On solutions of stochastic differential equations with parameters modelled by random sets

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Stochastic differential equations

$$dx_t = f(t, x_t)dt + G(t, x_t)dw_t$$

with

- $t_0 \leq t \leq \overline{t} < \infty$,
- $\{w_t\}_{t \geq t_0}$ being an m-dimensional Wiener process on a probability space $(\Omega_w, \Sigma_w, P_w)$,
- initial value x_{t_0} and coefficients f and G:

$$\begin{aligned} x_{t_0} &: \quad \Omega_w \to \mathbb{R}^d, & \omega_w &\mapsto x_{t_0}(\omega_w), \\ f &: \quad [t_0, \overline{t}] \times \mathbb{R}^d \to \mathbb{R}^d, & (t, x) &\mapsto f(t, x), \\ G &: \quad [t_0, \overline{t}] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}, & (t, x) &\mapsto G(t, x). \end{aligned}$$

$$dx_{t,a} = f(t, a, x_{t,a})dt + G(t, a, x_{t,a})dw_t$$

with uncertain parameters $a \in \mathbb{A} \subseteq \mathbb{R}^p$ and

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$$\begin{aligned} x_{t_0}: &\quad \mathbb{A} \times \Omega_w \to \mathbb{R}^d, & (a, \omega_w) &\mapsto x_{t_0,a}(\omega_w), \\ f: &\quad [t_0, \overline{t}] \times \mathbb{A} \times \mathbb{R}^d \to \mathbb{R}^d, & (t, a, x) &\mapsto f(t, a, x), \\ G: &\quad [t_0, \overline{t}] \times \mathbb{A} \times \mathbb{R}^d \to \mathbb{R}^{d \times m}, & (t, a, x) &\mapsto G(t, a, x). \end{aligned}$$

Solution processes

- Assume that for each $a \in \mathbb{A}$ the conditions for the existence of a unique solution $\{x_{t,a}\}_{t \in [t_0, \overline{t}]}$ are fulfilled.

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$$x : [t_0, \overline{t}] \times \mathbb{A} \times \Omega_w \longrightarrow \mathbb{R}^d, (t, a, \omega_w) \mapsto x_{t,a}(\omega_w)$$

which is a stochastic process on $[t_0, \overline{t}] \times \mathbb{A}$ and $(\Omega_w, \Sigma_w, P_w)$.

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  which is a stochastic process on $[t_0, \overline{t}] \times \mathbb{A}$ and $(\Omega_w, \Sigma_w, P_w)$.
- Under certain conditions it can be shown that x has a version
  – whose sample paths are continuous on $[t_0, \overline{t}] \times \mathbb{A}$,
  – which is $\mathcal{B}([t_0, \overline{t}]) \otimes \mathcal{B}(\mathbb{A}) \otimes \Sigma_w$-measurable.

Parameter uncertainty

- To model the uncertainty of a we use a random compact set

$$A : \Omega_{\mathbb{A}} \to \mathcal{K}'(\mathbb{A})$$

– on a probability space $(\Omega_{\mathbb{A}}, \Sigma_{\mathbb{A}}, P_{\mathbb{A}})$

– with non-empty compact values being subsets of $\mathbb{A}$

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- Defining measurability condition: for all $B \in \mathcal{B}(\mathbb{R}^d)$ it holds that

$$A^{-}(B) = \{\omega_{\mathbb{A}} : A(\omega_{\mathbb{A}}) \cap B \neq \emptyset\} \in \Sigma_{\mathbb{A}}.$$

Set-valued solution process

- Define the set-valued map

$$X : (t, \omega) \mapsto \{x_{t,a}(\omega_w) : a \in A(\omega_{\mathbb{A}})\}$$

where $(t, \omega) \in [t_0, \overline{t}] \times \Omega$ and Ω denotes the product space

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- **Is this a set-valued process (measurability)?**
- **What are its properties?**

Set-valued solution process

- X is a set-valued process on $[t_0, \overline{t}]$ and the completed probability space $(\Omega, \overline{\Sigma}^P, \overline{P})$ with values in $\mathcal{K}'(\mathbb{R}^d)$, i.e., for all $t \in [t_0, \overline{t}]$ and $B \in \mathcal{B}(\mathbb{R}^d)$ it holds that

$$X_t^-(B) = \{\omega : X_t(\omega) \cap B \neq \emptyset\} \in \overline{\Sigma}^P,$$

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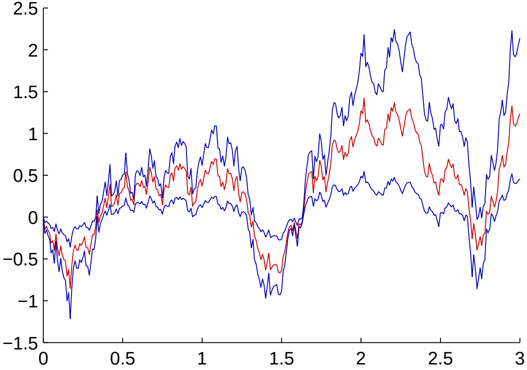
- all sample functions of X are continuous with respect to the Hausdorff-metric on $\mathcal{K}'(\mathbb{R}^d)$,
- X is measurable with respect to the product-σ-algebra $\mathcal{B}([t_0, \overline{t}]) \otimes \overline{\Sigma}^P$.

Picture of a sample path



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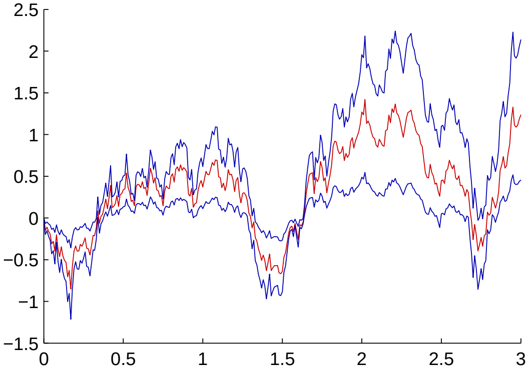

- blue lines: boundaries of sample path of set-valued process X

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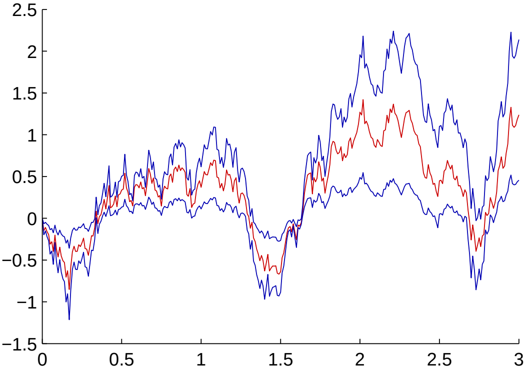
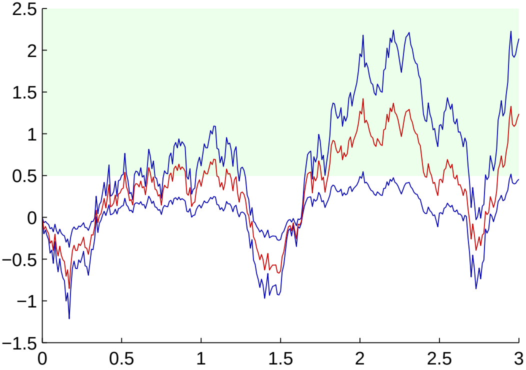


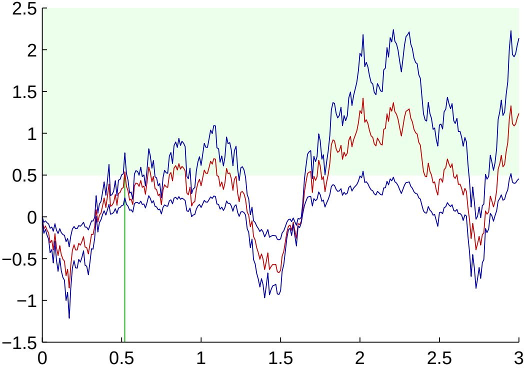
- blue lines: boundaries of sample path of set-valued process X
- red line: sample path of a selection ξ

First entrance and inclusion times



When does ξ enter B for the first time?

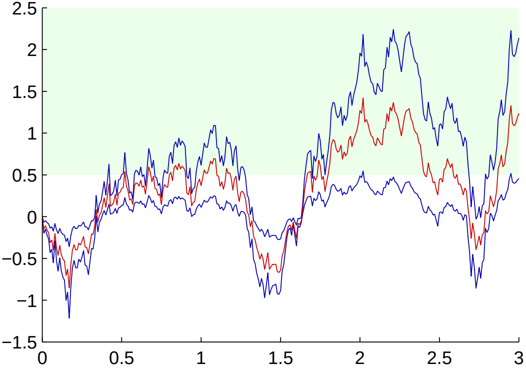
First entrance and inclusion times



When does ξ enter B for the first time?

$\rightarrow$ first entrance time $\tau_{\xi}^{B} : \omega \mapsto \inf\{t : \xi_t(\omega) \in B\}$

First entrance and inclusion times



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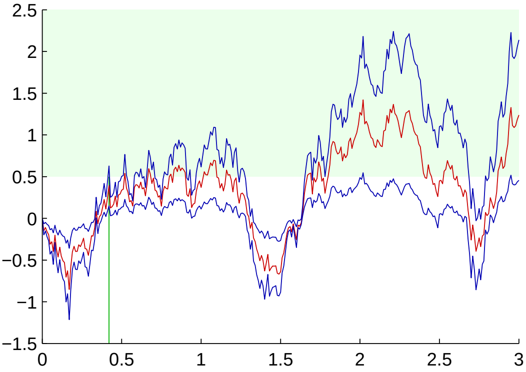

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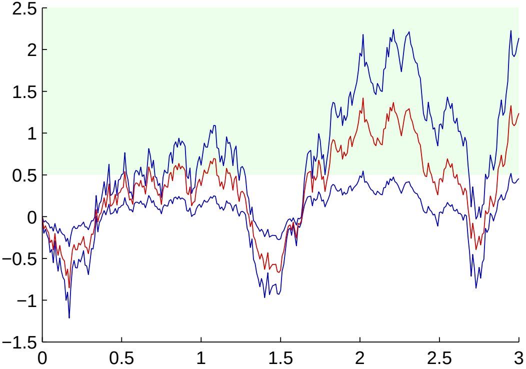

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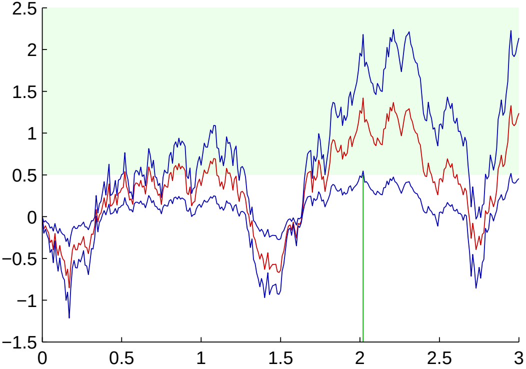

When is X contained in B for the first time?

$\rightarrow$ first inclusion time $\overline{\tau}^B : \omega \mapsto \inf\{t : X_t(\omega) \subseteq B\}$

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Under certain conditions $\underline{\tau}$ and $\overline{\tau}$

- are random variables on (Ω, Σ, P),
- are even stopping times with respect to an appropriate filtration,
- can be attained by first entrance times of selection processes of X.

Curious for more?

Then you should have a look at my poster!