Multivariate Models and Confidence Intervals:
A Local Random Set Approach

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Research Team (imprecise probabilities):

- Michael Oberguggenberger (head of unit)
- Bernhard Schmelzer (next presentation)
- Thomas Fetz

My research area:

- Propagating uncertainty through a mapping.
- Imprecise probabilities and independence.
- Starting: Bayesian Networks linked with Geographic Information Systems (GIS) in collaboration with civil engineers of our faculty.
Starting point: Non-parametric models, Tchebycheff

**Given:** Variable $X$ with $\mu = \text{E}(X)$ and $\sigma^2 = \text{V}(X)$ as sole information.

Generating a nested family $I = \{I_\alpha\}_{\alpha \in (0,1]}$ of confidence intervals

$$I_\alpha = \left[\mu - \frac{\sigma}{\sqrt{\alpha}}, \mu + \frac{\sigma}{\sqrt{\alpha}}\right], \quad \alpha \in (0,1]$$

using Tchebycheff’s inequality $P(|X - \mu| > \frac{\sigma}{\sqrt{\alpha}}) \leq \alpha$. 

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**Diagram:**

- $I_\alpha$ corresponds to the shaded region with $P(I_\alpha) \geq 1 - \alpha$.
- $I^c_\alpha$ is the complement with $P(I^c_\alpha) \leq \alpha$.
Questions

- What is it? Is it a random set or a fuzzy set?

- What is the upper probability $\overline{P}(A)$ of an event $A$? What is its interpretation with respect to confidence intervals?

- What happens in the multivariate case?
Looking at a single confidence interval

Equipping the two intervals $I_\alpha$ and $I^c_\alpha$ with weights

$$m(I_\alpha) = P(I_\alpha) \quad \text{and} \quad m(I^c_\alpha) = P(I^c_\alpha)$$

we get a local random set at level $\alpha$. 

\[\begin{array}{c}
\alpha \\
\hline
0 & 1 \\
\hline
\end{array}\]

\[\begin{array}{c}
I_\alpha \\
\hline
m(I_\alpha) \geq 1 - \alpha \\
\hline
m(I^c_\alpha) \leq \alpha \\
\hline
A \\
\end{array}\]
Looking at a single confidence interval

Equipping the two sets $I_\alpha$ and $I^c_\alpha$ with weights

$$m(I_\alpha) = P(I_\alpha) \quad \text{and} \quad m(I^c_\alpha) = P(I^c_\alpha)$$

we get a local random set at level $\alpha$.

The local upper probability $\overline{P}_\alpha(A)$ at level $\alpha$ for an event $A$ is

$$\overline{P}_\alpha(A) = m(I_\alpha) \chi(A \cap I_\alpha \neq \emptyset) + m(I^c_\alpha) \chi(A \cap I^c_\alpha \neq \emptyset).$$

($\chi$ indicator function)
Three different cases for an event $A$

<table>
<thead>
<tr>
<th>Case</th>
<th>Description</th>
<th>Cumulative Distribution $P_\alpha(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>$A \cap I_\alpha \neq \emptyset$</td>
<td>$P_\alpha(A) \in [0, \alpha]$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$A \cap I_\complement_\alpha \neq \emptyset$</td>
<td>$P_\alpha(A) \in [1-\alpha, 1]$</td>
</tr>
<tr>
<td>(iii)</td>
<td>$A \cap I_\alpha \neq \emptyset$ and $A \cap I_\complement_\alpha \neq \emptyset$</td>
<td>$P_\alpha(A) = 1$</td>
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\[
P_\alpha(A) = m(I_\alpha) \chi(A \cap I_\alpha \neq \emptyset) + m(I_\complement_\alpha) \chi(A \cap I_\complement_\alpha \neq \emptyset).
\]
### Three different cases for an event $A$

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$$
\overline{P}_\alpha(A) = m(I_\alpha) \chi(A \cap I_\alpha \neq \emptyset) + m(I^c_\alpha) \chi(A \cap I^c_\alpha \neq \emptyset).
$$

\[
\begin{align*}
\alpha &-\text{levels} \\
0 &\rightarrow m(I_\alpha) \geq 1 - \alpha \\
\alpha &\rightarrow m(I^c_\alpha) \leq \alpha \\
1 &\rightarrow P_\alpha(A) \in [1 - \alpha, 1]
\end{align*}
\]
Families of Confidence Intervals / The Univariate Case

Three different cases for an event $A$

(i) $A \cap I_\alpha = \emptyset$  \hspace{2cm} $\overline{P}_\alpha(A) \in [0, \alpha]$  

(ii) $A \cap I_c\alpha = \emptyset$  \hspace{2cm} $\overline{P}_\alpha(A) \in [1-\alpha, 1]$ 

(iii) $A \cap I_\alpha \neq \emptyset$ and $A \cap I_c\alpha \neq \emptyset$  \hspace{2cm} $\overline{P}_\alpha(A) = 1$

$$\overline{P}_\alpha(A) = m(I_\alpha) \chi(A \cap I_\alpha \neq \emptyset) + m(I_c\alpha) \chi(A \cap I_c\alpha \neq \emptyset).$$
Three different cases for an event $A$

<table>
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<tr>
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<td>$A \cap I_\alpha = \emptyset$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$A \cap I_\alpha^c = \emptyset$</td>
<td>$1$</td>
</tr>
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<td>$A \cap I_\alpha \neq \emptyset$ and $A \cap I_\alpha^c \neq \emptyset$</td>
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To avoid interval-valued $P_\alpha(A)$ we take always the upper bounds.
Most interesting case (i)

- $A$ has the role of the “bad and undesired” event.
- Meaning:

If $A$ is outside the confidence interval $I_\alpha$ at confidence level $1 - \alpha$, then we can say for sure that $A$ occurs only with probability $\alpha$, at most.
Formula for the upper probability $\overline{P}(A)$

\[
\overline{P}(A) = \inf_{\alpha \in (0,1]} \overline{P}_\alpha(A) = \inf_{\alpha \in (0,1]} \{ \alpha : I_\alpha \cap A = \emptyset \}
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Families of Confidence Intervals / The Univariate Case

Formula for the upper probability $\overline{P}(A)$

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Formula for the upper probability $\overline{P}(A)$

$$\overline{P}(A) = \inf_{\alpha \in (0,1]} \overline{P}_\alpha(A) = \inf_{\alpha \in (0,1]} \{ \alpha : I_\alpha \cap A = \emptyset \}$$

Interpretation of $I$ as random set or fuzzy set

$$\overline{P}(A) = \text{Plausibility}(A) = \text{Possibility}(A)$$

![Graph showing α-levels and the interval $I_{0.4}$]

$\mu - \sigma \quad \mu \quad \mu + \sigma$

$\overline{P}_{0.4}(A) = 0.4$
The Multivariate Case

Goal

A formula for the upper probability for given families $I_1, \ldots, I_n$ of confidence intervals similar to the univariate version.
The Multivariate Case

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A formula for the upper probability for given families $I_1, \ldots, I_n$ of confidence intervals similar to the univariate version.

Short preview

univariate

$$\overline{P}(A) = \inf_{\alpha \in (0,1]} \{ \alpha : I_\alpha \cap A = \emptyset \}$$

multivariate

$$\overline{P}_\ell^S(A) = \inf_{\alpha \in S} \{ \ell(\alpha) : J_\alpha \cap A = \emptyset \}$$
The Multivariate Case

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A formula for the upper probability for given families \( I_1, \ldots, I_n \) of confidence intervals similar to the univariate version.

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multivariate

\[
\overline{P}^S_\ell(A) = \inf_{\alpha \in S} \{ \ell(\alpha) : J_\alpha \cap A = \emptyset \}
\]

Possibilities of choice

1. For the set of confidence intervals considered to be combined.
2. For the weights used for the local joint random set.
Joint Confidence Set

Combination of marginal confidence intervals

\[ J = \{ J_\alpha \}_{\alpha \in S} \] is the family of all joint confidence sets

\[ J_\alpha = I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}, \quad I_{k,\alpha_k} \in I_k \]

with \( \alpha = (\alpha_1, \ldots, \alpha_n) \) depending on the set \( S \) of indices \( \alpha \):

1. Random set independence like: \( S = S_R = (0, 1]^n \).
2. Fuzzy set independence like: \( S = S_F = \{ \alpha \in (0, 1]^n : \alpha_1 = \cdots = \alpha_n \} \)
Joint Confidence Set

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Local upper probability

Event \( A \) with \( J_{\alpha} \cap A = \emptyset \).
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Local upper probability

Event \( A \) with \( J_\alpha \cap A = \emptyset \).

\[ P_\alpha(A) \leq P((I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n})^c) = 1 - m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}). \]
**Joint Confidence Set**

---

### Combination of marginal confidence intervals

\[ J = \{ J_\alpha \}_{\alpha \in S} \text{ is the family of all joint confidence sets} \]

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---

### Local upper probability

**Event** \( A \) with \( J_\alpha \cap A = \emptyset \).

\[
\overline{P}_\alpha(A) = \overline{P}( (I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n})^c ) = \\
= 1 - m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n})
\]

Worst case.
Random set independence

Product of the marginal weights: \( m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}) = \prod_{i=1}^{n} m_i(I_i,\alpha_i) \).

Leads to

\[
\bar{P}_{\alpha}(A) = 1 - \prod_{i=1}^{n} (1 - \alpha_i).
\]

Used if the uncertain variables are independent.
Joint Weight $m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n})$ / Local Upper Probability $\overline{P}_\alpha$

### Random set independence

Product of the marginal weights: $m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}) = \prod_{i=1}^{n} m_i(I_{i,\alpha_i})$.

Leads to

$$\overline{P}_\alpha(A) = 1 - \prod_{i=1}^{n} (1 - \alpha_i).$$

- Used if the uncertain variables are independent.

### Lower / upper bounds

Lower and upper bounds of Fréchet for the joint weights:

$$\max \left( \sum_{i=1}^{n} m(I_{i,\alpha_i}) - n + 1, 0 \right) \leq m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}) \leq \min_{i=1,...,n} m(I_{i,\alpha_i}).$$

Leads to

$$\max_{i=1,...,n} (\alpha_i) \leq \overline{P}_\alpha(A) \leq \min(\alpha_1 + \cdots + \alpha_n, 1).$$

- Used if nothing is known about interactions between the variables.
Levels of the Joint Confidence Set / Upper Probability

Level function $\ell(\alpha)$

The different approaches have only an influence on the level

$$\ell(\alpha) = \begin{cases} 
\max_{i=1,...,n} (\alpha_i) & \text{lower bound}, \\
1 - \prod_{i=1}^{n} (1 - \alpha_i) & \text{random set independence}, \\
\min(\alpha_1 + \cdots + \alpha_n, 1) & \text{upper bound}
\end{cases}$$

of the joint confidence set $J_\alpha$, but not on the confidence set itself.
Level function $\ell(\alpha)$

The different approaches have only an influence on the level

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\end{cases}$$

of the joint confidence set $J_\alpha$, but not on the confidence set itself.

Formula for the upper probability similar to the univariate case

$$\overline{P}_\ell^S(A) = \inf_{\alpha \in S} \{ \ell(\alpha) : J_\alpha \cap A = \emptyset \}.$$
Numerical Example

Beam bedded on two springs with uncertain spring constants

Criterion of failure of beam: \( g(\lambda_1, \lambda_2) \leq 0 \)

Failure function

\[
g(\lambda_1, \lambda_2) = M_{\text{yield}} - \max_{x \in [0,3]} |M(x, \lambda_1, \lambda_2)|
\]

- \( M(x, \lambda_1, \lambda_2) \) is the bending moment at \( x \) depending on \( \lambda_1, \lambda_2 \).
- \( M_{\text{yield}} = 12 \text{ kNm} \) is the elastic limit moment.
The upper probabilities of failure are the results at $g(\lambda_1, \lambda_2) = 0$. 
Ordering / Notations

Ordering of the upper probabilities

\[ \overline{P}_F(A) = \overline{P}_R^{\text{lower}}(A) = \overline{P}_U^{\text{lower}}(A) \]
\[ \overline{P}_R(A) \leq \overline{P}_R^{\text{indep}}(A) \leq \overline{P}_U^{\text{indep}}(A) \]
\[ \overline{P}_U(A) \leq \overline{P}_U^{\text{upper}}(A) \leq \overline{P}_U^{\text{upper}}(A) \]

Classical approaches

<table>
<thead>
<tr>
<th>Notation</th>
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</tr>
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<tbody>
<tr>
<td>( \overline{P}_F )</td>
<td>fuzzy set independence</td>
</tr>
<tr>
<td>( \overline{P}_R )</td>
<td>random set independence</td>
</tr>
<tr>
<td>( \overline{P}_U )</td>
<td>unknown interaction / Fréchet</td>
</tr>
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</table>
### Ordering / Notations

**Ordering of the upper probabilities**

\[
\overline{P}_F(A) = \overline{P}^R_{\text{lower}}(A) = \overline{P}^F_{\text{lower}}(A) \\
\overline{P}_R(A) \leq \overline{P}^R_{\text{indep}}(A) \leq \overline{P}^F_{\text{indep}}(A) \\
\overline{P}_U(A) \leq \overline{P}^R_{\text{upper}}(A) \leq \overline{P}^F_{\text{upper}}(A)
\]

### All possible combinations of confidence intervals, \( S = S_R \)

<table>
<thead>
<tr>
<th>Notation</th>
<th>level ( \ell(\alpha) )</th>
<th></th>
</tr>
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<tr>
<td>( \overline{P}^R_{\text{lower}} )</td>
<td>( \max_{i=1,\ldots,n} (\alpha_i) )</td>
<td>lower Fréchet bound</td>
</tr>
<tr>
<td>( \overline{P}^R_{\text{indep}} )</td>
<td>( 1 - \prod_{i=1}^{n} (1 - \alpha_i) )</td>
<td>random set independence</td>
</tr>
<tr>
<td>( \overline{P}^R_{\text{upper}} )</td>
<td>( \min (\sum_{i=1}^{n} \alpha_i, 1) )</td>
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### Ordering of the upper probabilities

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\overline{P}_F(A) = \overline{P}_{\text{lower}}(A) = \overline{P}_{\text{lower}}(A) \\
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\]

### Combinations of intervals of the same level $\alpha$ only, $S = S_F$

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<td>random set independence</td>
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<td>$\overline{P}_{\text{upper}}^F$</td>
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