On the Behavior of the Robust Bayesian Combination Operator and the Significance of Discounting

Alexander Karlsson, Ronnie Johansson, and Sten F. Andler

Informatics Research Centre
University of Skövde
Sweden
{alexander.karlsson, ronnie.johansson, sten.f.andler}@his.se

Abstract

We study the combination problem for credal sets via the robust Bayesian combination operator. We extend Walley's notion of degree of imprecision and introduce a measure for degree of conflict between two credal sets. We propose a new discounting operator to be used with the combination operator whenever intervals of reliability weights for the sources are available. We show that the result of the operators can be computed by utilizing the extreme points of the operand sets.

1. The Robust Bayesian Combination Operator

Definition 1. *The Robust Bayesian Combination (RBC) Operator [1, 2] :*

$$\mathcal{P}^1_X \otimes_{\mathcal{R}} \mathcal{P}^2_X \triangleq$$

$$CH\left\{\frac{p_i(X)p_j(X)}{\sum\limits_{x\in\Omega_X} p_i(x)p_j(x)} :\ p_i \in \mathcal{P}^1_X,\ p_j \in \mathcal{P}^2_X\right\},$$

where CH denotes the convex hull, the credal sets $\mathcal{P}^1_X$ and $\mathcal{P}^2_X$ are interpreted as strongly conditionally independent evidences, i.e., convex sets of normalized likelihoods that are strongly conditionally independent given X. The operator is undefined if there exists $p_i \in \mathcal{P}^1_X$ and $p_j \in \mathcal{P}^2_X$ such that $\sum_{x\in\Omega_X} p_i(x)p_j(x) = 0$.

Theorem 1.

$$\mathcal{P}^1_X \otimes_{\mathcal{R}} \mathcal{P}^2_X = ext(\mathcal{P}^1_X) \otimes_{\mathcal{R}} ext(\mathcal{P}^2_X),$$

where ext denotes the set of extreme points.

Proof. See paper. □

2. Imprecision and Conflict

Definition 2. *Degree of Imprecision:*

$$\mathcal{I}(\mathcal{P}_X) \triangleq \frac{1}{n}\sum_{x\in\Omega_X} \Delta(x)$$

where $\mathcal{P}_X \subseteq \mathbb{R}^n$, $n = |\Omega_X|$, and where $\Delta(x)$ is Walley's measure of degree of imprecision for a single event $x \in \Omega_X$:

$$\Delta(x) \triangleq \max_{p\in\mathcal{P}_X} p(x) - \min_{p\in\mathcal{P}_X} p(x)$$

Definition 3. *Degree of Conflict:*

$$\mathcal{K}(\mathcal{P}^1_X, \mathcal{P}^2_X) \triangleq \frac{\mathcal{H}(\mathcal{P}^1_X, \mathcal{P}^2_X)}{\sqrt{2}},$$

where the denominator is a constant corresponding to the diameter of the credal set containing all probability distributions for the variable X, and where $\mathcal{H}(\mathcal{P}^1_X, \mathcal{P}^2_X)$ is the Hausdorff distance defined as:

$$\mathcal{H}(\mathcal{P}^1_X, \mathcal{P}^2_X) \triangleq \max\left\{\overrightarrow{\mathcal{H}}(\mathcal{P}^1_X, \mathcal{P}^2_X), \overrightarrow{\mathcal{H}}(\mathcal{P}^2_X, \mathcal{P}^1_X)\right\},$$

where $\overrightarrow{\mathcal{H}}$ is the forward Hausdorff distance defined by:

$$\overrightarrow{\mathcal{H}}(\mathcal{F}_1, \mathcal{F}_2) \triangleq \max_{f_i\in\mathcal{F}_1}\left\{\min_{f_j\in\mathcal{F}_2} d(f_i, f_j)\right\},$$

where d denotes the Euclidean distance and $\mathcal{F}_1$ and $\mathcal{F}_2$ are general closed convex sets in $\mathbb{R}^n$.

3. The RBC Discounting Operator

Assume that the reliability of a source corresponds to an interval of reliability weigths $\mathcal{W}$. The source can then be discounted by utilizing *the RBC discounting operator*:

Definition 4. *The RBC Discounting Operator:*

$$\mathcal{D}(\mathcal{P}_X, \mathcal{W}) \triangleq CH\left\{wp + (1-w)p_u : w \in \mathcal{W}, p \in \mathcal{P}_X\right\},$$

where $\mathcal{P}_X \subseteq \mathbb{R}^n$, $\mathcal{W} \subseteq [0,1]$ is an interval of reliability weights, and $p_u \in \mathbb{R}^n$, $n = |\Omega_X|$, is the uniform distribution over Ω_X.

The RBC discounting operator collapses a credal set "towards" the uniform distribution.

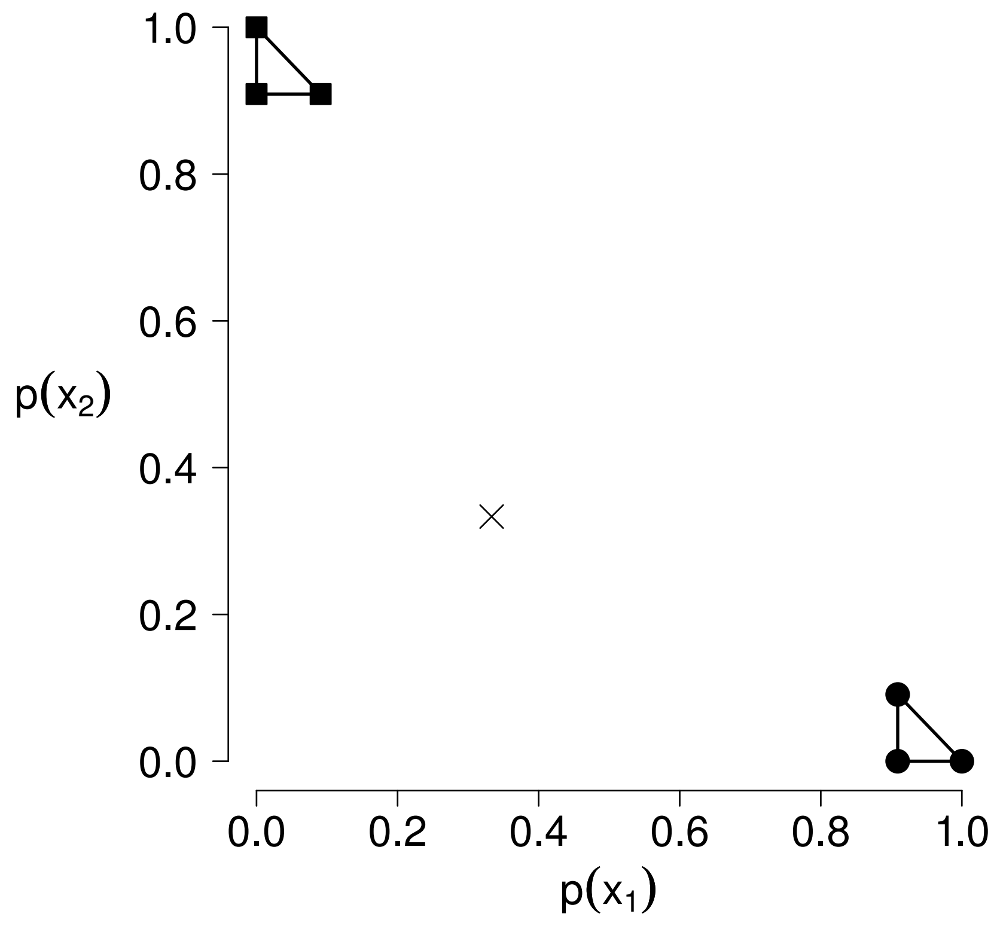
Theorem 2.

$$\mathcal{D}(\mathcal{P}_X, \mathcal{W}) = \mathcal{D}(ext\,(\mathcal{P}_X)\,, ext\,(\mathcal{W}))$$

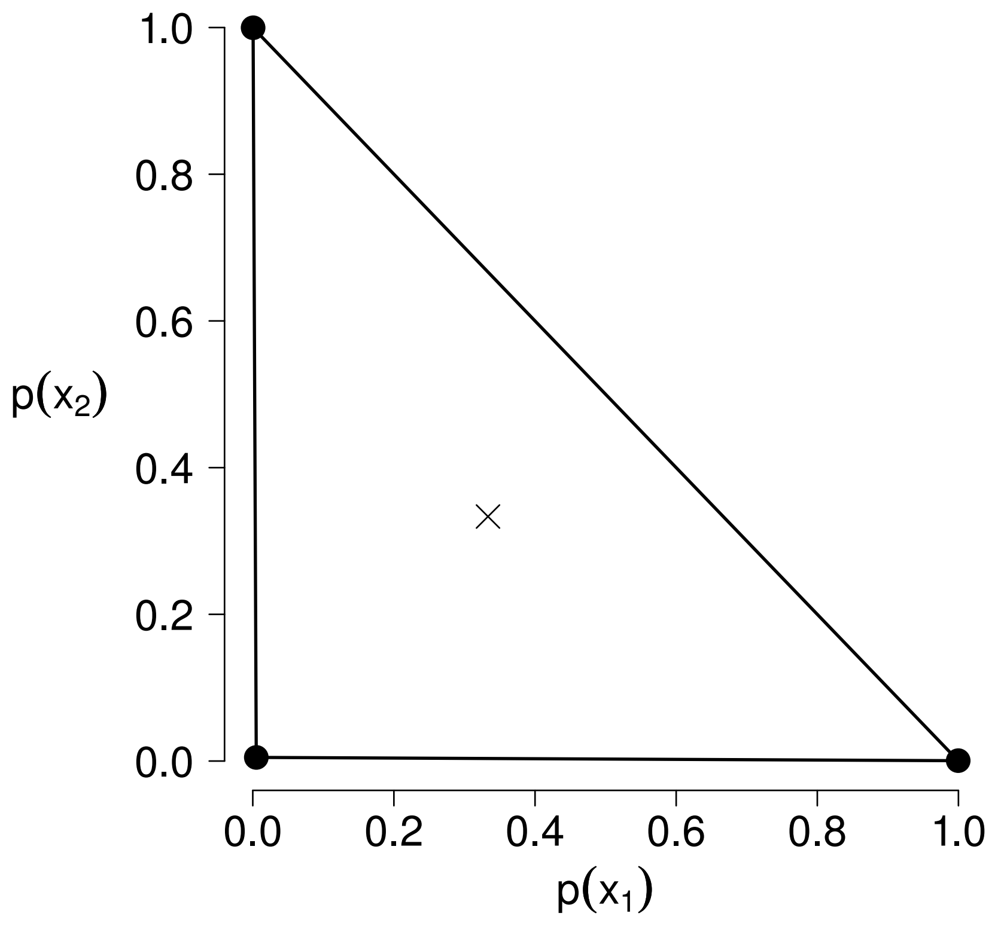
Proof. See paper. □

4. Example

Consider Fig. 1, where the result of combining the operands in Fig. 1(a), utilizing the RBC operator, is shown in Fig. 1(b).



(a) $\mathcal{P}^1_X$ (circles) and $\mathcal{P}^2_X$ (squares)



(b) $\mathcal{P}^{1,2}_X$

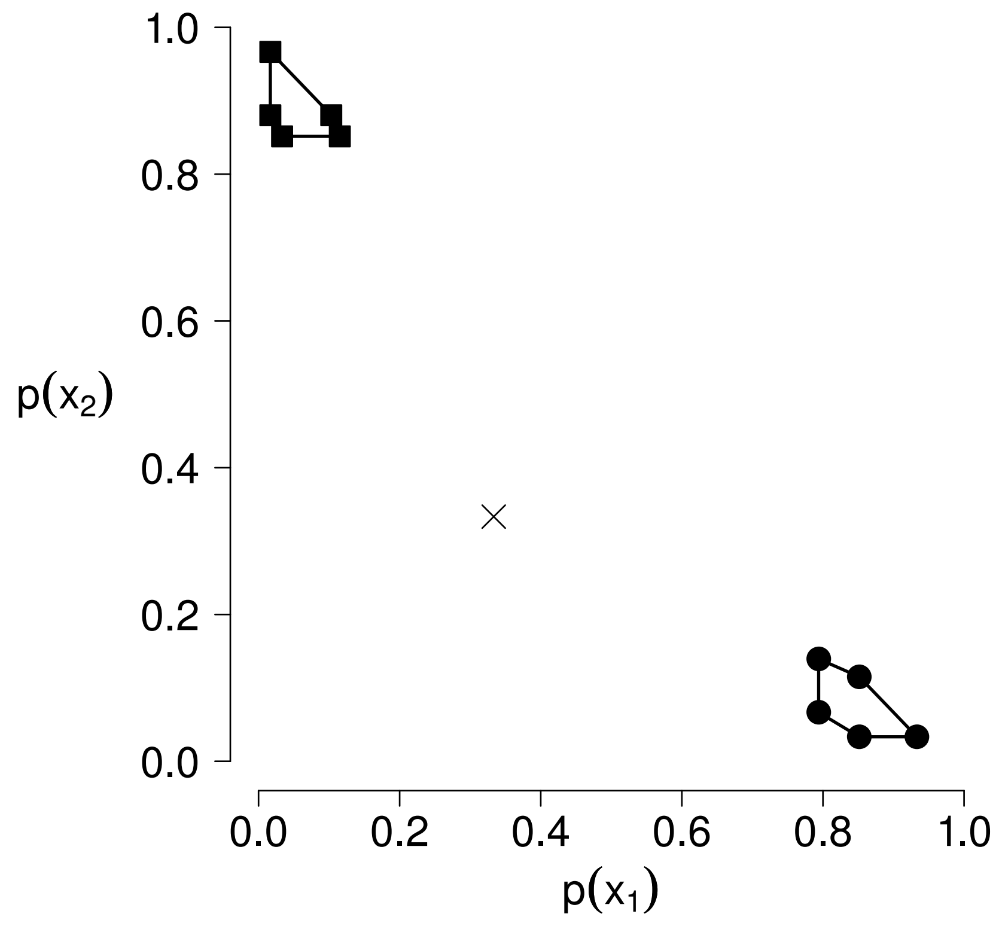
Figure 1: $\mathcal{P}^i_X$, $i \in \{1,2\}$, and $\mathcal{P}^{1,2}_X$

We see that there is a high degree of conflict, $\mathcal{K}(\mathcal{P}^1_X, \mathcal{P}^2_X) \approx 0.91$ and that the result $\mathcal{P}^{1,2}_X$ has a high degree of imprecision, $\mathcal{I}(\mathcal{P}^{1,2}_X) \approx 1$.

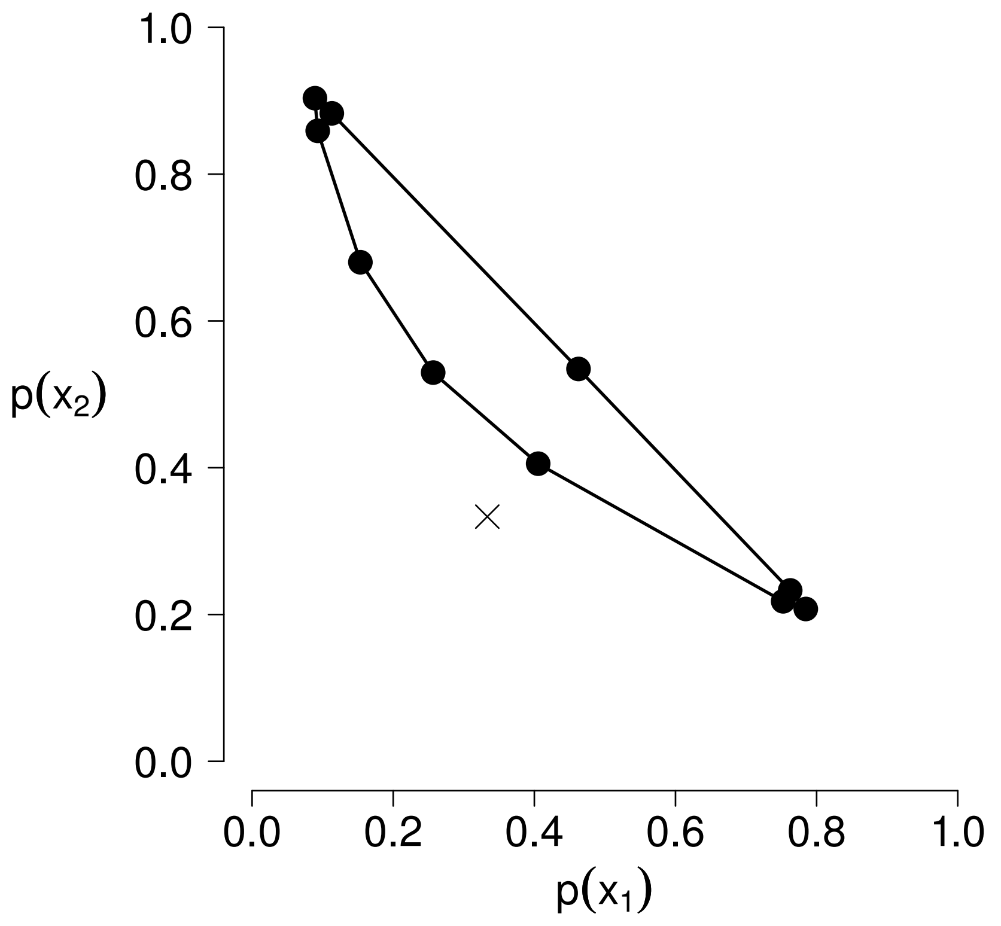
Now, assume that the following intervals of reliability weights regarding the sources are available:

$$\mathcal{W}_1 = [0.80, 0.90]$$
$$\mathcal{W}_2 = [0.90, 0.95]$$

The result of applying the RBC discounting operator on the operands in Fig. 1(a), utilizing the above set of reliability weights, is seen in Fig. 2, where we denote the resulting credal set as $\mathcal{P}^{1_d,2_d}_X$. We get $\mathcal{I}(\mathcal{P}^{1_d,2_d}_X) \approx 0.53$, hence, a significant difference compared to the non-discounted case in Fig. 1.



(a) $\mathcal{D}\left(\mathcal{P}^1_X, \mathcal{W}_1\right)$ (circles) and $\mathcal{D}\left(\mathcal{P}^2_X, \mathcal{W}_2\right)$ (squares)



(b) $\mathcal{P}^{1_d,2_d}_X$

Figure 2: *$\mathcal{D}\left(\mathcal{P}^i_X, \mathcal{W}_i\right)$, $i \in \{1,2\}$, where $\mathcal{P}^i_X$ are the sets shown in Fig. 1(a), and $\mathcal{P}^{1_d,2_d}_X$*

References

[1] Stefan Arnborg. Robust Bayesianism: Imprecise and paradoxical reasoning. In *Proceedings of the 7th International Conference on Information fusion*, 2004.

[2] Stefan Arnborg. Robust Bayesianism: Relation to evidence theory. *Journal of Advances in Information Fusion*, 1(1):63–74, 2006.

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