The role of generalized p-boxes in imprecise probability theories

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We have recently introduced generalized p-boxes [1], encompassing possibility distributions and classical p-boxes. Due to their simplicity and interpretation, they are promising models. Here, we study their computational aspects for various information processings, to evaluate their potential as practical uncertainty models.

Definitions

Variable assuming values on $X = \{x_1, \dots, x_N\}$. Two mappings are said to be **comonotonic** if there is a common permutation σ of $\{1, 2, \dots, N\}$ such that $f(x_{\sigma(1)}) \geq f(x_{\sigma(2)}) \geq \dots \geq f(x_{\sigma(N)})$ and $f'(x_{\sigma(1)}) \geq f'(x_{\sigma(2)}) \geq \dots \geq f'(x_{\sigma(N)})$.

Definition 1 (Gen. P-box) *A generalised p-box $[\underline{F}, \overline{F}]$ is a pair of comonotonic mappings $\underline{F}, \overline{F}$, $\underline{F} : \mathcal{X} \to [0,1]$ and $\overline{F} : \mathcal{X} \to [0,1]$ from $\mathcal{X}$ to $[0,1]$ such that $\underline{F} \leq \overline{F}$ and there is at least one element x in $\mathcal{X}$ for which $\overline{F}(x) = \underline{F}(x) = 1$, these bounds ensuring that $[\underline{F}, \overline{F}]$ will be a coherent lower probability.*

Induce weak order on $\mathcal{X}$ such that $x \leq_{[\underline{F},\overline{F}]} y$ iff $\overline{F}(x) \leq \overline{F}(y)$ and $\underline{F}(x) \leq \underline{F}(y)$. Elements $x_1, \dots, x_N$ are indexed such that $i < j \to \overline{F}(x_i) \leq \overline{F}(x_j)$ and $\underline{F}(x_i) \leq \underline{F}(x_j)$.

Definition 2 ($[\underline{F}, \overline{F}]$-connected subsets) *Subset $C \subseteq \mathcal{X}$ is called $[\underline{F}, \overline{F}]$-connected if it can be expressed as a union of consecutive elements x_k, that is*

$$C = \{x_k \in \mathcal{X} | 0 < i \leq k \leq j \leq N\}$$

Definition 3 ($\sqsubseteq_{[\underline{F},\overline{F}]}$-ordering) *$A = \{x_i, \dots, x_j\}, B = \{x_{i'}, \dots, x_{j'}\} \subseteq \mathcal{X}$ two $[\underline{F}, \overline{F}]$-connected sets. $\sqsubseteq_{[\underline{F},\overline{F}]}$-ordering between them defined as*

$$A \sqsubseteq_{[\underline{F},\overline{F}]} B \to \begin{cases} i \leq i' \\ j \leq j'. \end{cases}$$

Links with other models

A gen. p-box $[\underline{F}, \overline{F}]$ generates the following models:

- **Probability sets:** $\mathcal{P}_{[\underline{F},\overline{F}]} = \{P | \underline{F}(x_i) \leq P(\{x_1, \dots, x_i\}) \leq \overline{F}(x_i)\}$.
$\to$ classical p-boxes retrieved when $\mathcal{X} = \mathbb{R}$ and sets $A_i = (-\infty, x_i]$

- **Random sets:** RS mapping $m : \wp(\mathcal{X}) \to [0,1]$ s.t. $\sum_{E \in \wp(\mathcal{X})} m(E) = 1$. Associate to $A \subseteq X$ a lower meas. $Bel(A) = \sum_{E, E \subseteq A} m(E)$ and a set $\mathcal{P}_m = \{P | \forall A, P(A) \geq Bel(A)\}$.
Denote by $0 = \gamma_0 < \gamma_1 < \dots < \gamma_M = 1$ the M distinct values taken by $\overline{F}, \underline{F}$, then $[\underline{F}, \overline{F}]$ equivalent to the following random set, for $j = 1, \dots, M$

$$\begin{cases} E_j = \{x_i \in X | (\overline{F}(x_i) \geq \gamma_j) \wedge (\underline{F}(x_{i-1}) < \gamma_j)\}, \\ m(E_j) = \gamma_j - \gamma_{j-1}. \end{cases} \quad (1)$$

All E_j are $[\underline{F}, \overline{F}]$-connected and form a complete order w.r.t. $[\underline{F}, \overline{F}]$-ordering.

- **Possibility distributions:** mapping $\pi : X \to [0,1]$ generating, for $A \subseteq X$ an upper measure $\Pi(A) = \sup_{x \in A} \pi(x)$ and an associated set $\mathcal{P}_\pi = \{P | \forall A, P(A) \leq \Pi(A)\}$.
Gen. p-box modelled by pair of distributions $\pi_{\overline{F}}, \pi_{\underline{F}}$ s.t. for $i = 1, \dots, N$,

$$\pi_{\overline{F}}(x_i) = \overline{F}(x_i) \quad \text{and} \quad \pi_{\underline{F}}(x_i) = 1 - \underline{F}(x_{i-1})\}, \quad (2)$$

in the sense that $\mathcal{P}_{[\underline{F},\overline{F}]} = \mathcal{P}_{\pi_{\underline{F}}} \cap \mathcal{P}_{\pi_{\overline{F}}}$.
$\to$ the pair $[\pi_{\overline{F}}, 1 - \pi_{\underline{F}}]$ forms a **cloud** [3]

1. Computing probability bounds

Lower probability of a $[\underline{F}, \overline{F}]$-connected set $C = \{x_k \in \mathcal{X} | 0 < i \leq k \leq j \leq N\}$ is

$$\underline{P}(C) = \max\{0, \underline{F}(x_j) - \overline{F}(x_{i-1})\}.$$

With $\overline{F}(x_0) = \underline{F}(x_0) = 0$. **Lower probability additive on disjoint union E of $[\underline{F}, \overline{F}]$-connected set C_k** : $E = C_1 \cup \dots \cup C_M$

$$\underline{P}_{[\underline{F},\overline{F}]}(E) = \sum_{k=1}^{M} \underline{P}_{[\underline{F},\overline{F}]}(C_k).$$

Using boolean sub-algebra $\mathcal{H}$ induced by focal sets, we have, for any A, $\underline{P}_{[\underline{F},\overline{F}]}(A) = \underline{P}_{[\underline{F},\overline{F}]}(A_*)$ with A_* its maximal inner approx. in $\mathcal{H}$. If $A_* = C_1 \cup \dots \cup C_M$ and $C_i = \{x_{\underline{i}}, \dots, x_{\overline{i}}\}$, that

$$\underline{P}(A) = \sum_{i=1}^{M} \max\{0, \underline{F}(x_{\overline{i}}) - \overline{F}(x_{\underline{i}-1})\}.$$

Upper probabilities are obtained via duality $\overline{P}(A) = 1 - \underline{P}(A^c)$

example:

$$x_1 \leq_{[\underline{F},\overline{F}]} x_2 \leq_{[\underline{F},\overline{F}]} x_3 \simeq_{[\underline{F},\overline{F}]} x_4 \leq_{[\underline{F},\overline{F}]} x_5 \leq_{[\underline{F},\overline{F}]} x_6 \simeq_{[\underline{F},\overline{F}]} x_7$$

$$A = \{x_1, x_3, x_4, x_5, x_6\}$$

$$A_* = \{x_1\} \cup \{x_3, x_4, x_5\}$$

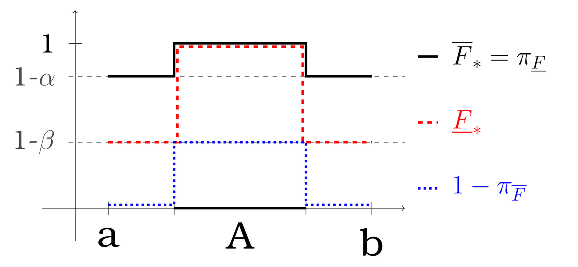
$$\underline{P}(A) = \underline{P}(A_*) = \max(0, \underline{F}(x_1) - \overline{F}(x_0)) + \max(0, \underline{F}(x_5) - \overline{F}(x_2))$$

2. Elicitation and representation

For $i = 1, \dots, N$, let denote α_i, β_i the bounds $\overline{F}(x_i)$, $\underline{F}(x_i)$ and A_i the sets $\{x_1, \dots, x_i\}$. A gen. p-box can then be **elicited as upper/lower confidence levels on nested sets/intervals**.

$\to$ Extend experts giving (imprecise) percentiles (p-boxes)
$\to$ Extend experts giving only lower bound (possibility dist.)

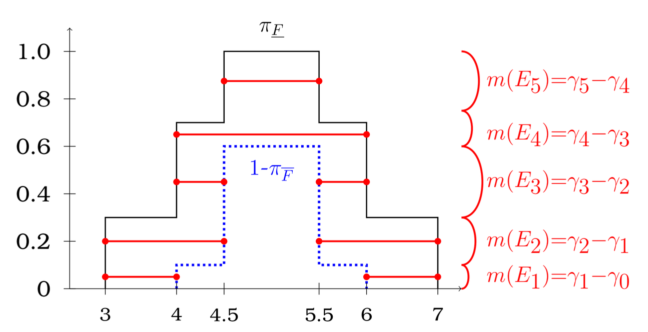
Example 1: Given a variable Θ with domain $[a, b]$ and the question "is Θ in A?", expert gives two bounds $\alpha \leq P(A) \leq \beta$



And $m(A) = \alpha$; $m([a, b]) = \beta - \alpha$; $m([a, b] \setminus A) = 1 - \beta$.

Example 2: An expert provide his opinion on a pH value in the form

$$0.3 \leq P(\text{pH} \in [4.5, 5.5]) \leq 0.6; \quad 0.7 \leq P(\text{pH} \in [4, 6]) \leq 0.9; \quad 1 \leq P(\text{pH} \in [3, 7]) \leq 1.$$



3. Conditioning

Given event $B = \{x_{b_1}, \dots, x_{b_M}\}$, two possible conditioning : Dempster's and Walley's

- **Dempster:** cond. upper measure, for $A \subseteq \mathcal{X}$

$$\overline{P}_{[B]}(A) = \frac{\overline{P}(A \cap B)}{\overline{P}(B)},$$

lower measure obtained by duality

Proposition 1 *$\underline{P}_{[\underline{F},\overline{F}]}$ induced by a gen. p-box. Lower measure* ***$\underline{P}_{[B]}$ obtained by Dempster's conditioning stems from a gen. p-box $[\underline{F}, \overline{F}]_{[B]}$*** *defined on $\mathcal{X} \cap B$ and yielding the restriction of $\leq_{[\underline{F},\overline{F}]}$ to elements $x \in B$.*

$\Rightarrow$ If $B_i = \{x_{b_1}, \dots, x_{b_i}\}$, sufficient to compute $\overline{P}_{[B]}(B_i)$, $\underline{P}_{[B]}(B_i)$ for $i = 1, \dots, M$. In particular, if B $[\underline{F}, \overline{F}]$-connected, then, for $i = 1, \dots, M$

$$\overline{P}_{[B]}(B_i) = \frac{\overline{F}(x_{b_i}) - \underline{F}(x_{b_1 - 1})}{\overline{F}(x_{b_M}) - \underline{F}(x_{b_1 - 1})} = \overline{F}_{[B]}(x_{b_i}),$$

$$\underline{P}_{[B]}(B_i) = \frac{\underline{F}(x_{b_i}) - \underline{F}(x_{b_1 - 1})}{\overline{F}(x_{b_M}) - \underline{F}(x_{b_1 - 1})} = \underline{F}_{[B]}(x_{b_i}).$$

- **Walley:** cond. lower measure, for $A \subseteq \mathcal{X}$

$$\underline{P}_{|B}(A) = \frac{\underline{P}(A \cap B)}{\underline{P}(A \cap B) + \overline{P}(A^c \cap B)}.$$

$\underline{P}_{|B}$ **cannot, in general, be modeled by a gen. p-box**

4. Propagation

Propagating through function f by computing image of sets, **3 methods giving $\neq$ random sets**:

- propagating $\alpha_i \leq P(f(A_i)) \leq \beta_i$ into $\alpha_i \leq P(f(A_i)) \leq \beta_i$

$$\left.\begin{matrix} \alpha_{i+1} > \theta \geq \alpha_i \\ \beta_{j+1} > \theta \geq \beta_j \end{matrix}\right\} \begin{matrix} m(f(A_{i+1}) \setminus f(A_j)) = \\ \min(\alpha_{i+1}, \beta_{j+1}) - \max(\alpha_i, \beta_j). \end{matrix} \Rightarrow \mathcal{P}_{f([\underline{F},\overline{F}])}$$

with $\theta \in [0,1]$. Gives **Inner approximation, low complexity.**

- Propagating sets E_i of equivalent random set

$$\left.\begin{matrix} \alpha_{i+1} > \theta \geq \alpha_i \\ \beta_{j+1} > \theta \geq \beta_j \end{matrix}\right\} \begin{matrix} m(f(A_{i+1} \setminus A_j)) = \\ \min(\alpha_{i+1}, \beta_{j+1}) - \max(\alpha_i, \beta_j); \end{matrix} \Rightarrow \mathcal{P}_{f((m,\mathcal{F}))}$$

with $\theta \in [0,1]$. Gives **exact result, high complexity.**

- Propagating separately $\pi_{\overline{F}}, \pi_{\underline{F}}$ separately t

$$\left.\begin{matrix} \alpha_{i+1} > \theta \geq \alpha_i \\ \beta_{j+1} > \theta \geq \beta_j \end{matrix}\right\} \begin{matrix} m(f(A_{i+1}) \setminus f(A_j^c)^c) = \\ \min(\alpha_{i+1}, \beta_{j+1}) - \max(\alpha_i, \beta_j). \end{matrix} \Rightarrow \mathcal{P}_{f(\pi_{\underline{F}}, \pi_{\overline{F}})}$$

with $\theta \in [0,1]$. Gives **outer approximation, Intermediate complexity.**

$$\mathcal{P}_{f([\underline{F},\overline{F}])} \subseteq \mathcal{P}_{f((m,\mathcal{F}))} \subseteq \mathcal{P}_{f(\pi_{\underline{F}}, \pi_{\overline{F}})},$$

The inclusions turning into equalities when f is injective (limiting in practical cases)

5. merging rules

S different gen. p-boxes $[\underline{F}, \overline{F}]_1, \dots, [\underline{F}, \overline{F}]_S$, said to form a comonotonic set if $\underline{F}_i, \overline{F}_i$, $i = 1, \dots, S$, are all comonotonic.

idempotent merging rules

- **Conjunction**: define $[\underline{F}, \overline{F}]_\cap$, for any $x \in \mathcal{X}$, as

$$\underline{F}_\cap(x) = \max_{i=1,S} \underline{F}_i(x) \text{ and } \overline{F}_\cap(x) = \min_{i=1,S} \overline{F}_i(x).$$

- **Disjunction**: define $[\underline{F}, \overline{F}]_\cup$, for any $x \in \mathcal{X}$, as

$$\underline{F}_\cup(x) = \min_{i=1,S} \underline{F}_i(x) \text{ and } \overline{F}_\cup(x) = \max_{i=1,S} \overline{F}_i(x).$$

- **Mean**: consider $\lambda_1, \dots, \lambda_S$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{S} \lambda_i = 1$.define $[\underline{F}, \overline{F}]_\Sigma$, for any $x \in \mathcal{X}$, as

$$\underline{F}_\Sigma(x) = \sum_{i=1}^{S} \lambda_i \underline{F}_i(x) \text{ and } \overline{F}_\Sigma(x) = \sum_{i=1}^{S} \lambda_i \overline{F}_i(x)$$

$\Rightarrow$ retrieve classical p-box merging and possibilistic merging as special cases

$\Rightarrow$ in general, merging result is not a gen. p-box but a cloud (lost of monotonicity).

Proposition 2 *Let $\mathcal{P}_{[\underline{F},\overline{F}]_1}, \dots, \mathcal{P}_{[\underline{F},\overline{F}]_S}$ be the sets of probabilities induced by $[\underline{F}, \overline{F}]_1, \dots, [\underline{F}, \overline{F}]_S$. Then, the following inclusions hold*

$$\mathcal{P}_{[\underline{F},\overline{F}]_\cap} \subseteq \bigcap_{i=1}^{S} \mathcal{P}_{[\underline{F},\overline{F}]_i}; \quad \mathcal{P}_{[\underline{F},\overline{F}]_\cup} \supseteq \bigcup_{i=1}^{S} \mathcal{P}_{[\underline{F},\overline{F}]_i},$$

first inclusion turning into an equality when generalised p-boxes form a comonotonic set.

non-idempotent merging rules

Extending idempotent rule by **using a t-norm $\top$ and its dual triangular conorm $\perp$**, possibly restricted to associative copulas. Disjunction and conjunction then respectively become

$$\underline{F}_\top(x) = \perp_{i=1,S} \underline{F}_i(x); \ \overline{F}_\top(x) = \top_{i=1,S} \overline{F}_i(x).$$

$$\underline{F}_\perp(x) = \top_{i=1,S} \underline{F}_i(x); \ \overline{F}_\perp(x) = \perp_{i=1,S} \overline{F}_i(x).$$

$\Rightarrow$ Allow taking source (in)dependencies into account ?
$\Rightarrow$ Inclusions of Proposition 2 remaining valids.
$\Rightarrow$ Equivalent to separately apply $\top$ (conjonction) or $\perp$ (disjunction) to $\pi_{\overline{F}}, \pi_{\underline{F}}$
$\Rightarrow$ Contrary to possibilistic case, no clear relation between $\top$=product and Dempster's rule of combination

Conclusions

- Generalized p-boxes **not very stable** under information processing (propagation, merging, ...), as result often not a generalized p-boxes, except under specific assumptions.
- **Main interests** lies in **uncertainty elicitation/representation**, thanks to their interpretation in terms of lower/upper confidence bounds, and in using them as **approximation alleviating computational burden**.
- Other potential interests: optimization under uncertainty using convex confidence regions [2], kernel signal-filtering with imprecise probabilities.

References

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