ satisfies the condition: $max\, \underline{G}_n | \mathcal{H}_n \geq 0$.
Let $A = (a_{hi})$ be a $m \times n-$matrix. Moreover, denote by $\mathbf{x}$ and $\mathbf{y}$, respectively, a row $m-$vector and a column $n-$vector. The vector $\mathbf{x} = (x_1, \dots, x_m)$ is said *semi-positive* if it is nonnegative and moreover $x_1 + \cdots + x_m \;>\; 0$. Then, we have ([6], Th. 2.10)

Theorem 6. Exactly one of the following alternatives holds.
Either the inequality $\mathbf{x}A \geq 0$ has a *semipositive* solution, or the inequality $A\mathbf{y} < 0$ has a *nonnegative* solution.

By applying Theorem 6, with

$$\begin{array}{l} x_h = \lambda_h \geq 0\,, \quad h \in J_m\,, \quad \sum_{h \in J_m} \lambda_h = 1\,, \\ y_k = \sigma_k \geq 0\,, \quad k \in J_n\,, \end{array}$$

and with $A = (a_{hi})$, where

$$a_{hi} = \left\{ \begin{array}{ll} 1 - l_i, & C_h \subseteq E_i H_i, \\ -l_i, & C_h \subseteq E_i^c H_i, \\ 0, & C_h \subseteq H_i^c, \end{array} \right.$$

we have

Theorem 7. The condition $max\, \underline{G}_n | \mathcal{H}_n \geq 0$ is satisfied if and only if the following system is solvable

$$\begin{array}{l} \sum_{E_j H_j} \lambda_r \geq l_j \sum_{H_j} \lambda_r\,, \quad j \in J_n\,; \\ \sum_{r \in J_m} \lambda_r = 1\,; \quad \lambda_r \geq 0\,, \quad r \in J_m\,. \end{array}$$

We observe that the assessment $P(E|H) \leq u$ is equivalent to $P(E^c|H) \geq 1 - u$; hence, an interval-valued assessment $([l_i, u_i],\ i \in J_n)$ on $\{E_i|H_i,\ i \in J_n\}$ can be represented as a lower probability $(l_i, 1 - u_i,\ i \in J_n)$ on $\{E_i|H_i, E_i^c|H_i,\ i \in J_n\}$. Therefore, Theorem 7 can be extended to the general case of interval-valued assessments. In this sense, the notions of g-coherent interval-valued assessments and AUL lower probability are equivalent and in what follows we will use interchangeably such terms.
We recall below two results which concern the problem of the g-coherent extension of interval-valued assessments ([1]).

Theorem 8. Let be given a g-coherent interval-valued assessment $X_n = ([l_i, u_i], i \in J_n)$ on $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$ and a further conditional event $E_{n+1}|H_{n+1}$. Then, there exists a suitable interval $[p_\circ, p^\circ]$ such that the interval-valued assessment $X_{n+1} = ([l_i, u_i], i \in J_{n+1})$, with $l_{n+1} = u_{n+1} = p_{n+1}$, on $\mathcal{F}_{n+1} = \{E_i|H_i,\ i \in J_{n+1}\}$, is g-coherent if and only if $p_{n+1} \in [p_\circ, p^\circ]$.

Theorem 9. Given a g-coherent interval-valued assessment $X_n = ([l_i, u_i],\, i \in J_n)$ on $\mathcal{F}_n = \{E_i|H_i,\, i \in J_n\}$, the extension $[l_{n+1}, u_{n+1}]$ of X_n to a conditional event $E_{n+1}|H_{n+1}$ is g-coherent if and only if $[l_{n+1}, u_{n+1}] \cap [p_\circ, p^\circ] \neq \emptyset$.

To determine the values $p_\circ, p^\circ$, a suitable algorithm has been given in [1].
By the same algorithm, starting with a g-coherent assessment X_n on $\mathcal{F}_n$, we can make its "least-committal" correction. In this way, we obtain the coherent (lower and upper) probability X_n^* on $\mathcal{F}_n$ which would be produced by applying the natural extension principle given in [12]. To obtain X_n^* we can apply n times this algorithm, by replacing each time $E_{n+1}|H_{n+1}$ by $E_j|H_j$, $j \in J_n$, using as probabilistic constraints the g-coherent assessment X_n.
We recall that a procedure to check coherence of an interval-valued conditional probability assessment and an algorithm for finding the best bounds for coherent extensions have been given in [11]. Moreover, an algorithm for computing the least-committal coherent correction of an imprecise assessment, also useful for inferential purposes, has been given in [10]. The problems of checking coherence and of the extension of lower-upper conditional probabilities have been studied also in [5]. Direct methods, which do not involve sequences of linear programming problems, have been proposed in [13].
As the above remarks suggest, each theoretical result obtained for g-coherent assessments can be suitably adapted to coherent ones.
In this paper we explicitly consider only the case of AUL (i.e. g-coherent) interval-valued assessments.

4 Some results on interval-valued assessments

In this section, among other results, we generalize Theorems 2, 3, and 4 to the case of interval-valued assessments. In the next result we prove that, if the system (4) associated with a pair $(\mathcal{F}_n, X_n)$ is solvable, then there exists solutions Λ's of such system which give positive values to the functions $\Phi_j(\Lambda) = \sum_{H_j} \lambda_r$ for every $j \in \Gamma_0 = J_n \setminus I_0$. This property will be exploited when proving Theorem 12. We have

Theorem 10. Given an interval-valued probability assessment $X_n = ([l_i, u_i],\ i \in J_n)$ on $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$, assume that system (4) is solvable. Then, there exists a solution $\Lambda = (\lambda_r,\, r \in J_m)$ of system (4) such that $\Phi_j(\Lambda) > 0\,,\ \forall\, j \in \Gamma_0$.

Proof. For each $i \in \Gamma_0$ it is max $\Phi_i > 0$; hence there exists a subset of the set of solutions S of system (4), which we denote by $\{\Lambda_i = (\lambda_r^{(i)},\, r \in J_m),\, i \in \Gamma_0\}$, such that $\Phi_i(\Lambda_i) > 0$, $i \in \Gamma_0$. Then, given any vector $\Lambda = (\lambda_r,\, r \in J_m) = \sum_{i \in \Gamma_0} x_i \Lambda_i$, with $\sum_{i \in \Gamma_0} x_i = 1$, $x_i > 0$, $\forall\, i \in \Gamma_0$, it is $\sum_{r \in J_m} \lambda_r = 1$, $\lambda_r \geq 0$, $\forall\, r \in J_m$. Moreover, for each $i \in \Gamma_0$ one has

$$l_j \sum_{H_j} \lambda_r^{(i)} \leq \sum_{E_j H_j} \lambda_r^{(i)} \ \leq\ u_j \sum_{H_j} \lambda_r^{(i)}\,,\ \ j \in J_n\,;$$

$$l_j \sum_{H_j} x_i \lambda_r^{(i)} \leq \sum_{E_j H_j} x_i \lambda_r^{(i)} \ \leq\ u_j \sum_{H_j} x_i \lambda_r^{(i)}\,,\ \ j \in J_n\,;$$

hence, for $j \in J_n$, it is

$$\textstyle\sum_{E_j H_j} \left(\sum_{i \in \Gamma_0} x_i \lambda_r^{(i)}\right) \ \geq\ l_j \sum_{H_j} \left(\sum_{i \in \Gamma_0} x_i \lambda_r^{(i)}\right)\,,$$

$$\textstyle\sum_{E_j H_j} \left(\sum_{i \in \Gamma_0} x_i \lambda_r^{(i)}\right) \ \leq\ u_j \sum_{H_j} \left(\sum_{i \in \Gamma_0} x_i \lambda_r^{(i)}\right)\,,$$

that is

$$l_j \sum_{H_j} \lambda_r \leq \sum_{E_j H_j} \lambda_r \ \leq\ u_j \sum_{H_j} \lambda_r\,,\ \ j \in J_n,$$

so that $\Lambda = (\lambda_r,\, r \in J_m)$ is a solution of system (4). Moreover,

$$\begin{aligned} &\Phi_j(\Lambda) = \Phi_j\left(\textstyle\sum_{i \in \Gamma_0} x_i \Lambda_i\right) = \\ &= \textstyle\sum_{i \in \Gamma_0} x_i \Phi_j(\Lambda_i) \geq x_j \Phi_j(\Lambda_j) > 0\,,\ \forall\, j \in \Gamma_0\,. \end{aligned} \tag{5}$$

□

In the next result we prove that the solvability of system (4) implies, for each $\Gamma \subseteq \Gamma_0$, the g-coherence of the sub-assessment X_Γ on $\mathcal{F}_\Gamma$. We have

Theorem 11. Given an interval-valued probability assessment $X_n = ([l_i, u_i],\ i \in J_n)$ on $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$, assume that the system (4) is solvable. Then, for every $\Gamma \subseteq \Gamma_0$, the sub-vector $X_\Gamma = ([l_i, u_i],\ i \in \Gamma)$ is a g-coherent interval-valued assessment on the sub-family $\mathcal{F}_\Gamma$.

Proof. Of course, g-coherence of X_{Γ_0} implies g-coherence of X_Γ too; so we only need to prove g-coherence of X_{Γ_0}. We distinguish two cases:
(i) the sub-assessment X_0 associated with I_0 is g-coherent;
(ii) the sub-assessment X_0 associated with I_0 is not g-coherent.

In the first case, by Theorem 5, X_n is g-coherent and hence X_{Γ_0} is g-coherent too.
In the second case, given any g-coherent sub-assessment $X_0^* = ([l_i^*, u_i^*], \, i \in I_0)$ on $\mathcal{F}_0$, by Theorem 5 the assessment

$$X_n^* = (X_{\Gamma_0}, X_0^*) = ([l_i, u_i], \, i \in \Gamma_0 \,;\, [l_i^*, u_i^*], \, i \in I_0)$$

on $\mathcal{F}_n$ is g-coherent and hence X_{Γ_0} is g-coherent too. $\square$

Given an assessment X_n on $\mathcal{F}_n$ and assuming system (4) solvable, let S' be a subset of the set S of solutions of (4). Recalling that $\Phi_j(\Lambda) = \sum_{H_j} \lambda_r$, where $\Lambda = (\lambda_r, \, r \in J_m)$, we set

$$I_{S'} = \{j \in J_n : \Phi_j(\Lambda) = 0, \, \forall \, \Lambda \in S'\} \,, \quad \Gamma_{S'} = J_n \backslash I_{S'} \,.$$

We denote by $X_{\Gamma_{S'}}$ (resp., $X_{I_{S'}}$) the sub-assessment of X_n associated with $\Gamma_{S'}$ (resp., $I_{S'}$). Obviously, $S' \subseteq S$ implies $\Gamma_{S'} \subseteq \Gamma_0$; hence, by Theorem 11, the sub-assessment $\mathcal{P}_{\Gamma_{S'}}$ is g-coherent. Notice that, by replacing $X_{I_{S'}}$ with any sub-assessment $X^*_{I_{S'}}$, the set S' is also a subset of the set of solutions of the system (4) associated with the assessment $X_n^* = (X_{\Gamma_{S'}}, X^*_{I_{S'}})$. Of course, the same remark holds in the particular case $S' = \{\Lambda\}$. Then, we have

Lemma 1. Given an interval-valued assessment X_n on $\mathcal{F}_n$, assume that system (4) is solvable. Then, given any subset $S' \subset S$ and any g-coherent assessment $X^*_{I_{S'}}$ on the sub-family $\mathcal{F}_{I_{S'}}$, the assessment $X_n^* = (X_{\Gamma_{S'}}, X^*_{I_{S'}})$ on $\mathcal{F}_n$ is g-coherent.

Proof. We observe that X_n^* is obtained by X_n by replacing $X_{I_{S'}}$ with $X^*_{I_{S'}}$ and that S' is also a subset of the set of solutions of the system (4) associated with X_n^*. Then, by applying Theorem 5 to the pair $(\mathcal{F}_n, X_n^*)$, system (4) is solvable and $I_0 \subseteq I_{S'}$. Moreover X_0, being a sub-assessment of $X^*_{I_{S'}}$, is g-coherent and hence X_n^* is g-coherent too. $\square$

By Lemma 1 it immediately follows that, if (4) is solvable and $X_{I_{S'}}$ is g-coherent, then X_n is g-coherent. Given a vector $\Delta = (\delta_i, \, i \in J_n) \in [0,1]^n$ and two interval-valued assessments

$$X'_n = ([l'_i, u'_i], \, i \in J_n) \,, \qquad X''_n = ([l''_i, u''_i], \, i \in J_n) \,,$$

by the symbol X_Δ we denote the interval-valued assessment $([l_i, u_i], \, i \in J_n)$ defined by

$$l_i = (1-\delta_i) l'_i + \delta_i l''_i \,, \quad u_i = (1-\delta_i) u'_i + \delta_i u''_i \,, \quad i \in J_n \,.$$

We set $\Delta_0 = (0, 0, \ldots, 0)$, $\Delta_1 = (1, 1, \ldots, 1)$; hence $X'_n = X_{\Delta_0}$, $X''_n = X_{\Delta_1}$. We denote by $\Im_n$ the set of g-coherent interval-valued assessments on $\mathcal{F}_n$. Then, the result below generalizes Theorem 4 to interval-valued assessments, by showing how to construct an infinite class of interval-valued assessments which are intermediate between X'_n, X''_n, i.e. convex combinations of them.

Theorem 12. Let be given two g-coherent interval-valued assessments $X'_n = ([l'_i, u'_i], \, i \in J_n)$, $X''_n = ([l''_i, u''_i], \, i \in J_n)$, on the family $\mathcal{F}_n = \{E_i | H_i, i \in J_n\}$. Then, we can construct an infinite class $\mathcal{X}$ of interval-valued probability assessments on $\mathcal{F}_n$ such that: (i) each $X_n \in \mathcal{X}$ is a convex combination between X'_n, X''_n; i.e., $X_n = X_\Delta$ for some $\Delta = (\delta_i, \, i \in J_n) \in [0,1]^n$; (ii) $\mathcal{X} \subseteq \Im_n$.

Proof. Using (2) adapted to imprecise assessments, we denote by I'_0 and I''_0 the subsets, associated respectively with X'_n and X''_n. From g-coherence of X'_n, X''_n, recalling Theorem 10, there exist two vectors

$$\Lambda'_0 = (\lambda'_r, r \in J_m) \,, \quad \Lambda''_0 = (\lambda''_r, r \in J_m) \,,$$

such that: $\Phi_j(\Lambda'_0) > 0 \,, \; \forall j \in \Gamma'_0 = J_n \setminus I'_0$, and $\Phi_j(\Lambda''_0) > 0 \,, \; \forall j \in \Gamma''_0 = J_n \setminus I''_0$. Given any number $\alpha_0 \in (0,1)$, let us consider the vector

$$\Lambda_0 = (\lambda_r, r \in J_m) = (1 - \alpha_0) \Lambda'_0 + \alpha_0 \Lambda''_0 \,. \qquad (6)$$

Of course, $\lambda_r = (1 - \alpha_0) \lambda'_r + \alpha_0 \lambda''_r$, $\forall \, r \in J_m$.
Defining $I^{(0)} = I'_0 \cap I''_0$, for each $j \in \Gamma^{(0)} = \Gamma'_0 \cup \Gamma''_0 = J_n \setminus I^{(0)}$ we have

$$\Phi_j(\Lambda_0) = \Phi_j[(1 - \alpha_0) \Lambda'_0 + \alpha_0 \Lambda''_0] =$$
$$= (1 - \alpha_0) \Phi_j(\Lambda'_0) + \alpha_0 \Phi_j(\Lambda''_0) > 0 \,,$$

with $\Phi_j(\Lambda_0) = 0, \, \forall \, j \in I^{(0)} = J_n \setminus \Gamma^{(0)}$. Moreover, from g-coherence of X'_n, X''_n, for each $i \in J_n$ we have

$$\begin{array}{c} l'_i \sum_{H_i} \lambda'_r \leq \sum_{E_i H_i} \lambda'_r \leq u'_i \sum_{H_i} \lambda'_r \,, \\ l''_i \sum_{H_i} \lambda''_r \leq \sum_{E_i H_i} \lambda''_r \leq u''_i \sum_{H_i} \lambda''_r \,. \end{array} \qquad (7)$$

Now, let us consider the interval-valued assessment $X_{\Gamma^{(0)}} = ([l_i, u_i], \, i \in \Gamma^{(0)})$, where

$$l_i = (1 - \delta_i^0) l'_i + \delta_i^0 l''_i \,, \; u_i = (1 - \delta_i^0) u'_i + \delta_i^0 u''_i \,, \qquad (8)$$

$$\delta_i^0 = \frac{\alpha_0 \sum_{H_i} \lambda''_r}{(1 - \alpha_0) \sum_{H_i} \lambda'_r + \alpha_0 \sum_{H_i} \lambda''_r} = \frac{\alpha_0 \sum_{H_i} \lambda''_r}{\sum_{H_i} \lambda_r} \,. \qquad (9)$$

From (7) and (9), for each $i \in \Gamma^{(0)}$ we have

$$\sum_{E_i H_i} \lambda_r = \sum_{E_i H_i} [(1 - \alpha_0) \lambda'_r + \alpha_0 \lambda''_r] =$$

$$= (1 - \alpha_0) \sum_{E_i H_i} \lambda'_r + \alpha_0 \sum_{E_i H_i} \lambda''_r \; \geq$$

$$\geq (1 - \alpha_0) l'_i \sum_{H_i} \lambda'_r + \alpha_0 l''_i \sum_{H_i} \lambda''_r =$$

$$= \left[\frac{(1-\alpha_0) \sum_{H_i} \lambda'_r}{\sum_{H_i} \lambda_r} \, l'_i + \frac{\alpha_0 \sum_{H_i} \lambda''_r}{\sum_{H_i} \lambda_r} \, l''_i \right] \sum_{H_i} \lambda_r =$$

$$= [(1 - \delta_i^0) l'_i + \delta_i^0 l''_i] \sum_{H_i} \lambda_r = l_i \sum_{H_i} \lambda_r \,.$$

By a similar reasoning $\sum_{E_iH_i} \lambda_r \leq u_i \sum_{H_i} \lambda_r$; hence, recalling (8),

$$l_i \sum_{H_i} \lambda_r \;\leq\; \sum_{E_iH_i} \lambda_r \;\leq\; u_i \sum_{H_i} \lambda_r \,, \;\; \forall\, i \in \Gamma^{(0)} \,.$$

Now, given any quantities

$$\delta_i^0 \in [0,1], \;\; i \in I^{(0)} = J_n \setminus \Gamma^{(0)} \,, \tag{10}$$

let us consider the assessment $X_n = ([l_i, u_i],\, i \in J_n)$, where, for each $i \in J_n$, it is

$$l_i = (1-\delta_i^0)l_i' + \delta_i^0 l_i'' ,\; u_i = (1-\delta_i^0)u_i' + \delta_i^0 u_i'' \,,$$

and where δ_i^0 is defined by (9) for $i \in \Gamma^{(0)}$ and by (10) for $i \in I^{(0)}$. We have

$$l_i \sum_{H_i} \lambda_r \;\leq\; \sum_{E_iH_i} \lambda_r \;\leq\; u_i \sum_{H_i} \lambda_r \,, \;\; \forall\, i \in J_n \,; \tag{11}$$

hence, Λ_0 is a solution of system (4) and, considering the set I_0 associated with X_n, as defined by (2) (adapted to imprecise assessments), we have $I_0 \subseteq I^{(0)}$, $\Gamma^{(0)} \subseteq \Gamma_0$; then, by Theorem 11, the assessment X_{Γ_0} on $\mathcal{F}_{\Gamma_0}$ is g-coherent (and hence $X_{\Gamma^{(0)}}$ is g-coherent too). Notice that $\delta_i^0 \geq 0$, $1-\delta_i^0 \geq 0$, $\forall\, i \in \Gamma_0$, with $\delta_i^0 > 0$, $1-\delta_i^0 > 0$, $\forall\, i \in \Gamma^{(0)}$; hence

$$\min\{l_i', l_i''\} \leq l_i \leq \max\{l_i', l_i''\} \,, \;\; \forall\, i \in \Gamma_0 \,,$$

$$\min\{u_i', u_i''\} \leq u_i \leq \max\{u_i', u_i''\} \,, \;\; \forall\, i \in \Gamma_0 \,,$$

with the inequalities strict for $i \in \Gamma^{(0)}$.

We denote, respectively, by $X_0', X_0'', \mathcal{F}_0$ the sub-assessments of X_n', X_n'' and the sub-family of $\mathcal{F}_n$ associated with I_0. Of course, from g-coherence of X_n' and X_n'', it follows that X_0' and X_0'', defined on $\mathcal{F}_0$, are g-coherent too.

Moreover, we denote, respectively, by I_1' and I_1'' the subsets associated with X_0' and X_0'', as defined by (2) (adapted to imprecise assessments).

Then, exploiting again Theorem 10, we iterate the above procedure by considering a pair of vectors $(\Lambda_1', \Lambda_1'')$ associated with X_0', X_0''. Given any number $\alpha_1 \in (0,1)$, we define a vector $\Lambda_1 = (1-\alpha_1)\Lambda_1' + \alpha_1\Lambda_1''$; then, we introduce , as in (9), suitable non negative coefficients δ_i^1, $i \in \Gamma_1$, with $\delta_i^1 > 0, \forall\, i \in \Gamma^{(1)}$. In this way, by Theorem 11, we construct a g-coherent assessment X_{Γ_1} defined on $\mathcal{F}_{\Gamma_1}$, where

$$\Gamma_1 \supseteq \Gamma^{(1)} = \Gamma_1' \cup \Gamma_1'' = I_0 \setminus I^{(1)} = I_0 \setminus (I_1' \cap I_1'') \,.$$

The g-coherence of the assessment $(X_{\Gamma_0}, X_{\Gamma_1})$ on $\mathcal{F}_{\Gamma_0} \cup \mathcal{F}_{\Gamma_1} = \mathcal{F}_{J_n \setminus I_1}$ is obtained by the following steps:
(a) let X_1 be any g-coherent assessment on the sub-family $\mathcal{F}_1$, associated with the subset I_1;
(b) then, by Theorem 5, the assessment (X_{Γ_1}, X_1) on $\mathcal{F}_0 = \mathcal{F}_{\Gamma_1} \cup \mathcal{F}_1$ is g-coherent;
(c) then, by Theorem 5, the assessment $X_n = (X_{\Gamma_0}, X_{\Gamma_1}, X_1)$ on $\mathcal{F}_n = \mathcal{F}_{\Gamma_0} \cup \mathcal{F}_{\Gamma_1} \cup \mathcal{F}_1$ is g-coherent;
(d) then, the sub-assessment $(X_{\Gamma_0}, X_{\Gamma_1})$ on $\mathcal{F}_{\Gamma_0} \cup \mathcal{F}_{\Gamma_1}$ is g-coherent.

By repeating the procedure for the triple $(X_1', X_1'', \mathcal{F}_1)$ associated with I_1, we determine a g-coherent probability assessment X_{Γ_2} defined on $\mathcal{F}_{\Gamma_2}$; and so on.

In this way, after $k+1$ steps, with $k \leq n-1$, we construct an interval-valued assessment

$$X_\Delta = (X_{\Gamma_0}, X_{\Gamma_1}, \ldots, X_{\Gamma_k})$$

on $\mathcal{F}_n$ which, by Theorems 5 and 11, is g-coherent.

In particular, we could construct g-coherent assessments on $\mathcal{F}_n$ of the kind $X_\Delta = (X_{\Gamma^{(0)}}, X_{\Gamma^{(1)}}, \ldots, X_{\Gamma^{(h)}})$, by applying Lemma 1 with $S' = \{\Lambda_j\}$, $j = 0, 1, \ldots, h$, where for each j the vector $\Lambda_j = (1-\alpha_j)\Lambda_j' + \alpha_j\Lambda_j''$ is obtained as in (6).

We remark that each assessment X_Δ is obtained by using the continuous parameters α_j, δ_i^j, $i \in \Gamma_j$, $j = 0, 1, \ldots, k$. Moreover, X_Δ is intermediate between X_n', X_n''; that is, X_Δ is a convex combination of X_n', X_n'' with coefficients the parameters δ_i^j, $i \in \Gamma_j$, $j = 0, 1, \ldots, k$.

We recall that the coefficients δ_i^j, $i \in \Gamma^{(j)}$, $j = 0, 1, \ldots, k$, are defined by using the continuous parameters $\alpha_0, \alpha_1, \ldots, \alpha_k$ and the vectors $\Lambda_0, \Lambda_1, \ldots, \Lambda_k$, as made in (9) for the coefficients δ_i^0, $i \in \Gamma_0$. Moreover, the parameters $\alpha_0, \alpha_1, \ldots, \alpha_k$ can assume any value in $(0,1)$ and, for each $j = 0, 1, \ldots, k$, we have

$$\lim_{\alpha_j \to 0} \delta_i^j = 0 \,, \quad \lim_{\alpha_j \to 1} \delta_i^j = 1 \,, \;\; \forall\, i \in \Gamma^{(j)} \,.$$

Finally, letting $\alpha_j \to 0$, $\delta_i^j \to 0$, and $\alpha_j \to 1$, $\delta_i^j \to 1\,, i \in \Gamma_j \setminus \Gamma^{(j)}$, $j = 0, 1, \ldots, k$, we obtain an infinite class $\mathcal{X}$ of AUL interval-valued assessments on $\mathcal{F}_n$, i.e. $\mathcal{X} \subseteq \Im_n$. Then, (under the ordinary topology of the space $\mathbb{R}^n$) we can write

$$\lim_{\Delta \to \Delta_0} X_\Delta = X_n' \,, \quad \lim_{\Delta \to \Delta_1} X_\Delta = X_n'' \,.$$

As is shown by the previous reasoning, we can move in a continuous way from X_n' to X_n''. By analogy with Theorem 4, we can say that X_n', X_n'' are *connected* by the interval-valued assessments contained in $\mathcal{X}$.

We also remark that, in general, we can find an infinite number (of sequences) of pair of solutions, like $(\Lambda_0', \Lambda_0''), \ldots, (\Lambda_k', \Lambda_k'')$; hence, we can construct an infinite number of classes like $\mathcal{X}$. □

We illustrate the previous result by the following

Example 1. *A Problem of Currency Exchange* (a similar problem is in [9]). Let $(A_B)_t$ denote the price of a unit of currency B in terms of a unit of currency A for the final trade that occurs in a currency market on day t. Consider the three currencies of the dollar,

\$, the pound sterling, $\pounds$, and the yen, Y. Consider the events $E_1 = (\$_{\pounds})_{t+1} \geq (\$_{\pounds})_t$, $E_2 = (\$_Y)_{t+1} \geq (\$_Y)_t$, $E_3 = (\pounds_Y)_{t+1} \geq (\pounds_Y)_t$, that is the events that the final trading price B in terms of A on day $t+1$ is at least as great as on day t. Given the family $\mathcal{F}_5 = \{E_1 \vee E_2, E_2, E_1|(E_1 \vee E_2), E_1|E_2, E_3|E_2\}$, suppose that two experts (say $\mathcal{E}_1$ and $\mathcal{E}_2$) assert (on $\mathcal{F}_5$) the following probability evaluations:

$$X'_5 = (0.8, 0.4, [0.75, 0.95], [0.45, 0.55], [0.4, 0.6])\,;$$

$$X''_5 = (0.9, 0.85, [0.65, 0.98], [0.60, 0.70], [0.5, 0.65])\,.$$

Such assessments are g-coherent, with $I'_0 = I''_0 = \emptyset$. Given any solution Λ'_0 of system (4) associated with X'_n, one has
$\Phi_1(\Lambda'_0) = \Phi_2(\Lambda'_0) = \sum_{\Omega} \lambda'_r = 1\,,$
$\Phi_3(\Lambda'_0) = \sum_{E_1 \vee E_2} \lambda'_r = 0.8\,,$
$\Phi_4(\Lambda'_0) = \Phi_5(\Lambda'_0) = \sum_{E_2} \lambda'_r = 0.4\,.$
Analogously, given any solution Λ''_0 of system (4) associated with X''_n, one has
$\Phi_1(\Lambda''_0) = \Phi_2(\Lambda''_0) = 1\,,\ \Phi_3(\Lambda''_0) = 0.9\,,$
$\Phi_4(\Lambda''_0) = \Phi_5(\Lambda''_0) = 0.85\,.$
Given any $\alpha_0 \in (0,1)$, let us consider the interval-valued assessment $X_\Delta = ([l_i, u_i],\, i \in J_5)$ on $\mathcal{F}_5$, defined by
$l_1 = u_1 = 0.8\,(1-\delta_1^0) + 0.9\,\delta_1^0,$
$l_2 = u_2 = 0.4\,(1-\delta_2^0) + 0.85\,\delta_2^0,$
$l_3 = 0.75\,(1-\delta_3^0) + 0.65\,\delta_3^0\,,\ u_3 = 0.95\,(1-\delta_3^0) + 0.98\,\delta_3^0,$
$l_4 = 0.45\,(1-\delta_4^0) + 0.60\,\delta_4^0\,,\ u_4 = 0.55\,(1-\delta_4^0) + 0.70\,\delta_4^0,$
$l_5 = 0.4\,(1-\delta_5^0) + 0.5\,\delta_5^0\,,\ u_5 = 0.6\,(1-\delta_5^0) + 0.65\,\delta_5^0,$
where $\delta_1^0 = \delta_2^0 = \alpha_0$, $\delta_3^0 = \frac{0.9\,\alpha_0}{0.8\,(1-\alpha_0)+0.9\,\alpha_0}$, $\delta_4^0 = \delta_5^0 = \frac{0.85\,\alpha_0}{0.4\,(1-\alpha_0)+0.85\,\alpha_0}$. It can be verified that X_Δ is g-coherent. Moreover, in this example, Δ only depends on α_0. Then, if we have the same confidence with both experts $\mathcal{E}_1$ and $\mathcal{E}_2$, we can choose $\alpha_0 = \frac{1}{2}$, by obtaining the following assessment on $\mathcal{F}_5$

$$X_\Delta = (0.85, 0.62, [0.69, 0.96], [0.55, 0.65], [0.46, 0.63])\,.$$

By Theorem 12 we obtain

Corollary 2. Let be given two g-coherent interval-valued assessments

$$X'_{n+1} = ([l_1, u_1], \ldots, [l_n, u_n], [p', p'])\,,$$
$$X''_{n+1} = ([l_1, u_1], \ldots, [l_n, u_n], [p'', p''])\,,$$

on $\mathcal{F}_{n+1} = \{E_i|H_i,\, i \in J_{n+1}\}$, with $p' < p''$. Then, for each $p \in [p', p'']$, the interval-valued assessment

$$X_{n+1} = ([l_1, u_1], \ldots, [l_n, u_n], [p, p])\,, \qquad (12)$$

on $\mathcal{F}_{n+1}$ is g-coherent.

Proof. By Theorem 12, an infinite class connecting X'_{n+1}, X''_{n+1} and containing all the assessments like (12) is given by

$$\mathcal{X} = \{X_\Delta,\, \Delta = (0, \ldots, 0, \delta_{n+1}),\, \delta_{n+1} \in [0,1]\}\,.$$

It can be easily verified that, for $\delta_{n+1} = \frac{p-p'}{p''-p'}$, one has $X_\Delta = ([l_1, u_1], \ldots, [l_n, u_n], [p, p])$. □

Remark 1. We recall that the notion of g-coherence and (the proof of) Theorem 5 are strictly related with the coherence principle of de Finetti. Moreover, Theorems 8 and 9 have been obtained in [1] with the aim of generalizing the fundamental theorem of de Finetti, even if they can be seen as sub-derivatives of the natural extension principle of Walley.
Then, along these lines, there are a natural interest and a deep motivation (at least in our de Finetti-based approach) in unifying Theorems 8 and 9, as made in the next result. As it will be seen, such result has a very simple proof and is directly based on Theorem 5 and on (Corollary 2 of) Theorem 12, which is our main result.

Theorem 13. Given a g-coherent interval-valued assessment $X_n = ([l_i, u_i],\, i \in J_n)$ on $\mathcal{F}_n = \{E_i|H_i,\, i \in J_n\}$ and a further conditional event $E_{n+1}|H_{n+1}$, there exists a suitable non empty interval $[p_0, p^0] \subseteq [0,1]$ such that the assessment $X_{n+1} = ([l_i, u_i],\, i \in J_{n+1})$ on $\mathcal{F}_{n+1} = \{E_i|H_i,\, i \in J_{n+1}\}$ is g-coherent if and only if $[l_{n+1}, u_{n+1}] \cap [p_0, p^0] \neq \emptyset$.

Proof. We denote by Π the set of values p such that

$$X_{n+1} = ([l_1, u_1], \ldots, [l_n, u_n], [p, p]\,,$$

is a g-coherent extension of X_n to $\mathcal{F}_{n+1}$. We first verify that Π is non empty. Let $\mathcal{D} = \{D_1, \ldots, D_s\}$ be the set of constituents generated by $\mathcal{F}_n \cup \{E_{n+1}|H_{n+1}\}$ and contained in $\mathcal{H}_{n+1} = H_1 \vee \cdots \vee H_{n+1}$. Considering the constituents $C_1, \ldots, C_m$ generated by $\mathcal{F}_n$ and contained in $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$, we observe that there exist disjoint subsets $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ of $\mathcal{D}$, such that

$$C_r = \bigvee_{D_t \subseteq C_r} D_t\,, \quad r \in J_m\,.$$

The system (4), with unknowns $\lambda_1, \ldots, \lambda_m$ associated with $C_1, \ldots, C_m$, can be written as a system with vector of unknowns $\Delta = (\delta_1, \ldots, \delta_s)$ associated with $D_1, \ldots, D_s$, by replacing each λ_r by $\sum_{D_t \subseteq C_r} \delta_t$. Then, we introduce the following extended system $\mathcal{S}'$, with a parameter $p \in [0,1]$,

$$\left\{\begin{array}{l} \sum_{E_{n+1}H_{n+1}} \delta_t = p \sum_{H_{n+1}} \delta_t\,, \\ l_j \sum_{H_j} \delta_t \leq \sum_{E_j H_j} \delta_t \leq u_j \sum_{H_j} \delta_t\,,\ j \in J_n\,, \\ \sum_{t \in J_s} \delta_t = 1\,, \quad \delta_t \geq 0\,,\ t \in J_s\,, \end{array}\right.$$

By suitably adapting p, with each solution of (4) we can associate (at least) a solution of $\mathcal{S}'$. Given any p, we denote by S' the set of solutions of $\mathcal{S}'$; moreover, we set $\Sigma^+ = \{\Delta \in S' : \sum_{H_{n+1}} \delta_r > 0\}$. We distinguish two cases: (i) there exists a value $p \in [0,1]$ such that $\Sigma^+ \neq \emptyset$; (ii) $\Sigma^+ = \emptyset$ for every

$p \in [0,1]$. In the first case, given any $\Delta \in \Sigma^+$, the assessment $X_{n+1} = ([l_1, u_1], \ldots, [l_n, u_n], [p,p]$, where $p = \frac{\sum_{E_{n+1}H_{n+1}} \delta_t}{\sum_{H_{n+1}} \delta_t}$, is a g-coherent extension of X_n to $E_{n+1}|H_{n+1}$; hence $\Pi \neq \emptyset$. In the second case, (using any value p) we determine the set $I'_0 = I_0 \cup \{n+1\} = \{j \in J_{n+1} : Max_{\Delta \in S'} \sum_{H_j} \delta_t > 0\}$, where $I_0 \subset J_n$. If $I_0 = \emptyset$, then by Theorem 5 the assessment X_{n+1} on $\mathcal{F}_{n+1}$ is g-coherent for every coherent assessment p on $E_{n+1}|H_{n+1}$; hence $\Pi \neq \emptyset$. If $I_0 \neq \emptyset$, we replace $(\mathcal{F}_n, X_n)$ by $(\mathcal{F}_0, X_0)$ by repeating the above reasoning. After a finite number of steps, we find a set $\Sigma^+ \neq \emptyset$; hence we conclude that $\Pi \neq \emptyset$. Defining $p_0 = \inf\, \Pi$, $p^0 = \sup\, \Pi$, by the closure property of the set of coherent probability assessments, we have $p_0 \in \Pi$, $p^0 \in \Pi$. Finally, by Corollary 2, we obtain $\Pi = [p_0, p^0]$. Then, it immediately follows that $[l_{n+1}, u_{n+1}]$ is a g-coherent extension of X_n if and only if $[l_{n+1}, u_{n+1}] \cap [p_0, p^0] \neq \emptyset$. □

We will now construct some other classes of g-coherent interval-valued probability assessments on a family $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$. Such construction could be useful, e.g., to conciliate possible discrepancies among different expert opinions, as shown in the following example. Let $\mathcal{F}_3$ be the family $\{E_1|H_1, E_2|H_2, E_3|H_3\}$ and X', X'' be two interval-valued assessments on $\mathcal{F}_3$ (made by two experts)

$$([a_1, b_1], [a_2, b_2], [a_3, b_3])\,, \quad ([\alpha_1, \beta_1], [\alpha_2, \beta_2], [\alpha_3, \beta_3])\,,$$

such that $b_1 < \alpha_1$ and $\beta_2 < a_2$; this implies

$$[a_1, b_1] \cap [\alpha_1, \beta_1] = [a_2, b_2] \cap [\alpha_2, \beta_2] = \emptyset\,.$$

Then, let us consider any assessment $X_3 = ([l_1, u_1], [l_2, u_2], [l_3, u_3])$ on $\mathcal{F}_3$ and the following conditions

(*) $[l_1, u_1] \subseteq [b_1, \alpha_1]$, $[a_i, b_i] \cup [\alpha_i, \beta_i] \subseteq [l_i, u_i]$, $i = 2, 3$;

(**) $[l_2, u_2] \subseteq [\beta_2, a_2]$, $[a_i, b_i] \cup [\alpha_i, \beta_i] \subseteq [l_i, u_i]$, $i = 1, 3$.

In the next theorem we prove that, if X', X'' are g-coherent, then any X_3 satisfying (*), or (**), is g-coherent too.

Given two coherent probability assessments on $\mathcal{F}_n$,

$$\mathcal{P}' = (p'_i, i \in J_n)\,, \quad \mathcal{P}'' = (p''_i, i \in J_n)\,,$$

and recalling (3), let be

$$[\mathcal{P}^m, \mathcal{P}^M] = [p^m_1, p^M_1] \times \cdots \times [p^m_n, p^M_n] = \\ = \{\mathcal{P} : \mathcal{P}^m \leq \mathcal{P} \leq \mathcal{P}^M\}\,.$$

Given any interval-valued assessment on $\mathcal{F}_n$, $X_n = ([l_i, u_i],\, i \in J_n)$, we denote the associated multi-interval by

$$\mathcal{I} = [l_1, u_1] \times \cdots \times [l_n, u_n]\,.$$

Of course, if $\mathcal{I} \cap \{\mathcal{P}', \mathcal{P}''\} \neq \emptyset$, then X_n is g-coherent. We have

Theorem 14. Given two g-coherent interval-valued probability assessments

$$X'_n = ([l'_i, u'_i],\, i \in J_n)\,, \quad X''_n = ([l''_i, u''_i],\, i \in J_n)\,,$$

on the family $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$, assume that, for a suitable non empty subset $\Gamma \subseteq J_n$, it holds

$$[l'_j, u'_j] \cap [l''_j, u''_j] = \emptyset\,, \ \ \forall j \in \Gamma\,.$$

Moreover, let be $\mathcal{X} = \bigcup_{j \in \Gamma} \mathcal{X}_j$, where, for each $j \in \Gamma$, $\mathcal{X}_j$ is the class of interval-valued probability assessments $X_n = ([l_i, u_i],\, i \in J_n)$ on $\mathcal{F}_n$ such that

$$[l_j, u_j] \subseteq [u^m_j, l^M_j]\,, \ \ [l'_i, u'_i] \cup [l''_i, u''_i] \subseteq [l_i, u_i]\,, \ \forall\, i \neq j\,,$$

where $u^m_j = min\,\{u'_j, u''_j\}$, $l^M_j = max\,\{l'_j, l''_j\}$. Then, for every $X_n \in \mathcal{X}$, X_n is g-coherent.

Proof. As X'_n, X''_n are g-coherent, there exist two coherent assessments $\mathcal{P}', \mathcal{P}''$ on $\mathcal{F}_n$ such that

$$l'_i \leq p'_i \leq u'_i\,, \quad l''_i \leq p''_i \leq u''_i\,, \quad i \in J_n\,;$$

hence, considering the multi-interval $[\mathcal{P}^m, \mathcal{P}^M]$, by Theorem 4 there exists a continuous curve $\mathcal{C}$, contained in the multi-interval $[\mathcal{P}^m, \mathcal{P}^M]$, which connects $\mathcal{P}', \mathcal{P}''$. Let $X_n = ([l_i, u_i],\, i \in J_n)$ be any interval-valued assessment in $\mathcal{X}_j$ and let $\mathcal{I}$ be the associated multi-interval. For any $p_j \in [p^m_j, p^M_j]$, we set

$$I_{p_j} = \{(p_1, \ldots, p_j, \ldots, p_n) : \, p_i \in [p^m_i, p^M_i], \ \forall\, i \neq j\}\,;$$

hence

$$[\mathcal{P}^m, \mathcal{P}^M] = \bigcup_{p_j \in [p^m_j, p^M_j]} I_{p_j}\,;$$

$$\mathcal{C} \cap I_{p_j} \neq \emptyset\,, \ \ \forall\, p_j \in [p^m_j, p^M_j]\,;$$

(notice that $\mathcal{C} \cap I_{p_j}$ is the intersection point of $\mathcal{C}$ and I_{p_j}). Moreover,

$$p^m_i \leq u^m_i\,, \quad p^M_i \geq l^M_i\,, \ \ \forall\, i \in J_n\,;$$

hence, by the hypotheses, one has

$$[l_j, u_j] \subseteq [p^m_j, p^M_j]\,; \quad [p^m_i, p^M_i] \subseteq [l_i, u_i]\,, \ \ \forall\, i \neq j\,.$$

Then, for every $p_j \in [l_j, u_j]$, it is $I_{p_j} \subseteq \mathcal{I}$, so that

$$\mathcal{C} \cap \mathcal{I} = \bigcup_{p_j \in [l_j, u_j]} (\mathcal{C} \cap I_{p_j}) \ \neq \ \emptyset\,,$$

and hence $X_n = ([l_i, u_i],\, i \in J_n)$ is g-coherent. □

Notice that each point $\mathcal{P} = \mathcal{C} \cap I_{p_j}$, $p_j \in [l_j, u_j]$, of the arc of curve $\mathcal{C} \cap \mathcal{I}$ is a coherent probability assessment on $\mathcal{F}_n$ consistent with X_n.

We illustrate the previous result by the following

Example 2. Recalling Example 1, let us consider the subfamily $\mathcal{F}_3 = \{E_1|(E_1 \vee E_2), E_1|E_2, E_3|E_2\}$ of $\mathcal{F}_5$. Then, let us consider the following g-coherent assessments on $\mathcal{F}_3$
$X_3' = ([\frac{65}{100}, \frac{85}{100}], [\frac{45}{100}, \frac{55}{100}], [\frac{4}{10}, \frac{6}{10}])$,

$X_3'' = ([\frac{65}{100}, \frac{98}{100}], [\frac{6}{10}, \frac{7}{10}], [\frac{5}{10}, \frac{65}{100}])$.
We observe that $[\frac{45}{100}, \frac{55}{100}] \cap [\frac{6}{10}, \frac{7}{10}] = \emptyset$; then, by Theorem 14, the assessment $X_3 = ([\frac{65}{100}, \frac{98}{100}], [\frac{55}{100}, \frac{6}{10}], [\frac{4}{10}, \frac{65}{100}])$ is g-coherent. To verify g-coherence of X_3, we observe that two precise assessments consistent, respectively, with X_3' and X_3'' are $\mathcal{P}' = (\frac{3}{4}, \frac{5}{10}, \frac{5}{10})$ and $\mathcal{P}'' = (\frac{241}{360}, \frac{65}{100}, \frac{55}{100})$. Hence $\mathcal{P}^m = (\frac{241}{360}, \frac{5}{10}, \frac{5}{10})$, $\mathcal{P}^M = (\frac{3}{4}, \frac{65}{100}, \frac{55}{100})$.
By Theorem 4, there exists a continuous curve $\mathcal{C} \subseteq \Pi_n$ such that $\mathcal{P}^m \leq \mathcal{P} \leq \mathcal{P}^M$, $\forall\, \mathcal{P} \in \mathcal{C}$. The idea of Theorem 14, is that as $\mathcal{C}$ is continuous, there exist coherent points (p_1, p_2, p_3)'s such that $p_1 \in [\frac{65}{100}, \frac{98}{100}]$, $p_2 \in [\frac{55}{100}, \frac{60}{100}]$ (the "interval" between $[0.45, 0.55]$ and $[0.60, 0.70]$), and $p_3 \in [\frac{4}{10}, \frac{65}{100}]$. Roughly speaking, while we "tighten" the interval associated with the 2nd event, we "enlarge" the intervals associated with the 1st and the 3rd event. For example, the precise assessment $(\frac{7}{10}, \frac{58}{100}, \frac{5}{10})$ on $\mathcal{F}_3$ is coherent and verifies the thesis of Theorem 14.

5 Totally coherent interval-valued assessments

In this section we consider the notion of total coherence; then, we give a result on totally coherent imprecise probability assessments and we examine its relationship with a necessary and sufficient condition for total coherence of interval-valued assessments. Let Π_n be the set of coherent probability assessments $\mathcal{P}_n = (p_1, \ldots, p_n)$ on $\mathcal{F}_n = \{E_i|H_i,\, i \in J_n\}$. An imprecise probability assessment on $\mathcal{F}_n$, represented by a subset $S_n \subseteq [0,1]^n$, is g-coherent (i.e., avoiding uniform loss) if and only if $S_n \cap \Pi_n \neq \emptyset$. We say that the imprecise assessment is totally coherent if and only if $S_n \subseteq \Pi_n$. In these cases, we also say that the set S_n is g-coherent (resp., totally coherent).
Before giving the next result, we illustrate it in a particular case. Let be given a family $\mathcal{F}_3 = \{E_1|H_1, E_2|H_2, E_3|H_3\}$, a subset Φ of the unit square $[0,1]^2$, and two functions g, f defined on Φ, with $0 \leq g(x,y) \leq f(x,y) \leq 1, \forall\, (x,y) \in \Phi$. Given any $\alpha \in [0,1]$, let γ_α be the function $\alpha g + (1-\alpha) f$. We denote, respectively, by $\Sigma_g, \Sigma_f, \Sigma_{\gamma_\alpha}$ the surfaces associated with the functions g, f, γ_α, and by S the set $\bigcup_{\alpha \in [0,1]} \Sigma_{\gamma_\alpha}$. Then, in the next theorem we prove that totally coherence of the set $\Sigma_g \cup \Sigma_f$ implies totally coherence of the set Σ_{γ_α} and is equivalent to totally coherence of the set S.
Given a subset $\Gamma_{n-1} = \{i_1, \ldots, i_{n-1}\} \subset J_n$, we denote by Π_{n-1} the set of coherent assessments on $\mathcal{F}_{n-1} = \{E_i|H_i,\, i \in \Gamma_{n-1}\}$. Then, we have

Theorem 15. Let be given two functions $f(\pi)$, $g(\pi)$, where $\pi = (p_i,\, i \in \Gamma_{n-1})$, defined on a set $\Phi \subseteq \Pi_{n-1}$, with $0 \leq g(\pi) \leq f(\pi) \leq 1$, $\forall\, \pi \in \Phi$. Moreover, let be

$$\mathcal{K} = \{\gamma : g(\pi) \leq \gamma(\pi) \leq f(\pi),\, \forall\, \pi \in \Phi\}\,;$$

$$\Sigma_\gamma = \{(\pi, \gamma(\pi)) : \pi \in \Phi\}\,,\ \ \gamma \in \mathcal{K}\,;$$

$$\Sigma_{\gamma_\alpha} = \{(\pi, \gamma_\alpha(\pi)) : \pi \in \Phi\}\,,\ \alpha \in [0,1]\,,$$

$$\gamma_\alpha = \alpha g + (1-\alpha) f\,,\ \alpha \in [0,1]\,,$$

$$S = \bigcup_{\alpha \in [0,1]} \Sigma_{\gamma_\alpha} = \{(\pi, p_n) : \pi \in \Phi,\, g(\pi) \leq p_n \leq f(\pi)\}\,.$$

Then, one has:
(i) $\Sigma_g \cup \Sigma_f \subseteq \Pi_n \implies \Sigma_\gamma \subseteq \Pi_n\,,\ \forall\, \gamma \in \mathcal{K}$;
(ii) $S \subseteq \Pi_n \iff \Sigma_g \cup \Sigma_f \subseteq \Pi_n$.

Proof. (i) assume that $\Sigma_g \cup \Sigma_f \subseteq \Pi_n$; then, let us consider any $\gamma \in \mathcal{K}$. Given any $\mathcal{P} = (\pi, \gamma(\pi)) \in \Sigma_\gamma$, let S_π be the segment with vertices

$$\mathcal{P}_g = (\pi, g(\pi)) \in \Sigma_g\,,\quad \mathcal{P}_f = (\pi, f(\pi)) \in \Sigma_f\,.$$

We have $\mathcal{P} \in S_\pi$ and, by Corollary 1, $\mathcal{S}_\pi \subseteq \Pi_n$; then $\mathcal{P} \in \Pi_n$, $\forall\, \mathcal{P} \in \Sigma_\gamma$; hence $\Sigma_\gamma \subseteq \Pi_n$.
(ii) of course $S \subseteq \Pi_n \implies \Sigma_g \cup \Sigma_f \subseteq \Pi_n$.
Conversely, assume that $\Sigma_g \cup \Sigma_f \subseteq \Pi_n$. We observe that, for each $\alpha \in [0,1]$, it is $\gamma_\alpha \in \mathcal{K}$; then $\Sigma_{\gamma_\alpha} \subseteq \Pi_n$, $\forall\, \alpha \in [0,1]$; hence $S = \bigcup_{\alpha \in [0,1]} \Sigma_{\gamma_\alpha} \subseteq \Pi_n$. $\square$

We remark that in general the checking for total coherence of (arbitrary) sets like S, or Σ_g, or Σ_f, may be intractable. Let be given an interval-valued assessment $X_n = ([l_1, u_1], \ldots, [l_n, u_n])$ on $\mathcal{F}_n$ and the associated multi-interval and set of vertices

$$\mathcal{I} = [l_1, u_1] \times \cdots \times [l_n, u_n]\,,\quad \mathcal{V} = \{l_1, u_1\} \times \cdots \times \{l_n, u_n\}\,.$$

We recall below a necessary and sufficient condition of total coherence for X_n ([8]), which amounts to checking coherence of all vertices of $\mathcal{I}$.

Theorem 16. Given an interval-valued assessment $X_n = ([l_1, u_1], \ldots, [l_n, u_n])$ on $\mathcal{F}_n$, one has:

$$\mathcal{I} \subseteq \Pi_n \iff \mathcal{V} \subseteq \Pi_n\,. \tag{13}$$

As an application of Theorem 15, we sketch below an alternative proof of Theorem 16. We set $\mathcal{V} = \mathcal{V}_g \cup \mathcal{V}_f$, where

$$\mathcal{V}_g = \{l_1, u_1\} \times \cdots \times \{l_{n-1}, u_{n-1}\} \times \{l_n\}\,,$$

$$\mathcal{V}_f = \{l_1, u_1\} \times \cdots \times \{l_{n-1}, u_{n-1}\} \times \{u_n\}\,.$$

Of course, $\mathcal{I} \subseteq \Pi_n$ implies $\mathcal{V} \subseteq \Pi_n$. Conversely, assume that $\mathcal{V} \subseteq \Pi_n$ and (by induction) that (13) holds for the multi-interval $[l_1, u_1] \times \cdots \times [l_{n-1}, u_{n-1}]$. Then, applying Theorem 15 with $g(\pi) = l_n$, $f(\pi) = u_n$, and $\Phi = [l_1, u_1] \times \cdots \times [l_{n-1}, u_{n-1}]$, we have

$$\Sigma_g = \{(\pi, l_n) : \pi \in \Phi\}\,, \quad \Sigma_f = \{(\pi, u_n) : \pi \in \Phi\}\,,$$

$$\gamma_\alpha(\pi) = \alpha\, l_n + (1-\alpha)\, u_n\,, \quad S = \bigcup_{\alpha \in [0,1]} \Sigma_{\gamma_\alpha} = \mathcal{I}\,.$$

As $\mathcal{V}_g \subseteq \Pi_n$ (resp., $\mathcal{V}_f \subseteq \Pi_n$), by the inductive hypothesis one has $\Sigma_g \subseteq \Pi_n$ (resp., $\Sigma_f \subseteq \Pi_n$), so that $\Sigma_g \cup \Sigma_f \subseteq \Pi_n$; hence $\mathcal{I} \subseteq \Pi_n$.

6 Conclusions

We have examined interval-valued probability assessments on finite families of conditional events. Our approach has been based on the notion of g-coherence which coincides with Walley's AUL property. We have generalized some recent results on precise probability assessments to the case of interval-valued assessments. In particular, we have generalized a connection property of the set Π_n of precise coherent conditional probability assessments to the case of interval-valued assessments. More precisely, we have proven that, with any pair of AUL interval-valued assessments X'_n, X''_n we can associate an infinite class $\mathcal{X}$ of AUL interval-valued imprecise assessments connecting X'_n and X''_n. Then, exploiting such result, we have examined the extension of g-coherent imprecise assessments. We have also examined a method to construct classes of g-coherent interval-valued assessments, which could be useful to conciliate possible discrepancies between different opinions of experts. Finally, we have given a result on totally coherent set-valued probability assessments, by examining its relationship with a necessary and sufficient condition for total coherence of interval-valued assessments. We remark that the results obtained for AUL assessments can be suitably adapted to coherent ones. This study could be deepened in a further work.

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