On solutions of stochastic differential equations with parameters modelled by random sets

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Introduction

Motivated by the investigation of mechanical systems under stochastic excitations we consider stochastic differential equations whose initial value and integrands depend on some uncertain parameters $a = (a_1, \dots, a_p) \in \mathbb{A} \subseteq \mathbb{R}^p$, that is,

$$dx_{t,a} = f(t, a, x_{t,a})dt + G(t, a, x_{t,a})dw_t \qquad (1)$$

with initial value $x_{t_0,a}$ where $t_0 \le t \le \bar{t} < \infty$, w_t denotes an m-dimensional Wiener process on a probability space $(\Omega_w, \Sigma_w, P_w)$ and

$$\begin{array}{rlcl} x_{t_0}: & \mathbb{A} \times \Omega_w & \to & \mathbb{R}^d, \\ f: & [t_0, \bar{t}] \times \mathbb{A} \times \mathbb{R}^d & \to & \mathbb{R}^d, \\ G: & [t_0, \bar{t}] \times \mathbb{A} \times \mathbb{R}^d & \to & \mathbb{R}^{d\times m}. \end{array}$$

The uncertainty of a shall be modelled by random compact sets which under certain conditions leads to compact set-valued processes. Furthermore analogues of first entrance times for set-valued processes are introduced.

Stochastic differential equations with random set parameters

We suppose that for each $a \in \mathbb{A}$ the conditions for existence and uniqueness (see e.g. [Arnold]) of solutions to Equation (1) are fulfilled. Hence, we get a family of solution processes which can be interpreted as a stochastic process on $[t_0, \bar{t}] \times \mathbb{A}$:

$$x : [t_0, \bar{t}] \times \mathbb{A} \times \Omega_w \to \mathbb{R}^d, (t, a, \omega_w) \mapsto x_{t,a}(\omega_w) \quad (2)$$

Under certain conditions it can be shown that the process defined by (2) satisfies the inequality

$$\mathrm{E}(\|x_{s,a} - x_{t,b}\|^{2n}) \le C \left\| \begin{pmatrix} s-t \\ a-b \end{pmatrix} \right\|^n$$

for some $n \ge p + 2$ from which one can conclude that there is a $\mathcal{B}([t_0, \bar{t}]) \otimes \mathcal{B}(\mathbb{A}) \otimes \Sigma_w$-measurable version of x which is continuous on $[t_0, \bar{t}] \times \mathbb{A}$ for all $\omega_w \in \Omega_w$.

Random set parameters

The uncertainty of the parameter a in Equation (1) shall be modelled by a random compact set

$$A : \Omega_{\mathbb{A}} \to \mathcal{K}'(\mathbb{A})$$

where $(\Omega_{\mathbb{A}}, \Sigma_{\mathbb{A}}, P_{\mathbb{A}})$ is a probability space and $\mathcal{K}'(\mathbb{A})$ denotes the set of all non-empty compact subsets of $\mathbb{R}^p$ that are also subsets of $\mathbb{A}$ endowed with the Hausdorff metric. By definition for each $B \in \mathcal{B}(\mathbb{A})$ it holds that

$$A^-(B) = \{\omega_{\mathbb{A}} : A(\omega_{\mathbb{A}}) \cap B \ne \emptyset\} \in \Sigma_{\mathbb{A}}.$$

By $\mathcal{S}(A)$ we denote the set of measurable selections $\alpha : \Omega_{\mathbb{A}} \to \mathbb{A}$ of A which means that $\alpha(\omega_{\mathbb{A}}) \in A(\omega_{\mathbb{A}})$ for all $\omega_{\mathbb{A}} \in \Omega_{\mathbb{A}}$.

If x is the process (2) which is assumed to be measurable and continuous then for each $\alpha \in \mathcal{S}(A)$ the map

$$\begin{array}{rlcl} \xi^\alpha : & [t_0, \bar{t}] \times \Omega_{\mathbb{A}} \times \Omega_w & \to & \mathbb{R}^d \\ & (t, \omega_{\mathbb{A}}, \omega_w) & \mapsto & x(t, \alpha(\omega_{\mathbb{A}}), \omega_w) \end{array}$$

is a measurable and continuous process on $[t_0, \bar{t}]$ and the product space

$$(\Omega, \Sigma, P) = (\Omega_{\mathbb{A}} \times \Omega_w, \Sigma_{\mathbb{A}} \otimes \Sigma_w, P_{\mathbb{A}} \otimes P_w).$$

The set-valued solution process

Let us define a set-valued function X by

$$X : (t, \omega) \mapsto \{x_{t,a}(\omega_w) : a \in A(\omega_{\mathbb{A}})\} \qquad (3)$$

where $(t, \omega) \in [t_0, \bar{t}] \times \Omega$. By using the measurable selections ξ^α of X and applying the Fundamental Measurability Theorem for multifunctions one can show that

- X is a set-valued process on $[t_0, \bar{t}]$ and the completed probability space $(\Omega, \overline{\Sigma}^P, \overline{P})$ with values in $\mathcal{K}'(\mathbb{R}^d)$, i.e., for all $t \in [t_0, \bar{t}]$ and $B \in \mathcal{B}(\mathbb{R}^d)$ it holds that
$$X_t^-(B) = \{\omega : X_t(\omega) \cap B \ne \emptyset\} \in \overline{\Sigma}^P,$$
- all sample functions of X are continuous with respect to the Hausdorff-metric on $\mathcal{K}'(\mathbb{R}^d)$,
- X is measurable with respect to the product-σ-algebra $\mathcal{B}([t_0, \bar{t}]) \otimes \overline{\Sigma}^P$.

First entrance and inclusion times

First entrance times are often used to assess the reliability of a system described by a stochastic process.

Analogues for a continuous set-valued process $\{X_t\}_{t\in[t_0,\bar{t}]}$ with values in $\mathcal{K}'(\mathbb{R}^d)$ can be defined by

$$\begin{array}{l} \underline{\tau}^B : \Omega \to [t_0, \bar{t}], \quad \omega \mapsto \inf\{t : X_t(\omega) \cap B \ne \emptyset\}, \\ \overline{\tau}^B : \Omega \to [t_0, \bar{t}], \quad \omega \mapsto \inf\{t : X_t(\omega) \subseteq B\}. \end{array}$$

If the infimum does not exist, we set $\underline{\tau}^B(\omega) = \bar{t}$ or $\overline{\tau}^B(\omega) = \bar{t}$, respectively. We call $\underline{\tau}^B$ the first entrance time of X into B, and we call $\overline{\tau}^B$ the first inclusion time of X in B. Considering the natural filtration $\{\Sigma_t\}_{t\in[t_0,\bar{t}]}$ of X defined by

$$\Sigma_t = \sigma(X_s^-(C) : s \in [t_0, t], C \in \mathcal{B}(\mathbb{R}^d))$$

one can show that $\underline{\tau}^B$ and $\overline{\tau}^B$ are stopping times w.r.t. the standardized natural filtration (which is right-continuous and contains all subsets of measure-zero sets of Σ) if B is an open or a closed subset of $\mathbb{R}^d$.

Relations to selections

An interesting question is if $\underline{\tau}^B$ and $\overline{\tau}^B$ can be attained by first entrance times of selections of X. For $\xi \in \mathcal{S}(X)$ and $B \subseteq \mathbb{R}^d$ consider the first entrance time of ξ into B:

$$\tau_\xi^B : \Omega \to [t_0, \bar{t}], \quad \omega \mapsto \inf\{t : \xi_t(\omega) \in B\}$$

Then for all $\omega \in \Omega$ it holds that

$$\begin{array}{rcl} \inf\limits_{\xi\in\mathcal{S}(X)} \tau_\xi^B(\omega) & = & \underline{\tau}^B(\omega), \\ \sup\limits_{\xi\in\mathcal{S}(X)} \tau_\xi^B(\omega) & \le & \overline{\tau}^B(\omega). \end{array}$$

If (Ω, Σ, P) is complete and B is open then for all $\omega \in \Omega$ the second inequality becomes an equality.

For a set-valued process defined by (3) we can consider for each $\alpha \in \mathcal{S}(A)$ and $a \in \mathbb{A}$ the special entrance times

$$\begin{array}{rcl} \tau_\alpha^B : \omega & \mapsto & \inf\{t \in [t_0, \bar{t}] : x_{t,\alpha(\omega_{\mathbb{A}})}(\omega_w) \in B\}, \\ \tau_a^B : \omega_w & \mapsto & \inf\{t \in [t_0, \bar{t}] : x_{t,a}(\omega_w) \in B\}. \end{array}$$

where $B \subseteq \mathbb{R}^d$. Then for all $\omega \in \Omega$ it holds that

$$\begin{array}{l} \inf\limits_{a\in A(\omega_{\mathbb{A}})} \tau_a^B(\omega_w) = \inf\limits_{\alpha\in\mathcal{S}(A)} \tau_\alpha^B(\omega) = \inf\limits_{\xi\in\mathcal{S}(X)} \tau_\xi^B(\omega), \\ \sup\limits_{a\in A(\omega_{\mathbb{A}})} \tau_a^B(\omega_w) = \sup\limits_{\alpha\in\mathcal{S}(A)} \tau_\alpha^B(\omega) \le \sup\limits_{\xi\in\mathcal{S}(X)} \tau_\xi^B(\omega). \end{array}$$

Example

We consider the so-called Ornstein-Uhlenbeck process which is the solution of the Langevin equation

$$dx_t = -a_1 x_t dt + a_2 dw_t$$

with initial value $x_0 = 0$ ($d = m = 1, t_0 = 0$) and parameters $a_1 > 0$ and $a_2 \in \mathbb{R}$ whose uncertainty is modelled by a random set A with four focal elements which are listed in the table below together with their weights. Furthermore we consider a selection α of A.

i	A_i	α_i	P_i
1	$[1,3] \times [0.5,1.5]$	$(1.7, 1.1)$	$2/15$
2	$[1,3] \times [1,2]$	$(2.3, 1.5)$	$4/15$
3	$[2,4] \times [0.5,1.5]$	$(3.0, 0.9)$	$1/5$
4	$[2,4] \times [1,2]$	$(3.2, 1.4)$	$2/5$

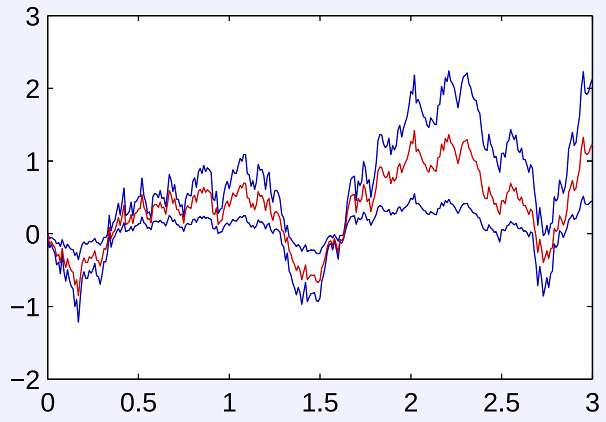

Figure 1: Sample path of X (boundaries in blue lines) and ξ^α (red line) on the time interval $[0, 3]$.

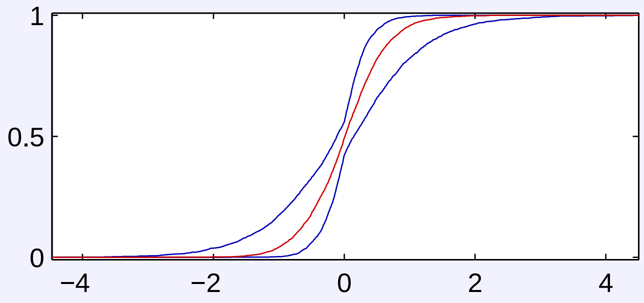

Figure 2: Probability box of X_t (blue lines) and CDF of ξ_t^α (red line) at time $t = 10$.

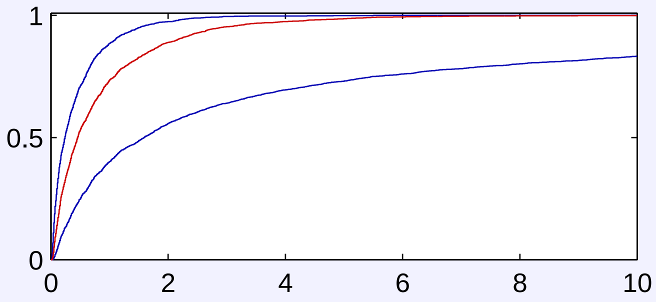

Figure 3: CDFs of $\underline{\tau}^B$ (upper blue line), $\overline{\tau}^B$ (lower blue line) and τ_α^B (red line), $B = (0.5, \infty)$.