Sets of Desirable Gambles and Credal Sets

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Gambles

- We have an uncertain taking values on a finite set $\Omega$
- A gamble is a mapping $X : \Omega \rightarrow \mathbb{R}$
- $X(\omega)$ is the reward if $X = \omega$
- Some gambles are clearly desirable for us (for example if $X(\omega) > 0$, $\forall \omega$) and other are undesirable (for example if $X(\omega) < 0$, $\forall \omega$)

Example

- Consider the result of football match with $\Omega = \{0 - 0, 1 - 0, 0 - 1, 1 - 1, 2 - 0, \ldots, 15 - 15\}$
- A gamble $X_1(1 - 0) = 10$, $X_1(r) = -1$, otherwise
- Another example could be $X_2(i - j) = 1$, if $i > j$, $X_2(i - j) = -1$, if $i < j$ and 0 otherwise.
- If we believe in 'draw' we could accept: $X_3(i - i) = 1$, $X_3(i - j) = -1$, $i \neq j$
Coherent Set of Desirable Gambles

D1. $0 \not\in D$,
D2. if $X \in \mathcal{L}$ and $X > 0$ then $X \in D$,
D3. if $X \in D$ and $c \in \mathbb{R}^+$ then $cX \in D$,
D4. if $X \in D$ and $Y \in D$ then $X + Y \in D$.

Basic Consistency Condition
A set of desirable gambles $D$ avoids partial loss if and only if $0 \not\in D$.

We should not accept: $f(i - j) = -1$ if $i > j$ and 0 otherwise.

Closed Set of Gambles
A set of desirable gambles $D$ is closed if D2, D3, and D4 are verified.

Almost Desirable Gambles
D1’ $\forall X \in D^*$, we have $\sup X \geq 0$
D2 If $X > 0$, then $X \in D^*$
D3 If $X \in D^*$ and $\lambda > 0$ then $\lambda X \in D^*$
D4 If $X_1, X_2 \in D^*$ then $X_1 + X_2 \in D^*$
D5 If $X + \epsilon \in D^*$, $\forall \epsilon > 0$ then $X \in D^*$

Basic Consistency Condition
A set of almost desirable gambles $D^*$ avoids sure loss if and only if $\forall X \in D$ such that $\sup X \geq 0$.

Desirable vs Almost Desirable Gambles
Desirable Gambles

$D_1$  $D_2$  $D_3$  $D_4$  $D_5$

Almost Desirable Gambles

Desirable Gambles are a more general model.
Desirable vs. almost desirable gambles

Let us consider the gambles:
\[ X_\epsilon(i - j) = \epsilon \text{ if } i - j \neq 15 - 15, \quad X_\epsilon(15 - 15) = -1 \]

- It is possible that all the gambles \( X_\epsilon \) are desirable.
- If they are almost desirable, then the gamble:
  \[ X_0(i - j) = 0 \text{ if } i - j \neq 15 - 15, \quad X_0(15 - 15) = -1 \]
  is almost desirable.
- Almost desirable gambles avoids uniform loss, but not partial loss.

Strictly Desirable Gambles

D2 If \( X > 0 \), then \( \mathcal{D} \)
D3 If \( X \in \mathcal{D} \) and \( \lambda > 0 \) then \( \lambda X \in \mathcal{D} \)
D4 If \( X_1, X_2 \in \mathcal{D} \) then \( X_1 + X_2 \in \mathcal{D} \)
D5' If \( X \in \mathcal{D} \) then either \( X > 0 \) or \( \exists \epsilon > 0, \ X - \epsilon \in \mathcal{D} \)

Basic Consistency Condition: A set of desirable gambles \( \mathcal{D} \) avoids partial loss (\( 0 \notin \mathcal{D} \))

Upper and Lower Previsions and Desirable Gambles

- The lower prevision of gamble \( X \) is
  \[ \underline{P}(X) = \sup\{\alpha : X - \alpha \in \mathcal{D}\} \]
  The supremum of the buying prices.
- The upper prevision of gamble \( X \) is
  \[ \overline{P}(X) = \inf\{\alpha : -X + \alpha \in \mathcal{D}\} \]
  The infimum of the selling prices.

Credal Sets and Desirable Gambles

- A set of desirable gambles \( \mathcal{D} \) defines a credal set:
  \[ \mathcal{P}_D = \{P : P[X] \geq 0, \forall X \in \mathcal{D}\} \]
- A set of desirable gambles \( \mathcal{D} \) and the set of almost desirable gambles \( \mathcal{D}^* \) define the same credal set
- A credal set \( \mathcal{P} \) defines a set of almost desirable gambles:
  \[ \mathcal{D}_P^* = \{X : P[X] \geq 0, \forall P \in \mathcal{P}\} \]
- But several sets of desirable gambles can be associated:
  \[ \mathcal{D}_P = \{X : P[X] > 0, \forall P \in \mathcal{P}\} \cup \{X : X > 0\} \]
  \[ \mathcal{D}'' = \{X : P[X] \geq 0, \forall P \in \mathcal{P}, \exists P \in \mathcal{P} P[X] > 0\} \cup \{X : X > 0\} \]
Graphical Representation: Credal Set

\[ E_P[X] \geq 0, \forall P \in \mathcal{P} \]

\[ \Omega = \{\omega_1, \omega_2, \omega_3\} \]

Desirable Gamble and Strictly Desirable  
Almost Desirable Gamble, Desirable?  
Non Desirable Gamble

Conditioning

If we have a set of desirable gambles \( \mathcal{D} \) and we observe event \( B \), the conditional set of desirable gambles given \( B \) is given by:

\[ \mathcal{D}_B = \{X : X.I_B \in \mathcal{D}\} \cup \{X : X > 0\} \]

Example

I we accept a gamble \( X(\text{Win}) = 1, \ X(\text{Loss}) = -1, \ X(\text{Draw}) = 0, \) if we know that Draw has not happened, then we should accept any gamble: \( Y(\text{Win}) = 1, \ Y(\text{Loss}) = -1, \ Y(\text{Draw}) = \alpha \)

In fact, all the conditional information is in \( \mathcal{D} \).
If $P(B) > 0$, then the credal set associated to the conditional set $\mathcal{D}$ is uniquely determined with independence of what happens with gambles in the frontier.

Conditioning: Lower Probability equal to 0

If $P(B) = 0$, all the gambles with $X(D) = 0.0$ are in the frontier. The credal set does not contain information about the conditioning.
Conditioning: Lower Probability equal to 0

\[ B = \{ \text{Win, Loss} \} \]

This situation is compatible with accepting as desirable the gambles:

\[
\begin{align*}
X(D) &= 1, & X(W) &= -1, & X(L) &= -1 \\
Y(D) &= 0, & Y(W) &= 1.2, & Y(L) &= -1 \\
Z(D) &= 0, & Z(W) &= -1, & Z(L) &= 1.2
\end{align*}
\]

But it is also compatible with gambles \( \{X, Y + \epsilon, Z + \epsilon\} \).
In this case, the conditioning is very wide: natural extension.

The case \( \overline{P}(B) = 0 \)

- Imagine that we have \( \omega_1 = \text{'There are less than 30 goals'}; \omega_2 = \text{'Win or Draw with 30 goals or more in total'}; \omega_3 = \text{'Loss with 30 goals or more in total'} \).
- It is possible that we accept any gamble with
  \[
  X(\omega_1) = \epsilon, \quad X(\omega_2) = -1, \quad X(\omega_3) = -1
  \]
- If \( B = \{\omega_2, \omega_3\}, \overline{P}(B) = P(B) = 0 \).
- The conditioning will depend of which gambles
  \[
  g(\omega_1) = 0, \quad g(\omega_2) = \alpha_1, \quad g(\omega_3) = \alpha_2
  \]
I have an urn with *Red*, *Blue*, *White* balls.
I know that there is exactly the same number of Blue and White balls.
This situation can be represented by the convex set of probability distributions:

<table>
<thead>
<tr>
<th></th>
<th>Red</th>
<th>Blue</th>
<th>White</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$P_2$</td>
<td>0</td>
<td>0.5</td>
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</tbody>
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If the set of desirable gambles is:

$$ D' = \{ X : E_P[X] > 0, \forall P \in \mathcal{P} \} $$

then, if we know that a ball randomly selected from the urn is not red, then conditional to this information, the gamble $X(Blue) = 2$, $X(White) = -1$ is not accepted.

This does not seem reasonable. I should accept any gamble in which $X(Blue) + X(White) > 0$. 

Regular Extension

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Natural Extension
**Regular Extension**

Desirable gambles, regular extension is obtained assuming if \( P(B) > 0 \) or:

\[
X \in D^* \text{ and } -X \notin D^* \Rightarrow X \in D.
\]

**Natural Extension - Encoding sets of gambles**

If \( \mathcal{F} \) is a set of gambles, its natural extension \( \overline{\mathcal{F}} \) is the set of gambles obtained from \( \mathcal{F} \) applying axioms A2, A3, and A4 (the minimum set of gambles containing \( \mathcal{F} \) and verifying these axioms.

**Finitely Generated Sets of Gambles**

A set of almost desirable gambles \( D \) is **finitely generated** if \( D = D_0 \) where \( D_0 \) is finite.

This definition is not appropriate for desirable gambles. We could not represent \( P(B) = 0 \). Which is equivalent to the acceptance of gambles \( \epsilon.l_B = -l_B \) for any \( \epsilon \).
Basic Reasoning Tasks

1. to determine whether the natural extension \( \overline{F} \) is coherent (i.e. \( 0 \notin \overline{F} \)),
2. given \( X \), to determine whether \( X \in F \),
3. given \( X \) and \( B \subseteq \Omega \), to compute \( P(X|B) \) and \( \overline{P}(X|B) \) under \( \overline{F} \) when this set is coherent.

**Theorem**

If \( F \) is an arbitrary set of gambles such that \( \overline{F} \) is coherent, then \( X \in F \) if and only if \( F \cup \{-X\} \) is not coherent.

\( \epsilon \)-set representation

A basic set of gambles is a set of gambles \( F_{X,B} = \{ X + \epsilon B : \epsilon > 0 \} \), where \( X \) is an arbitrary gamble and \( B \subseteq \Omega \), denoted as \((X, B)\).

\( \epsilon \)-set representation: \( F \) the union of: \((X_1, B_1), \ldots, (X_k, B_k)\)

**Representation of Conditional Probabilities**

\[ P(X|B) = c \] is represented by means of \((X - c)B, B)\)

\[ \overline{P}(X|B) = c \] is represented by means of \((c - X)B, B)\)

**Checking Consistency**

\( F \) generated by \((X_1, B_1), \ldots, (X_k, B_k)\): system in \( \lambda_i \) and \( \epsilon \) has no solution:

\[
\sum_{i=1}^{k} \lambda_i (X_i + \epsilon B_i) \leq 0 \\
\lambda_i \geq 0, \quad \epsilon > 0
\]


1. Set \( I = \{1, \ldots, k\} \)
2. Solve

\[
\text{sup} \sum_{i=1}^{k} \tau_i \\
\text{s.t.} \quad \sum_{i}(\lambda_i X_i + \tau_i B_i) \leq 0 \\
\lambda_i \geq 0, \quad 0 \leq \tau_i \leq 1
\]
3. Let \( I' = \{i | \tau_i = 1\} \) in the optimal solution
4. If \( I' = \emptyset \), then Return(Consistency)
5. If \( I' = I \neq \emptyset \) then Return(Nonconsistency)
6. else \( I = I' \) and goto 2

To compute \( P(X|B) \)

\[
\text{sup} \alpha \\
\text{s.t.} \\
\sum_{i=1}^{k} \lambda_i (X_i + \epsilon B_i) \leq (X - \alpha)B \\
\epsilon > 0, \lambda_i \geq 0
\]
Maximal Sets of Gambles

**Definition**

We will say that a set of gambles $\mathcal{D}$ is *maximal* if it is coherent and there does not exist any $X \notin \mathcal{D}$ such that $\mathcal{D} \cup \{X\}$ is coherent.

**Lemma**

If $\mathcal{D}$ is coherent and $-X \notin \mathcal{D}$, $X \neq 0$, then $\mathcal{D} \cup \{X\}$ is coherent.

**Theorem**

A coherent set of gambles $\mathcal{D}$ is maximal if and only if $X \in \mathcal{D}$ xor $-X \in \mathcal{D}$, for all $X \in \mathcal{L}$, $X \neq 0$.

**Lemma**

Let $\mathcal{D}$ be a maximal set of gambles and let $P$ and $\overline{P}$ be respectively the lower and the upper previsions associated to it. Then $P(B) = \overline{P}(B)$, $\forall B \subseteq \Omega$.

**Definition**

If we have a sequence of nested sets $
\Omega = C_0 \supset C_1 \supset \cdots \supset C_n = \emptyset$, and $B \subseteq \Omega$, then the *layer* of $B$ with respect to this sequence, will be the minimum value of $i$ such that $B \cap (C_i \setminus C_{i+1}) \neq \emptyset$. It will be denoted by $\text{layer}(B)$.

**Theorem**

If $\mathcal{D}$ is maximal then there is a sequence of nested sets $
\Omega = C_0 \supset C_1 \supset \cdots \supset C_n = \emptyset$ and a sequence of probability measures $P_0, \ldots, P_{n-1}$ satisfying the following conditions:

1. for each probability $P_i$, $P_i(C_i \setminus C_{i+1}) = 1$, $P_i(\omega) > 0$ for any $\omega \in C_i \setminus C_{i+1}$,
2. for each $A \subseteq B \subseteq \Omega$, if $i = \text{layer}(B)$, then $P(A|B) = \overline{P}(A|B) = P_i(A|B)$, where $P(A|B)$ and $\overline{P}(A|B)$ are the lower and upper probabilities computed from $\mathcal{D}_B$.

Coletti and Scozzafava (2002)
Theorem

There exists at least one maximal set of gambles containing a coherent set.

Theorem

If $\mathcal{D}$ is coherent, then $\mathcal{D} = \bigcap_{i \in I} \mathcal{D}_i$, where $\mathcal{D}_i$ are maximal coherent gambles containing $\mathcal{D}$.

Correspondence (Sequences of probabilities $\leftrightarrow$ Maximal coherent sets) non one-to-one

$\Omega = \{\omega_1, \omega_2\}$ and $P_0(\omega_1) = P_0(\omega_2) = 0.5$.

Any gamble with $X(\omega_1) + X(\omega_2) > 0$ is desirable.

Given $Y(\omega_1) = 1, Y(\omega_2) = -1$.

We can have $Y$ desirable xor $-Y$ desirable.

Alternative model one-to-one:

D1”. If $X \in \mathcal{D}$, then there is $\epsilon > 0$, such that $-X + \epsilon \text{supp}(X) \notin \mathcal{D}$.

More Work

- More general representation schemes?
- Algorithms for them?
- Local computation
- Independence and local computation