Duality Between Maximization of Expected Utility and Minimization of Relative Entropy When Probabilities are Imprecise

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Markets & Imprecise Probabilities

Imprecise probabilities naturally arise in the analysis of financial markets under uncertainty wherever those markets are incomplete, which is to say, virtually everywhere. A market is incomplete if some assets have distinct bid and ask prices (or are not priced at all) because of caution or lack of information on the part of buyers and sellers and/or because of transaction costs. The simplest case of a market under uncertainty is one in which assets are purchased at time 0 and sold at time 1, and the uncertainty about asset prices at time 1 is modeled by a finite set of states. Any financial asset in such a market can be constructed from a portfolio of "Arrow securities," where an Arrow security is an asset whose payoff is $1 in a given state and zero otherwise. The bid and ask prices for a state-i Arrow security can viewed as lower and upper probabilities assigned to state i by the representative agent. Bid and ask prices for more complex assets (which may yield arbitrary payoffs in different states) establish other linear inequality constraints on the probability distribution of the representative agent, so that in general the imprecise beliefs of the representative agent are described by a convex polytope of distributions that is the intersection of all the constraints. This set is non-empty if and only if there are no arbitrage opportunities in the market, a result that is known as the "fundamental theorem of asset pricing" but which was introduced much earlier by de Finetti as the "fundamental theorem of subjective probability." The problem we consider is that of an investor whose (precise) subjective probability distribution is p and who invests optimally in a market where the imprecise probabilities of the representative agent are described by a convex set *Q* that is disjoint from p.

Generalized Divergence Measures

Divergence measures have been widely used in statistics for many years as (seemingly) utility-free measures of the value of information. More recently it has been recognized that certain divergences are interpretable as measures of expected utility gains that are available to decision makers who have opportunities to bet against less-well-informed opponents or to invest in financial markets, as will be shown here. In particular, we give attention to two commonly used parametric families of divergence measures, which both generalize the well-known Kullback-Leibler (KL) measure. The power and pseudospherical divergences are defined for all $\beta \in$ R and are given by

$$D_\beta^{\mathbf{P}}(\mathbf{p}\|\mathbf{q}) \equiv \frac{E_{\mathbf{p}}[(\mathbf{p}/\mathbf{q})^{\beta-1}]-1}{\beta(\beta-1)} \qquad D_\beta^{\mathbf{S}}(\mathbf{p}\|\mathbf{q}) \equiv \frac{\left(E_{\mathbf{p}}[(\mathbf{p}/\mathbf{q})^{\beta-1}]\right)^{1/\beta}-1}{\beta-1}$$

The cases of $\beta = -1$, 0, 1/2 , 1, and 2 are of special interest. At β=1, the power divergence between p and q is equal to the KL divergence, and at β=0 it is the reverse KL divergence. The case β=1/2 has perfect symmetry, and reduces to a measure which is proportional to the squared Hellinger distance. At β=2, it reduces to (a multiple of) the Chi-square divergence. The pseudospherical divergence is a nonlinear transformation of the power divergence, hence it can also be expressed as a function of other well-known divergences for special cases of β.

Novel Parameterization of Generalized Power Utility Functions

In the optimization problems to be discussed, the decision maker's utility function will be assumed to be drawn from the most commonly used parametric family of utility functions, namely the generalized power family that includes the exponential and logarithmic utility functions as limiting cases. We introduce a new parameterization of this family, which we call normalized linear risk tolerance (LRT) utility functions:

$$u_\beta(x) \equiv \frac{1}{\beta-1}((1+\beta x)^{(\beta-1)/\beta}-1) \text{ if } \beta x > -1$$

and -∞ otherwise, for all $\beta \in$ R. The advantages of this normalization are that (a) it is a natural one for modeling utility gains and losses relative to the status quo rather than relative to some hypothetical zero-point of wealth at which utility goes to minus-infinity, and (b) for fixed x, u(x) is a continuous function of β on the entire real line, so that it sweeps out the widest possible spectrum of local risk attitudes.

Main Result

The expected-utility maximization problem faced by a risk-averse decision maker in an incomplete financial market where the beliefs of the representative agent are described by a convex set of imprecise risk-neutral probabilities is dual to the problem of finding the point in the latter set which minimizes a generalized divergence w.r.t. the decision maker's distribution. In particular:

(a) In an incomplete, single-period market, maximization of expected linear-risk-tolerance (LRT) utility with risk tolerance coefficient β, i.e.

$$U_\beta^{\mathbf{S}}(\mathbf{p}\|\mathbf{Q}) \equiv \max_{\mathbf{x}\in\mathbb{R}^n} E_{\mathbf{p}}[u_\beta(\mathbf{x})] \text{ subject to } \mathbf{Qx} \le \mathbf{0}$$

is dual to minimization of the pseudospherical divergence of order β between the decision maker's subjective distribution p and a risk neutral distribution q consistent with asset prices. That is, the corresponding dual problem is:

$$D_\beta^{\mathbf{S}}(\mathbf{p}\|\mathbf{Q}) \equiv \min_{\mathbf{z}\in\triangle^k} D_\beta^{\mathbf{S}}(\mathbf{p}\|\mathbf{z}^T\mathbf{Q})$$

where Q is the matrix whose rows are the extremal risk-neutral distributions of the representative agent. The optimal objective values of these two problems are the same and the optimal values of the decision variables in one problem are equal to the normalized optimal values of the Lagrange multipliers in the other.

(b) In an incomplete, two-period market, maximization of expected quasilinear linear-risk-tolerance utility with second-period risk tolerance coefficient β, i.e.

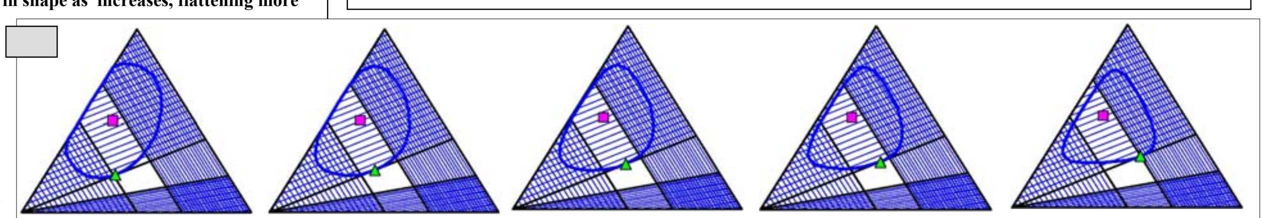
$$U_\beta^{\mathbf{P}}(\mathbf{p}\|\mathbf{Q}) \equiv \max_{y\in\mathbb{R},\mathbf{x}\in\mathbb{R}^n} E_{\mathbf{p}}[u_\beta(\mathbf{x})] - y \text{ subject to } \mathbf{Qx} \le y\mathbf{1}$$

is equivalent to minimization of the power divergence of order β between the decision maker's subjective distribution p and a risk neutral distribution q consistent with asset prices. Their optimal objective values are the same and the optimal values of the decision variables in one problem are equal to the normalized optimal values of the Lagrange multipliers in the other. That is, the corresponding dual problem is:

$$D_\beta^{\mathbf{P}}(\mathbf{p}\|\mathbf{Q}) \equiv \min_{\mathbf{z}\in\triangle^k} D_\beta^{\mathbf{P}}(\mathbf{p}\|\mathbf{z}^T\mathbf{Q})$$

Illustration

To visualize the preceding results, consider a simple example in which there are three states and (only) lower and upper bounds of 0.3 and 0.5 are given for the probability of state 1 and lower and upper bounds of 0.6 and 0.8 are given for the conditional probability of state 3 given not-state-1. The set Q of probability distributions that satisfies these constraints is the unshaded quadrilateral in the lower center of the simplex. Let the reference distribution be p = (0.35, 0.5, 0.15), which is the square dot in the upper left. The figures below show the solution of the dual problem of finding the element of Q that minimizes the pseudo-spherical or power divergence between itself and p for $\beta = -1$, 0, 0.5, 1, and 2. The triangular dot is the minimum-divergence solution, and the contour (level curve) that passes through it is also shown. In this case, the solution moves from the left to the right of the upper edge of the quadrilateral as β increases from -1 to 2. Also, the contours become more triangular in shape as increases, flattening more near the edges of the simplex, because as q approaches an edge of the simplex, q_i goes to zero for some *i*, and the term $(p_i/q_i)^{-1}$ in the divergence calculation blows up faster for larger values of β as that edge is approached.



$\beta = -1$ $\qquad$ $\beta = 0$ $\qquad$ $\beta = 0.5$ $\qquad$ $\beta = 1$ $\qquad$ $\beta = 2$

Discussion

A financial market under uncertainty provides one of the purest and most economically important examples of a situation in which subjective beliefs – in this case those of a risk neutral representative agent with whom individual investors may trade – are represented by imprecise probabilities that are subject to direct measurement. The measurement process, which consists of setting bid and ask prices for portfolios of Arrow securities, is essentially the same operational method of eliciting subjective probabilities that was introduced by de Finetti, and it naturally leads to a representation of beliefs in the form of a convex polytope of probability distributions. Here, we have considered the decision problem faced by a risk-averse investor in such a market when her risk preferences are represented by a utility function drawn from the generalized power family, which is the family most commonly used in finance theory and applied decision analysis. Under a natural (but novel) parameterization of the generalized power utility function, the investor's optimal expected utility is equal to the minimum of a generalized divergence between her own distribution and the nearest element of the polytope that characterizes the imprecise beliefs of the representative agent, where the generalized divergence is drawn from a parametric family that generalizes the Kullback-Lielber divergence. We have also pointed out connections with recent developments in the use of generalized divergences in robust Bayesian statistics. These results highlight the interconnections among information theory, Bayesian statistics, decision analysis, and finance theory with respect to the program of modeling imprecise probabilities.