Information Processing under Imprecise Risk with the Hurwicz criterion

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Abstract

An agent has Hurwicz criterion with pessimism-optimism index α under imprecise risk and adopts the root dictatorship version of McClennen's Resolute Choice in sequential decision situations, i.e. evaluates strategies at the root of the decision tree by the Hurwicz criterion and enforces the best strategy, thus behaving in a dynamically consistent manner. We address two questions raised by this type of behavior: (i) is information processed correctly? and (ii) to what extent do unrealized outcomes influence decisions (non-consequentialism)? Partial answers are provided by studying: (i) the random sampling of a binary variable, and finding the influence of the pessimism-optimism index to be decreasing with the sample size, and the optimal decision rule to asymptotically only depend on the relative frequencies observed; and (ii) an insurance problem in which the agent chooses his coverage at period two after observing the period one outcome (*accident* or *no accident*); when no accident happened, a seemingly irrelevant data - the first period deductible level- is found to be able to influence the second period insurance choice. We analyse this result in relation with the existence and value of the pessimism-optimism degree.

Keywords. Imprecise risk, Hurwicz criterion, resolute choice, non-consequentialism, learning

1 Introduction

This paper deals with the impact of information on the decisions of an agent whose beliefs concerning the events are imprecise and whose preferences are not in accordance with the Subjective Expected Utility (SEU) model. Precisely, we assume that preferences are representable by the Hurwicz criterion: the value of a decision is a weighted sum of its lowest possible expected value (pessimistic evaluation) and of its highest one (optimistic evaluation).

It is well known that a SEU maximizer has dynamically consistent preferences: future decisions which seem the best today will still be judged the best tomorrow; this justifies the determination of the optimal strategy by backward induction (sophisticated choice). Preferences as modelled by the Hurwicz criterion no longer verify this consistency property. Thus, sophisticated choice no longer guarantees a rational behavior: the selected strategy may well be dominated.

An alternative to sophisticated choice which ensures rationality is the version of McClennen's Resolute Choice (1990) where the best strategy at the root is continued at every node (root dictatorship). We adopt this model here: strategies are evaluated at the root of the decision tree by the Hurwicz criterion; the enforcement of the best strategy all along the tree automatically guarantees dynamic consistency.

The use of Resolute Choice in an imprecise probability environment raises a first important question: is information processed correctly in this model? The existence of phenomena such as dilation (ambiguity increase with new information, cf. Seidenfeld, Wasserman (1993)) makes the answer unclear. We provide a positive answer in a particular case by considering a situation where data are provided by the random sampling of a binary variable and decisions are bets on future values of that variable. This decision problem is closely related to simple hypothesis testing.

Optimal decision rules turn out to be based on observed frequencies (just as likelihood ratio tests) and the influence of the degree of pessimism fades progressively when samples become larger.

A distinctive, controversial feature of Resolute Choice is non-consequentialism: decisions may depend on seemingly irrelevant data such as unrealized outcomes. Since this is a theoretical result, the question arises whether this phenomenon is widespread in real world decision problems or not. As a first field of investigation, we have chosen multi-period insurance contracting which constitutes an active research domain (Dionne, Doherty, Fombaron 2000). In this domain, up to now, the environment has invariably been described as a situation of risk (subjective or frequentist probabilities) and the model used is EU theory. However, for some risks, due to lacking or conflicting data, this assumption is highly irrealistic which is our justification for introducing imprecise risk in the case of a two-period insurance problem in which an individual has to chose his coverage for the second period after observing the first period outcome (*loss*, *no loss*). We apply Hurwicz's criterion together with a Resolute Choice behavior and determine to which extent unrealized outcomes influence optimal decisions. It turns out that such an influence indeed exists but only to a limited extent and for individuals who are neither extremely pessimistic, nor extremely optimistic.

2 Dynamic decision making in imprecise probabilities framework

2.1 Imprecise Risk

When facing common, general or personal, hazards, and in particular insurable hazards, most agents do not have a precise idea of their likelihoods. Statistics may be inexistent, unavailable or just neglected by the agent; also, important individual variations can exist. Thus, whatever the reasons, an agent may prove to be unable to ascribe specific probabilities to the relevant events in a significant manner.

On the other hand, he may feel more comfortable with associating with each event E a probability interval, $[P^-(E), P^+(E)]$; for instance, typical intervals would be: $[0.01, 0.10]$ for an event he considers as "very unlikely to happen but not impossible"; $[0.10, 0.30]$ for an event he judges "rather unlikely to happen"; and their union $[0.01, 0.30]$ for an event he just thinks "unlikely to happen".

If the agent moreover believes that there is a *true* probability P_0 on the events (which he is just not able to identify), these judgments are submitted to consistency rules, such as $P^+(E) \geq 1 - P^+(E^c)$ for complementary events E and E^c; this circumscribes P_0 to $\mathcal{P} = \{P : \text{for all } E,\ P(E) \in [P^-(E), P^+(E)]\}$, a subset of $\mathcal{L}$, set of all probabilities on the event set.

Such an agent uses an *imprecise probability* representation of uncertainty and, accordingly, makes decisions under *imprecise risk*.

2.2 The Hurwicz decision criterion

Various theories have been proposed for modelling decision making under imprecise risk. The most popular one (but not the only one, see § 2.3.4.) combines existing theories applying to the limiting cases of risk and complete ignorance.

(i) Under *risk*, the standard criterion is Expected Utility (EU). A decision maker (DM), believing the true probability to be P_0, ascribes to a decision δ value

$$U_{P_0}(\delta) = E_{P_0} u(\delta) = \sum_x u(x) P_0(\delta^{-1}(x))$$

i.e., the expectation of the utilities of the outcomes x that δ may bring about depending on which event $\delta^{-1}(x)$ obtains;

(ii) Under *complete ignorance*, Hurwicz's criterion, proposed as early as 1951, ascribes to a decision δ a value which is a weighted sum of its worst and best possible outcomes, $\alpha m_\delta + (1-\alpha) M_\delta$; parameter α being interpreted as a degree of pessimism.

Suppose now that *complete ignorance* prevails in $\mathcal{P}$ and consider a DM for whom being only able to locate probability P_0 in a set $\mathcal{P}$ amounts to being uncertain about which of the values $U_P(\delta)$, P in $\mathcal{P}$, is the correct one. Then, this DM will look at the worse and best possible evaluations and, according to its degree of pessimism, will put more or less weight on the former or the later, which is expressed by the following formula:

$$V(\delta) = \alpha \inf_{P \in \mathcal{P}} E_P u(\delta) + (1-\alpha) \sup_{P \in \mathcal{P}} E_P u(\delta) \quad (1)$$

This criterion being the natural extension of the Hurwicz one to imprecise risk, we will preserve its denomination of "Hurwicz criterion". In a decision making context, the interest of a preference model depends crucially on its ability to induce *economically rational behavior*, which includes invulnerability to Dutch books and money-pumps (Schick 1986, Diecidue, Wakker 2002) in situations involving sequential choices. Obviously, economic rationality cannot be guaranteed by a criterion which does not increase with dominance - is not *monotone* - in some sense.

Under suitable topological assumptions ($\mathcal{P}$ a compact subset of a separable space), Hurwicz's criterion satisfies strict and weak monotonicity properties. If the expected utility of decision δ is strictly higher than that of decision d for every probability measure, i.e., $E_P u(\delta) > E_P u(d)$ for all $P \in \mathcal{P}$ (strict pointwise dominance on $\mathcal{P}$), then $\inf_{P \in \mathcal{P}} E_P u(\delta) > \inf_{P \in \mathcal{P}} E_P u(d)$, $\sup_{P \in \mathcal{P}} E_P u(\delta) > \sup_{P \in \mathcal{P}} E_P u(d)$, and finally $V(\delta) > V(d)$; moreover, the weaker relation, $E_P u(\delta) \geq E_P u(d)$ for all $P \in \mathcal{P}$, implies $V(\delta) \geq V(d)$. In particular, if decision δ performs strictly better (resp. better) than decision d whatever happens, i.e., $u(\delta(e)) > (\geq)\ u(d(e))$ for every event e on which both δ and d are constant, then $E_P u(\delta) > (\geq)\ E_P u(d)$ for all $P \in \mathcal{P}$, hence $V(\delta) > (\geq)\ V(d)$.

On the other hand, if $E_P u(\delta) \geq E_P u(d)$ for all $P \in \mathcal{P}$, with $E_P u(\delta) > E_P u(d)$ for some $P \in \mathcal{P}$, it may none- theless happen that $\inf_{P\in\mathcal{P}} E_P u(\delta) = \inf_{P\in\mathcal{P}} E_P u(d)$ and $\sup_{P\in\mathcal{P}} E_P u(\delta) = \sup_{P\in\mathcal{P}} E_P u(d)$, hence that $V(\delta) = V(d)$; in particular, $u(\delta(e)) \geq u(d(e))$ for every e, plus $u(\delta(e)) > u(d(e))$ for some e, do not imply $V(\delta) > V(d)$. Note however that for every $\varepsilon > 0$, $V(\delta) > V(d-\varepsilon)$ and $V(\delta+\varepsilon) > V(d)$ will hold; thus, although not monotone, Hurwicz's criterion is, in a straightforward sense, $\varepsilon-$*monotone*.

These monotonicity properties are sufficient to make the model behave satisfactorily in one-shot decision problems. Multiple decision situations are a different matter, as illustrated in the following subsection.

2.3 Problems with dynamic decision making and the Resolute Choice solution

2.3.1 An illustrative example

Consider a DM who at time 1 (node A of the decision tree in Fig.1) has to choose between two decisions, Up_1 and $Down_1$; then, at time 2 (node B), provided he has chosen Up_1 and event E obtains, he has again a choice, Up_2 or $Down_2$, his gain further depending on the realization or not of some events, G or G^c and H or H^c; if at time 1 he has chosen Up_1 and event E^c obtains, or has chosen $Down_1$, there is no other choice to make. Gains are indicated next to the corresponding leaves of the tree.

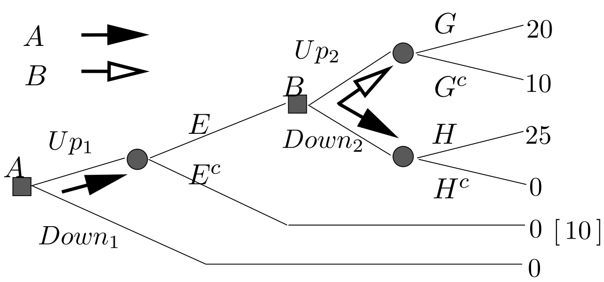

Figure 1: Dynamically inconsistent preferences

The DM's criterion is Hurwicz's, with the same parameters u and α, at both decision nodes, A and B. For the sake of simplicity we assume $\alpha = 1/2$, risk-neutrality ($u(x) = 2x$ for all x), and complete ignorance on the algebra of events generated by E, G and H; thus, $\mathcal{P} = \mathcal{L}$ and a strategy (at A), as well as a substrategy (at B), δ, giving outcomes $\delta(e)$ on events e has value $V(\delta) = \inf_e \delta(e) + \sup_e \delta(e)$.

At node A, the values of the three available strategies, $(Up_1, Up_2), (Up_1, Down_2)$, and $Down_1$ ((Up_1, Up_2) means Up_1 at node A; then Up_2 at node B if E happens; etc.) are, respectively, $V(Up_1, Up_2) = 20; V(Up_1, Down_2) = 25; V(Down_1) = 0$; thus the DM prefers $(Up_1, Down_2)$ to (Up_1, Up_2) (and to $Down_1$) in A.

However at node B he prefers substrategy (decision) Up_2 to substrategy $Down_2$ since $V(Up_2) = 30 > V(Down_2) = 25$; thus, if he takes decision Up_1 in A and event E happens, then, once arrived in B, he no longer considers $Down_2$ to be the best feasible action; his preferences are not *dynamically consistent*.

2.3.2 Resolute Choice

What are the decisions actually made by a DM with a logical mind, who is able to anticipate on his future actions (*sophistication*, as opposed to *myopia*), and is aware that his preferences are not dynamically consistent? Roughly, one can think of two different patterns of behavior.

(i) If his future choices are always dictated by his future preferences, then the DM should use *backward induction* in the decision tree: at each given decision node, knowing which substrategies would be triggered by each of his feasible actions, he can evaluate and compare them, according to his criterion, and choose the best available action. Coping locally in that way with his preferential inconsistencies unfortunately does not warrant him at the end (when arrived at the root of the tree) the selection of a strategy possessing a valuable global property. Indeed, going back to the example, the DM would be willing to pay up to 5 units to have the tree pruned and edge Up_2 suppressed in B. Consider then the augmented tree in which a new subtree offers this possibility to the DM; strategy $(Up_1, Down_2)$, which is still materially feasible, clearly strictly dominates the additional strategy, which is nonetheless chosen by the backward induction procedure. In general, the use of that behavioral procedure is always a potential source of unnecessary waste: it is not economically rational.

How can any waste be avoided? There is a straightforward way:

(ii) If the strategy which is judged best according to preferences at the root node is actually played, then, the criterion being used only once as in one-shot decision problems, the monotonicity of Hurwicz's criterion guarantees economic rationality. This dictatorship of the root node preferences means of course that future choices do not have to bear any relation with future preferences. More generally and less drastically, *Resolute Choice* (McClennen, 1990, p.260) only requires the achievement of a compromise strategy reflecting both present and future preferences; in McClennen's terms: "the theory of resolute choice is predicated on the notion that the single agent who is faced with making decisions over time can achieve a cooperative arrangement between his present self and his relevant future selves that satisfies the principle of intrapersonal optimality". Resolute Choice is not just a theoretical construct; it can be implemented in an operational way (see Jaffray-Nielsen 2006).

2.3.3 Non-consequentialism and unrealized outcomes

A feature of Resolute Choice is *non-consequentialism*: the choice at a given decision node, being induced by a strategy which depends on all the data in the decision tree, may in particular depend on those data which are outside the subtree rooted at that node; these elements are known as *unrealized outcomes.*

In Fig.1, if the best strategy in A, $(Up_1, Down_2)$, can be imposed, $Down_2$ is played in B. Modify now a single outcome, at the leaf following Up_1 and E^c, by changing 0 into 10; the best strategy in A is now (Up_1, Up_2) and Up_2 is played in B accordingly; thus the action taken in B depends on a unrealized outcome, the outcome at a leaf that is not part of the subtree rooted at B.

For an illuminating discussion of consequentialism see Machina (1989). Let us just note for the moment that, since, as seen above, economic rationality cannot provide arguments against non-consequentialism, any defense of consequentialism must rely on a different conception of rationality.

2.3.4 Alternative approaches

Resolute Choice should not be confused with consequentialist approaches to dynamic decision making, which have recourse to recursive models (see e.g. Epstein, Schneider 2003); such models are straightforwardly dynamically consistent and backward induction remains valid; on the other hand, economic rationality is not necessarily satisfied. Neither is it in the non-consequentialist approach, preserving a weak form od dynamic consistency of Hanany, Klibanoff (2006).

Another approach to dynamic decision making under uncertainty, called E-admissibility, has been suggested by Levi (1974) and discussed by Seidenfeld (2004). It works by first selecting all the last stage Bayes rules and then moving backwards repeating this selection stage by stage. In order to uniquely select a strategy in the remaining set, a secondary criterion, applied at the root node, is used. While more discriminating than Resolute Choice with root dictatorship, E-admissibility (with a suitable secondary criterion) still guarantees normative qualities such as nonnegative value of information.

Note that E-admissibility is a non-consequentialist solution in general. However, de Cooman and Troffaes (2005) prove the validity of dynamic programming (which amounts to consequentialism) in the particular case of sequential decision making in the absence of conditional decisions.

3 Learning with Resolute Choice

An urn contains red and black balls; the proportion of red balls is *either* p^- *or* p^+, where $0 < p^- < p^+ < 1$. The DM is told that: $n + 1$ balls are going to be drawn one by one from the urn, with replacement; that he can make bets on the color of the $(n + 1)^{th}$ being red; and that his decision of betting or not can be conditioned on the outcome of the n first draws. When betting, his stake is m and he will receive gain M if the $(n + 1)^{th}$ is red. We assume $p^- < \frac{m}{M} < p^+$.

The DM conditions his bets on the outcomes of the n first draws by just specifying a *betting rule* $K_n \subseteq \{0, 1, .., n\}$, "$k \in K_n$" meaning: "if k balls among the n first drawn are red, bet (on red) at the $(n + 1)^{th}$ draw".

One denotes $k_n = \min_{k \in K_n} k$.

The DM uses the Hurwicz criterion, is risk neutral ($u(x) = x$) and is resolute; he chooses his betting rule when learning the sample size n and before the observations begin.

We are interested in the evolution of the optimal betting rule when n tends to infinity.

A *betting behavior* is a sequence $(K_n)_{n\in\mathbb{N}}$. Betting behavior $(K_n)_{n\in\mathbb{N}}$ *weakly dominates* betting behavior $(K'_n)_{n\in\mathbb{N}}$ if for all $n \in \mathbb{N}$, $V(K_n) \geq V(K'_n)$; if, moreover, $V(K_n) > V(K'_n)$ for some value of $n \in \mathbb{N}$, then $(K_n)_{n\in\mathbb{N}}$ *dominates* $(K'_n)_{n\in\mathbb{N}}$. A betting behavior which is not dominated by any other is *admissible.* A betting behavior which weakly dominates all the others is *optimal.*

A betting behavior $(K_n)_{n\in\mathbb{N}}$ will be called *consistent* when its betting rules are all of the form $K_n = \{k_n, k_n + 1, .., n\}$ (i.e., betting if and only if at least

k_n red balls have been drawn).

Lemma 1 For a fixed n, let betting rules K_n and K'_n only differ in the case where k red balls are drawn: $k \in K_n$; $K'_n = K_n \backslash \{k\}$; then

$$V(K_n) > [=] V(K'_n) \Longleftrightarrow \frac{k}{n} > [=] L + \frac{1}{n} R$$

with $L = \dfrac{\ln \dfrac{1-p^-}{1-p^+}}{\ln \dfrac{p^+(1-p^-)}{(1-p^+)p^-}}$ and

$$R = \frac{\ln \left[\dfrac{\alpha}{1-\alpha} \times \dfrac{m - p^- M}{p^+ M - m} \right]}{\ln \dfrac{p^+(1-p^-)}{(1-p^+)p^-}}$$

N.B. The proofs of Lemma 1 and of the other results can be found in Jaffray, Jeleva (2007).

The following proposition is a direct application of Lemma 1.

Proposition 1 Consider betting behavior $(K_n)_{n \in \mathbb{N}}$, and let $k_n = \min_{k \in K_n} k$.

A necessary condition for the admissibility of $(K_n)_{n \in \mathbb{N}}$ is that
$\dfrac{k_n}{n} \to_{n \to \infty} L$
with L defined in lemma 1.

Proposition 2 The consistent betting behavior, $(K^*_n)_{n \in \mathbb{N}}$ where $K^*_n = \{k^*_n, k^*_n + 1, k^*_n + 2, ..., n\}$, and for each n, k^*_n is the smallest integer such that

$\dfrac{k^*_n}{n} \geq L + \dfrac{1}{n} R$ with L and R defined in emma 1.

is an *optimal betting behavior.*

Note that expression $\left[\dfrac{p^+}{1-p^+} \times \dfrac{1-p^-}{p^-} \right]^k \times \left[\dfrac{1-p^+}{1-p^-} \right]^n$ is a likelihood ratio; in fact the monotonicity properties of the Hurwicz criterion make likelihood ratio (possibly random) tests an admissible family as in the standard statistical decision theory (Neyman-Pearson lemma). For related results concerning hypothesis testing with imprecise probabilities on the parameter space, see Jaffray, Saïd (1994).

Note also that expression R , defined in emma 1, has a strong similarity with the term that would appear in a Bayesian model, which is $\dfrac{\ln \left[\dfrac{\pi}{1-\pi} \times \dfrac{m - p^- M}{p^+ M - m} \right]}{\ln \dfrac{p^+(1-p^-)}{(1-p^+)p^-}}$, with π the prior probability of p^- being the true proportion of red balls.

Let us finally emphasize the fact that, although all betting decisions are made only on the basis of a single *ex ante* evaluation, data are taken into account in a sensible way: for high values of n, the DM acts as if he used relative frequencies as estimators of probabilities; however, for smaller n, the degree of pessimism has some influence on the bets through the term R.

4 An application of Resolute Choice to Two-period Insurance Demand

In this section, we study a two-period insurance problem in which an individual has to choose his coverage at period 2 after observing the period 1 outcome ([*a*] *loss* [*occurred*] or *no loss* [*occurred*]).

An individual with initial wealth W faces a risk with a unique amount of potential loss $L < W$. This situation can be represented by a random variable X: if E is the event *loss (occurs)* and E^c the event *no loss*, $X(\omega) = L$ for $\omega \in E$ and $X(\omega) = 0$ for $\omega \in E^c$. The individual's information and/or beliefs allow him to assert that the probability of loss occurrence during a year is between p^- and p^+. The set of probability distributions which are consistent with the available information is:

$$\mathcal{P} = \left\{ P \in \mathcal{L} : P(E) \in \left[p^-, p^+ \right] \right\} \qquad (2)$$

where $\mathcal{L}$ denotes the set of all probability distributions on the relevant support.

Two periods of time are considered: in the first period, the individual has no insurance choice to make; for instance, he rents a car, and an insurance coverage with a deductible $K \leq L$ is automatically included in the contract. In the second period however, the individual has to decide if he will subscribe an insurance contract or not, for instance he will buy a car and has to decide whether or not he will take a theft insurance (which is not mandatory). We assume that only one insurance contract is available: it corresponds to full coverage and the premium is $\Pi < L$.

We assume that the individual needs to decide immediately, at the beginning of the first period, what his insurance policy will be; the reason may be, for instance, that he still has then other opportunities beside renting-then-buying a car and that their comparisons require accurate evaluations, or that he has to plan out his expenses in advance.

Individual preferences are represented by the Hurwicz criterion: a decision $\delta : \Omega \to \mathbb{R}$ is evaluated by functional V of formula (1) where u is a strictly increasing function.

In the simpler, one period situation, where there is no previous experience of loss, the set of strategies D contains only two elements, denoted: d, the individual subscribes an insurance contract, and $\bar{d}$, the individual does not buy any insurance. According to (1), these decisions have the following values:

$$V(d) = u(W - \Pi)$$
$$V(\bar{d}) = (\alpha p^+ + (1-\alpha) p^-)\, u(W - L) + (1 - \alpha p^+ - (1-\alpha) p^-)\, u(W)$$

and the decision to buy coverage depends on the pessimism-optimism index α and on the information precision in the following way:

$$V(d) \geq V(\bar{d}) \Leftrightarrow \alpha (p^+ - p^-) \geq \frac{u(W) - u(W-\Pi)}{u(W) - u(W-L)} - p^-.$$

Thus, a higher degree of pessimism and a greater imprecision both act in favor of the decision to buy insurance coverage.

4.1 Decisions evaluation

We now turn to the evaluation of the decisions of an individual who acquires additional information related to a period one potential loss. His decisions can then be conditioned on the realization of the loss in the first period. Our goal is to determine the influence of the first period loss realization on the second period decision as well as the impact of α on that decision. We further assume probabilistic independence of the successive events, i.e., that for any given probability $p \in [0,1]$, with E_i denoting the event "loss in period i", if $P(E_1) = p$ then $P(E_2/E_1) = p$ as well, hence $P(E_2) = p$ and $P(E_1 \cap E_2) = p^2$.

A strategy is now characterized by a pair of decisions: the first one conditional on the realization of E_1, and the second one on the realization of E_1^c. The set of possible strategies D consists then in four pairs of decisions: $D = \{dd,\ d\bar{d},\ \bar{d}d,\ \bar{d}\bar{d}\}$, where $dd = \{d$ if E_1, d if $E_1^c\}$, $d\bar{d} = \{d$ if E_1, $\bar{d}$ if $E_1^c\}$, ... The decision tree corresponding to this problem is given in Fig.2.

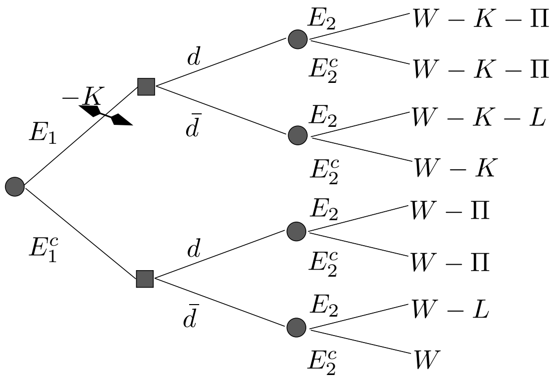

Figure 2: Insurance Demand Tree

The evaluations of the strategies at the beginning of period one by the Hurwicz criterion are given in the following proposition. This evaluation requires the determination of the probabilities in $[p^-, p^+]$ at which the lowest and highest expected utility are achieved. It turns out that these probabilities may well differ from p^+ and p^- and depend on the strategy.

Proposition 3 If Π, K, L, p^-, p^+ are such that:

- $u(W - L - K) \leq \frac{1}{2p^-}\left[u(W-\Pi) + (2p^- - 1) u(W-K)\right]$
- $p^* = \frac{1}{2} + \frac{u(W) - u(W-\Pi-K)}{2[u(W) - u(W-L)]}$
  verifies $p^* \in [p^-, p^+]$ and $p^* > \frac{1}{2}(p^- + p^+)$,

then the available decisions are evaluated as follows:

$V(dd) = A(p^+, p^-) u(W - \Pi - K) + (1 - A(p^+, p^-))\, u(W - \Pi)$;

$V(d\bar{d}) = A(p^*, p^-) u(W - \Pi - K) + B(p^*, p^-) u(W - L) + C(1 - p^*, 1 - p^-) u(W)$;

$V(\bar{d}d) = C(p^+, p^-) u(W - L - K) + B(p^+, p^-) u(W - K) + A(1 - p^+, 1 - p^-) u(W - \Pi)$

$V(\bar{d}\bar{d}) = C(p^+, p^-) u(W - L - K) + B(p^+, p^-) \times (u(W-K) + u(W-L)) + C(1 - p^+, 1 - p^-) u(W)$

where

$A(p, q) = \alpha p + (1-\alpha) q,$
$B(p, q) = \alpha p(1-p) + (1-\alpha) q(1-q),$
$C(p, q) = \alpha p^2 + (1-\alpha) q^2.$

In very ambiguous situations, the requirements above are not too restrictive; for instance, in the limiting case of complete ignorance, that is, for $[p^-, p^+] = [0, 1]$, these conditions reduce to $\Pi > K$.

From now on, we assume that these conditions are satisfied.

Note that the pessimistic evaluation of strategy $d\bar{d}$ is not achieved at the upper probability bound p^+: with p^* smaller than p^+ but close to it, the advantage of incurring period 1 loss K with the smaller probability p^* is not compensated by the disadvantage of incurring period 2 loss L with probability $(1 - p^*)p^*$ greather than $(1 - p^+)p^+$.

Let us now turn to a specific feature of the model: the relevance of unrealized outcomes.

Consider strategies dd and $d\bar{d}$. They differ by the

decision that follows the period 1 *no loss* event. The utilities involved in the direct comparison of these conditional decisions do not depend on K, and its value would be irrelevant in a consequentialist approach. However, with our criterion, $V(dd) - V\left(d\bar{d}\right) =$
$\alpha\left(p^{+} - p^{*}\right) u\left(W - \Pi - K\right) +$
$\left(1 - \alpha p^{+} - (1-\alpha)p^{-}\right) u\left(W - \Pi\right)$
$- \left(\alpha p^{*}(1-p^{*}) + (1-\alpha)p^{-}(1-p^{-})\right) u\left(W - L\right) -$
$\left(\alpha(1-p^{*})^{2} + (1-\alpha)(1-p^{-})^{2}\right) u(W)$

The sign of the previous expression is indeterminate and depends on the value of K, which influences both the lowest utility $u\left(W - \Pi - K\right)$ and p^{*}. More precisely, the influence of K on the discrepancy between $V(dd)$ and $V\left(d\bar{d}\right)$ increases with the pessimism-optimism index α, since

$\frac{d\left[V(dd) - V\left(d\bar{d}\right)\right]}{dK} =$
$\alpha\{-\frac{dp^{*}}{dK}u\left(W - \Pi - K\right) - \left(p^{+} - p^{*}\right) u'\left(W - \Pi - K\right)$
$+ \left(2p^{*} - 1\right) \frac{dp^{*}}{dK} u(W - L) + 2\left(1 - p^{*}\right) \frac{dp^{*}}{dK} u(W)\}$

The reason why the comparison of $V(dd)$ and $V\left(d\bar{d}\right)$ depends on the irrelevant outcome K is that the Hurwicz criterion is a limiting form of a rank dependent utility (RDU) criterion and that in RDU theory (Quiggin 1982) the decision weight associated with a consequence depends on the rank of this consequence in the set of consequences of a given decision. Decisions dd and $d\bar{d}$ have $W - \Pi - K$ as a common consequence but while with dd, $W - \Pi - K$ is the worst consequence, this is no longer the case with $d\bar{d}$ for which it is $W - L$. Consequently, the decision weight of $u\left(W - \Pi - K\right)$ is not the same in the evaluation of dd and $d\bar{d}$, even if this consequence is obtained for the same event (E_1) with both decisions. Thus, the second period preference between insurance or not in the case where no loss occurred in the first period may depend on the deductible level which the individual would have paid had loss occurred.

4.2 A numerical example

The following example illustrates the impact of K on the optimal strategy[1].

We consider an individual with initial wealth $W =$ 1 000 000 who faces the risk of a loss of amount $L =$ 40 000. Loss probability at each period, p, belongs to $[0.01,\ 0.7]$. The insurance premium for full coverage is $\Pi = 4\ 000$. The utility function is assumed to be in the CRRA class (with constant relative risk aversion) that is $u(x) = \dfrac{x^{1-R}}{1-R}$; here, we take $R = 2$.

The sign of $V(dd) - V\left(d\bar{d}\right)$ depends on α and K as follows:

[1]Numerical results are obtained with Mathematica 4.1.

- for $\alpha \in [0, 0.22[$, $V(dd) - V\left(d\bar{d}\right) < 0$ for any $K \in [0, 40\ 000]$;
- for $\alpha \in [0.22,\ 0.29[$, there exist $K^{*} < 40\ 000$ such that $V(dd) - V\left(d\bar{d}\right) \leq 0$ for $K \leq K^{*}$ and $V(dd) - V\left(d\bar{d}\right) > 0$ for $K > K^{*}$;
- for $\alpha \in [0.29, 0.33[$, there exist K^{*} and K^{**} with $0 < K^{*} < K^{**} < 40\ 000$ such that $V(dd) - V\left(d\bar{d}\right) < 0$ for $K^{*} < K < K^{**}$ and $V(dd) - V\left(d\bar{d}\right) \geq 0$ for $K \leq K^{*}$ and $K \geq K^{**}$;
- for $\alpha \in [0.33, 1]$, $V(dd) - V\left(d\bar{d}\right) > 0$ for any $K \in [0, 40\ 000]$.

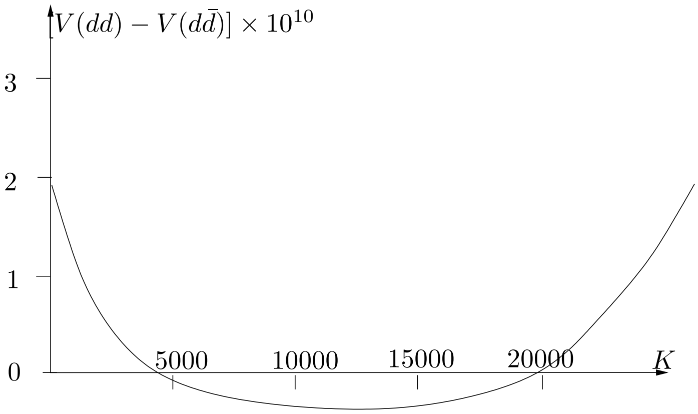

Figure 3: Choice Dependence on K for $\alpha = 0.31$

Let us now study the dependence of the optimal strategy on K and α.

- We start by comparing $d\bar{d}$ and $\bar{d}d$: $V\left(d\bar{d}\right) - V\left(\bar{d}d\right)$is a linear function of α; moreover,

  for $\alpha = 0$, as well as for $\alpha = 1$, $V\left(d\bar{d}\right) - V\left(\bar{d}d\right) > 0$ for any $K \in [0, 40000]$; thus, for any $\alpha \in [0, 1]$, $d\bar{d}$ is preferred to $\bar{d}d$.
- The same result is obtained for $d\bar{d}$ when compared with $\bar{d}\bar{d}$.
- The choice between dd and $\bar{d}\bar{d}$ depends on α in the following way:

  $$V(dd) - V(\bar{d}\bar{d}) < 0 \text{ for } \alpha \in [0, 0.003[;$$
  $$V(dd) - V(\bar{d}\bar{d}) > 0 \text{ for } \alpha \in [0.003, 1].$$

Thus, for any $K \in [0, 40000]$, strategies $\bar{d}\bar{d}$ and $\bar{d}d$ are dominated so that, the best strategy is always either dd or $d\bar{d}$.

This dominance is due to the low insurance premium Π that corresponds here to a probability estimation of 0.1. In consequence, individuals prefer either to fully insure in any case (if they are pessimistic enough) and thus benefit from the low premium, or to adapt their decision to the observed loss. Fig.4 shows the optimal strategy as a function of K and α. It appears that the optimal decision results from a trade-off between the attractivity of low price insurance and that of information depending decisions. For strong optimists, the information effect dominates, whereas for strong pessimists, the full coverage effect dominates. For intermediate values of α however, the deductible value K may influence choice: a high value of K can even influence all decisions by lowering the individual's expected wealth perspectives and acting in favor of full coverage.

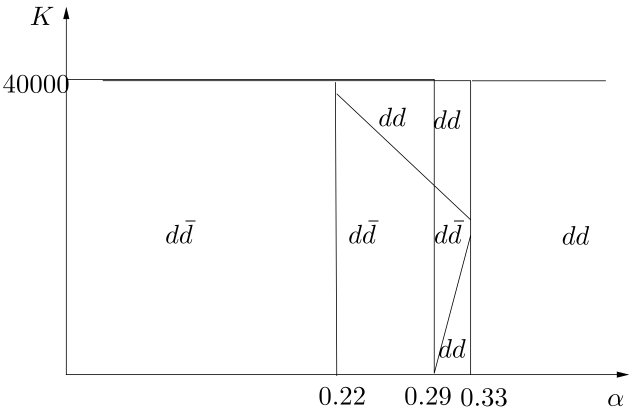

Figure 4: Choice Dependence on K and α

4.3 Optimal strategy for risk-neutral individuals

To emphasize the impact of the pessimism index α on the optimal insurance strategies, we now consider the case when $u(x) = x$. This allows us to isolate the influence of ambiguity attitude, characterized here by α, from that of the risk attitude, characterized by u.

Proposition 4 Consider a two period insurance problem, where the individual's imprecise information on the loss probability is given by an interval $[p^-, p^+]$ with $p^- < \frac{1}{2} < p^+$ and the insurance premium Π for full coverage is such that $\Pi \in [p^-L, p^+L]$. The preferences of the individual are characterized by the Hurwicz criterion with $u(x) = x$. Then, he orders the different available strategies in the following way:

- $d\bar{d} \succsim \bar{d}d$ for any $\alpha \in [0, 1]$;
- $dd \succsim \bar{d}\bar{d} \Leftrightarrow \alpha \geq \alpha^*$ with $\alpha^* = \frac{(\Pi - p^- L)}{(p^+ - p^-)L}$ where $\alpha^* < 1$;
- if $K = 0$, $dd \succsim d\bar{d} \Leftrightarrow \alpha \geq \alpha^{**}$ with $\alpha^{**} = \frac{(1-p^-)(\Pi - p^- L)}{(p^* - p^-)(\Pi - p^- L + L(1-p^*))}$ where $\alpha^{**} < 1$; if $K > 0$, both $dd \succsim d\bar{d}$ and $d\bar{d} \succsim dd$ are possible depending on the value of K.
- $d\bar{d} \succsim \bar{d}\bar{d} \Leftrightarrow \alpha \geq \alpha^{***}$ with $\alpha^{***} = \frac{p^-(\Pi - p^- L)}{(p^+ - p^*)(K+L) + (p^* - p^-)[L(p^* + p^-) - \Pi]}$ where $\alpha^{***} < 1$.

This proposition allows to determine, for $K = 0$ the impact of the pessimism index on the individual's optimal strategy. More precisely, in this case, $\alpha^{***} < \alpha^* < \alpha^{**}$ and $\bar{d}\bar{d}$ is the optimal strategy for $\alpha \in [0, \alpha^{***}[$, $d\bar{d}$ is the optimal strategy for $\alpha \in]\alpha^{***}, \alpha^{**}[$ and dd is the optimal strategy for $\alpha \in]\alpha^{**}, 1]$. For $\alpha = \alpha^{***}$, the individual is indifferent between $\bar{d}\bar{d}$ and $d\bar{d}$, and for $\alpha = \alpha^{**}$, he is indifferent between $d\bar{d}$ and dd.

To sum up, in this model, neither a very optimistic individual (α close to 0) nor a very pessimistic one (α close to 1) takes advantage of the information: his decisions do not depend on his period 1 observation. The reason is that, strong pessimists are trying above all to avoid the lowest possible consequences, which are here $W - L - K$ if E_1 and $W - L$ if E_1^c; choosing dd is the strategy that makes it possible. The opposite is true for strong optimists: they will prefer the decisions that allow the higher possible consequences, which are here $W - K$ if E_1 and W if E_1^c.

For moderate individuals, choice is less straightforward: for them, it is valuable both to avoid $W - L - K$ if E_1 (which however means renouncing to get $W - K$) and to preserve the possibility to obtain W if E_1^c (which however means risking to get $W - L$); this is only possible with $d\bar{d}$, and trade-offs, which depend on all the parameters (in particular on Π) may favor this strategy.

5 Conclusion

The preceding results demonstrate the operational tractability of the Resolute Choice dynamic adaptation of the Hurwicz criterion for decision making under imprecise risk. This model is able to process information correctly; in particular, for large samples, choices made show that the true probabilities are learned correctly although implicitly.

Also, the puzzling influence of unrealized outcomes appears as rather limited (only concerns individuals whose pessimism index belongs to a small range) and does not seem to lead to counter-intuitive decisions. It is moreover interesting to note that sensitivity to unrealized outcomes being excluded by Expected Utility theory, the Resolute Choice model has a flexibility that makes it attractive for descriptive purposes.

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