Multi-criteria decision making with a special type of information about importance of groups of criteria

Lev V. Utkin
St.Petersburg Forest Technical Academy
lev.utkin@mail.ru

Abstract

An axiomatic approach for solving a multi-criteria decision making problem is studied in the paper, which generally allows reducing a set of Pareto optimal solutions. The information about criteria in the problem is represented as the decision maker judgments of a special type. The judgments have a clear behavior interpretation and can be used in various decision problems. It is shown in the paper how to combine the judgments and to use them for reducing the Pareto set when they are provided by several decision makers. Two global criteria of decision making are introduced for comparing of decision alternatives. The first criterion based on the lower expectation, the second one is based on determining the belief and plausibility functions in the framework of Dempster-Shafer theory and uses the "threshold" probability for the final decision making. The numerical examples illustrate the proposed approach.

1. The MCDM problem statement

THERE is a set of DA's $\mathbb{X} = \{X_1, ..., X_n\}$ consisting of n elements. Moreover, there is a set of criteria $\mathbb{C} = \{C_1, ..., C_r\}$ consisting of r elements, $r \geq 2$. For every DA, say the k-th DA, we can write the value of the i-th criterion $C_i(X_k)$ briefly denoted x_{ki}, $k = 1, ..., n$, $i = 1, ..., r$. The i-th DA is characterized by the vector $X_i = (x_{i1}, ..., x_{ir})$.

Definition 1 *$X \in \mathbb{X}$ dominates $Y \in \mathbb{X}$, denoted $X \succ Y$ iff $\forall i = 1, ..., r$, $x_i \geq y_i$ with at least one strict inequality.*

Definition 2 *$Y \in \mathbb{X}$ is a Pareto optimal alternative, also called an efficient alternative, iff $\nexists X \in \mathbb{X}$ such that $X \succ Y$. The set of all Pareto optimal alternatives in $\mathbb{X}$ or Pareto set is denoted $\mathcal{P}(\mathbb{X})$.*

2. Noghin's relative importance of criteria

Axiom 1 *The preference relation $\succ$ is invariant with respect to positive linear transformation.*

The main idea of Noghin's theory is to compare criteria by means of parameters.

Definition 3 *Let $i, j \in N = \{1, 2, ..., r\}$, $i \neq j$. We say that the i-th criterion is more important than the j-th criterion with two positive parameters w_i and w_j if for any two vectors $X, Y \in \mathbb{X}$ such that*

$$x_i > y_i,\ x_j < y_j,\ x_k = y_k,\ \forall k \in N \backslash \{i, j\},$$

$$x_i - y_i = w_i,\ x_j - y_j = -w_j,$$

the relationship $X \succ Y$ is valid.

The DM is willing to pay w_j units for the j-th criterion in order to get w_i units for the i-th criterion.
The relative importance coefficient is defined as

$$\theta_{ij} = \frac{w_j}{w_i + w_j}.$$

Introduce the following vector

$$W_{ij} = (0, ..., 0, w_i, 0, ..., -w_j, 0, ..., 0),$$

whose $r - 2$ elements are zero, the i-th element is w_i, the j-th element is $-w_j$.

Theorem 1 (Noghin [1]) *Let the i-th criterion be more important than the j-th criterion with the pair of positive parameters w_i and w_j. Then for any nonempty set of optimal vectors $\mathrm{Opt}\mathbb{X}$, it follows that*

$$\mathrm{Opt}\mathbb{X} \subseteq \mathcal{P}^*(\mathbb{X}) \subseteq \mathcal{P}(\mathbb{X}),$$

where $\mathcal{P}(\mathbb{X})$ is a set of Pareto-optimal vectors with respect to criteria $\mathbb{C} = \{C_1, ..., C_r\}$; $\mathcal{P}^*(\mathbb{X})$ is a set of Pareto-optimal vectors with respect to criteria $\mathbb{C}^* = \{C_1^*, ..., C_r^*\}$ such that

$$C_j^* = w_j C_i + w_i C_j,\ C_k^* = C_k,\ k \neq j.$$

3. Sets of desirable gambles

For $X, Y \in \mathcal{L}$, write $X \geq Y$ to mean that $X(\omega) \geq Y(\omega)$ for all $\omega \in \Omega$, and write $X > Y$ to mean $X \geq Y$ and $X(\omega) > Y(\omega)$ for some $\omega \in \Omega$. A set of desirable gambles [2], denoted by $\mathcal{D}$, is a subset of $\mathcal{L}$. A set of desirable gambles is said to be *coherent* when it satisfies the four axioms:

D1. $0 \notin \mathcal{D}$.
D2. if $X \in \mathcal{L}$ and $X > 0$, then $X \in \mathcal{D}$.
D3. if $X \in \mathcal{D}$ and $c \in \mathbb{R}_+$, then $cX \in \mathcal{D}$.
D4. if $X \in \mathcal{D}$ and $Y \in \mathcal{D}$, then $X + Y \in \mathcal{D}$.

Thus a coherent set of desirable gambles is a convex cone of gambles that contains all positive gambles ($X > 0$) but not the zero gamble. Consequence of the axioms: If $X \in \mathcal{D}$ and $Y \geq X$, then $Y \in \mathcal{D}$.

Walley states that there is a one-to-one correspondence between coherent sets of desirable gambles and coherent partial preference orderings, defined by $X \succ Y$ if and only if $X - Y \in \mathcal{D}$. This is very important statement which allows to find the same correspondence between the framework of desirable gambles and Noghin's theory.

$$\mathcal{D} = \{X \in \mathcal{L} : X > 0 \text{ or } \mathbb{E}_P(X) > 0, \forall P \in \mathcal{M}\}. \quad (1)$$

or

$$\mathcal{D} = \{X \in \mathcal{L} : X > 0 \text{ or } \underline{\mathbb{E}}_{\mathcal{M}}(X) > 0\}. \quad (2)$$

Then $\mathcal{D}$ is coherent and $\mathcal{M}$ can be recovered from it by

$$\mathcal{M} = \{P : \mathbb{E}_P(X) \geq 0,\ \forall X \in \mathcal{D}\}. \quad (3)$$

Now we reformulate Noghin's theorem and prove it in terms of desirable gambles.

Let X and Y be two DA's. We will denote below the vector $Z = X - Y$ and its components $z_k = x_k - y_k$ for short.

Theorem 2 *The preference $X \succ Y$ is valid if $X^* > Y^*$ and $W_{ij} \in \mathcal{D}$. Here $X^* = (x_1^*, ..., x_r^*)$ and $Y^* = (y_1^*, ..., y_r^*)$ such that*

$$x_j^* = w_j x_i + w_i x_j,\ x_k^* = x_k,\ k \neq j,$$
$$y_j^* = w_j y_i + w_i y_j,\ y_k^* = y_k,\ k \neq j.$$

4. Groups of the most important or preferable criteria

4.1 Simple comparison judgments

First we consider simple comparison judgments of the form: "I do not know which criterion is the most important, but this criterion belongs to the subset $B \subseteq \mathbb{C}$". Here the degree v is assumed to be unknown. Suppose that the unknown important criterion has the number k and the subset B contains t elements with numbers from the index set B^0. Denote $B^1 = N \backslash B^0$, $N = \{1, ..., r\}$. Then we can provide $r - t$ judgments:

"The k-th criterion is more important than the j-th criterion from $\mathbb{C} \backslash B$ with the pair of positive parameters $w_k = 1$ and $w_j = 1$".

Here $k \in B^0$ and $j \in B^1$. So, every judgment produces the gamble

$$W_{kj} = (0, ..., 0, 1_k, 0, ..., -1_j, 0, ..., 0), \quad (4)$$

such that $W_{kj} \succ 0_r$, $k \in B^0$, $j \in B^1$.

Proposition 1 *Given the additional information in the form (4), the preference $X \succ Y$ is valid if the condition*

$$z_k + \sum_{j \in L} z_j \geq 0$$

is valid for all $L \subseteq B^1$ and $z_i \geq 0$ for all $i \in B^0$.

The lower expectation of the gamble $Z = X - Y$ under conditions $W_{kj} \succ 0_r$, $j \in D_l^1$, denoted $\underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(Z)$ is of the form

$$\underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(Z) = \min_{L \subseteq D_l^1} \mathbb{E}_P(Z)$$
$$= \min_{L \subseteq D_l^1} \frac{1}{q_L + 1} \left(z_k + \sum_{j \in L} z_j \right). \quad (5)$$

Here L is a subset of D_l^1; q_L is the number of elements in L ($q_L = \mathrm{card}(L)$).

4.1.1 The first global criterion

The preference $X \succ Y$ is valid if $\underline{\mathbb{E}}_{\mathcal{P}} \mathbb{E}_P (X - Y) > 0$. Here $\mathcal{P}$ is a set of probability distributions defined on the partition of $\mathcal{M}$ produced by the given information in the form of preferences $D_l \succ \mathbb{C}$ with BPA's $m(D_l)$, $l = 1, ..., t$.

Proposition 2 *Suppose that there is a set of t judgments of the form $D_l \succ \mathbb{C}$ with the corresponding BPA's $m(D_l)$, $l = 1, ..., t$. The preference $X \succ Y$ is valid in accordance with the first global criterion if the condition*

$$\underline{\mathbb{E}}_{\mathcal{P}} \mathbb{E}_P (X - Y) = \sum_{l=1}^{t} m(D_l) \min_{k \in D_l^0} \underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(X - Y) \geq 0$$

is valid. Here $\underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(X - Y)$ is defined from (5).

4.1.2 The second global criterion

According to this criterion, we can say about the preference $X \succ Y$ with some "threshold" or confident probability which lies between the belief and plausibility functions.

Proposition 3 *Suppose that there is a set of t judgments of the form $D_l \succ \mathbb{C}$ with the corresponding BPA's $m(D_l)$, $l = 1, ..., t$. The preference $X \succ Y$ is valid in accordance with the second global criterion with a probability belonging to the interval with the following bounds*

$$\mathrm{Bel}(X - Y \in \mathcal{D}) = \sum_{l \in R} m(D_l),$$

$$\mathrm{Pl}(X - Y \in \mathcal{D}) = \sum_{l \in G} m(D_l),$$

where R is a set of indices such that for every $l \in R$, there holds

$$\min_{k \in D_l^0} \underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(X - Y) > 0,$$

G is a set of indices such that for every $l \in G$, there holds

$$\max_{k \in D_l^0} \underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(X - Y) > 0.$$

Here $\underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(X - Y)$ is defined from (5).

4.2 General case

"The k-th criterion from B is more important than the j-th criterion from $\mathbb{C} \backslash B$ with the pair of positive parameters w_k and w_j".

Assume for example that $\mathbb{C} = \{C_1, C_2, C_3\}$, $B = \{C_1, C_2\}$, and $\mathbb{C} \backslash B = \{C_3\}$. Then the corresponding judgment of a DM might also have the form: "I'm willing to pay w_3 for C_3 in order to get w_1 for C_1. I'm also willing to pay w_3 for C_3 in order to get w_2 for C_2. However, I do not know what is better. "

Proposition 4 *Given the additional information in the form W_{ij}, the preference $X \succ Y$ is valid if the condition*

$$z_k + \sum_{j \in L} \frac{w_k}{w_j} z_j \geq 0$$

is valid for all $L \subseteq B^1$ and $z_i \geq 0$ for all $i \in B^0$.

Corollary 1 *Suppose that $B^1 = \{k\}$ and $B^0 = \{j\}$. Then the preference $X \succ Y$ is valid if the conditions*

$$w_j z_k + w_k z_j \geq 0,\ z_i \geq 0,\ \forall i \neq j,$$

are valid.

Corollary 2 *If there are judgments of one DM ($l = 1$, $D_1 = D$) with the BPA $m(D_l) = 1$, then the preference $X \succ Y$ is valid in accordance with the first global criterion if the conditions*

$$\min_{k \in D^0} \{z_k + T w_k\} \geq 0$$

are valid. Here

$$T = \min_{L \subseteq D^1} \sum_{j \in L} z_j w_j^{-1}.$$

References

[1] V.D. Noghin. Relative importance of criteria: a quantitative approach. *Journal of Multi-Criteria Decision Analysis*, 6:355–363, 1997.

[2] P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.