

Interpretation and Computation of α -Junctions for Combining Belief Functions

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Motivation

- The α -junctions : a family of combination rules for belief functions introduced axiomatically by Ph. Smets in 1997.
 - It basically represents the set of associative, commutative and linear operators for belief functions with a neutral element.
 - Particular cases of α -junctions : the unnormalized Dempster's rule (conjunctive rule), the disjunctive rule, the exclusive disjunctive rule and its De Morgan dual.
 - A family never exploited !
 - It suffered until now from two main limitations :
 - no interpretation in the general case ;
 - difficult to compute (very laborious definition).
- Propose new ideas to overcome these limitations.

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Overview

- 1 Belief Function Theory
- 2 The α -Junctions : Basic Notions
- 3 Interpretation
- 4 Computation

Mass and belief functions

Let Ω be a finite set of answers to a question Q .

Definition (Mass function)

A mass function is a mapping $m : 2^\Omega \rightarrow [0, 1]$ s.t.

$$\sum_{A \subseteq \Omega} m(A) = 1.$$

Definition (Belief function)

$$bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B), \quad \forall A \subseteq \Omega.$$

Two classical combination rules

Definition (Unnormalized Dempster's rule (conjunctive rule))

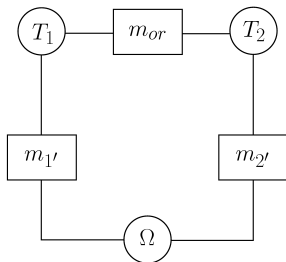
$$m_{1\oplus 2}(A) = \sum_{B \cap C = A} m_1(B)m_2(C), \quad \forall A \subseteq \Omega.$$

Definition (Disjunctive rule)

$$m_{1\cup 2}(A) = \sum_{B \cup C = A} m_1(B)m_2(C), \quad \forall A \subseteq \Omega.$$

Properties :

- commutativity, associativity ;
- neutral element :
 - the vacuous mass function ($m(\Omega) = 1$ noted m_Ω) for \oplus ;
 - the or-vacuous mass function ($m(\emptyset) = 1$ noted m_\emptyset) for \cup .

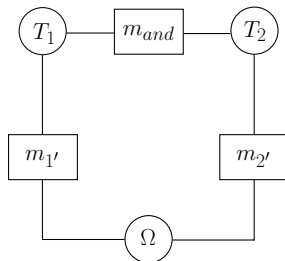
Valuation network for \cup 

- $T_i = \{t_i, f_i\}$: truthfulness of S_i , $i = 1, 2$.
- $m_{or} =$ “at least one of the two sources tells the truth” :

$$m_{or}^{T_1 \times T_2} (\{(t_1, t_2), (t_1, f_2), (f_1, t_2)\}) = 1$$

- $m_{i'}^{\Omega \times T_i}$ is built from the information m_i^Ω provided by S_i and **takes into account the truthfulness** of S_i :

$$m_{i'}^{\Omega \times T_i} (A \times \{t_i\} \cup \bar{A} \times \{f_i\}) = m_i^\Omega(A), \forall A \subseteq \Omega.$$

Valuation network for \cap 

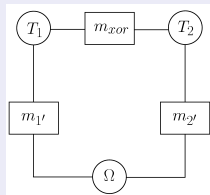
- $m_{and} =$ “Both sources tell the truth” :

$$m_{and}^{T_1 \times T_2} (\{(t_1, t_2)\}) = 1.$$

Two other rules

Exclusive disjunctive rule \oplus

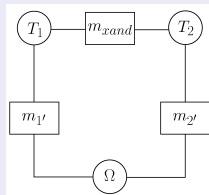
$$m_{1\oplus 2}(A) = \sum_{B \cup C = A} m_1(B)m_2(C)$$



m_{xor} = “exactly one of the two sources tells the truth”

Exclusive conjunctive rule \ominus

$$m_{1\ominus 2}(A) = \sum_{B \cap C = A} m_1(B)m_2(C)$$



m_{xand} = “either all or none of the sources tell the truth”

- Commutativity, associativity, m_Ω neutral for \oplus , m_\emptyset neutral for \ominus

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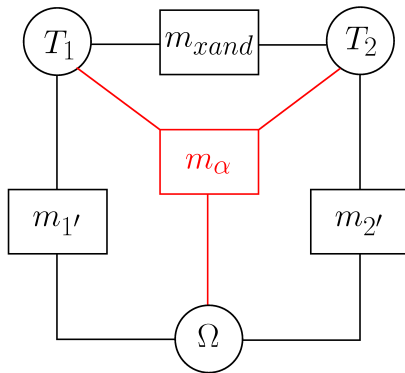
The α -junctions : basic notions

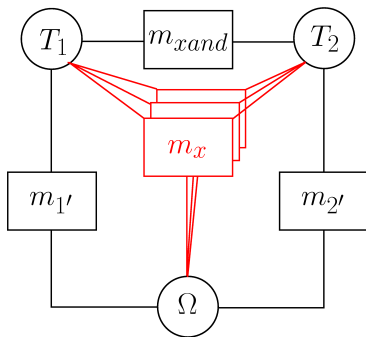
- The **set of associative, commutative and linear operators** for belief functions **with a neutral element**.
- Smets showed that these operators depend on **a parameter $\alpha \in [0, 1]$** and the neutral element.
- He also showed that the neutral element can only be :
 - either $m_\Omega \rightarrow$ **α -conjunctions** noted \odot^α ;
 - or $m_\emptyset \rightarrow$ **α -disjunctions** noted \oplus^α .
- Particular cases of α -junctions :
 - when $\alpha = 1$, $\odot^\alpha = \odot$ and $\oplus^\alpha = \oplus$.
 - when $\alpha = 0$, $\odot^\alpha = \odot$ and $\oplus^\alpha = \oplus$.
- When $\alpha \in (0, 1)$, no interpretation and no simple expression.

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Interpretation of the α -conjunctions



Interpretation of the α -conjunctions (cont'd)

m_α can be obtained from distinct items of evidence for all $x \in \Omega$ indicating that if $\omega = x$, then there is a degree of belief α that at least one of the sources tells the truth :

$$bel^{T_1 \times T_2}[x](\{(t_1, t_2), (f_1, t_2), (t_1, f_2)\}) = \alpha.$$

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4 new ways

- 1 Valuation network.
- 2 Generalization of the classical expression of \circledast .
- 3 Generalization of the conditioning operation.
- 4 Generalization of the commonality function :
 - Let $q : 2^\Omega \rightarrow [0, 1]$ defined by $q(A) = \sum_{B \supseteq A} m(B)$.
 - $q_1 \circledast_2 = q_1 \cdot q_2$
 - $q = Q \cdot m$ and $m = Q^{-1} \cdot q$.

- Matrix Q when $\Omega = \{a, b\}$:

	\emptyset	a	b	ab
\emptyset	1	1	1	1
a	.	1	.	1
b	.	.	1	1
ab	.	.	.	1

- **Kronecker product** : $Q^{i+1} = \mathbf{Kron} \left(\begin{bmatrix} 1 & 1 \\ . & 1 \end{bmatrix}, Q^i \right), Q^1 = 1$.

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α -commonality function

- Smets showed that we have

$$g_1 \circledast^{\alpha} g_2 = g_1 \cdot g_2$$

with $g = G \cdot m$.

- However, the definition of G provided by Smets was very complex.

Theorem

$$G^{j+1} = \text{Kron} \left(\begin{bmatrix} 1 & 1 \\ \alpha - 1 & 1 \end{bmatrix}, G^j \right), G^1 = 1.$$

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Theorem

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Computation using the α -commonality function

- 1 Construction of the matrix G by the preceding theorem.
- 2 $g_1 = G \cdot m_1$ and $g_2 = G \cdot m_2$.
- 3 $g_1 \oplus^{\alpha} g_2 = g_1 \cdot g_2$.
- 4 $m_1 \oplus^{\alpha} m_2 = G^{-1} \cdot g_1 \oplus^{\alpha} g_2$.

As simple as Dempster's rule !

Summary and Perspectives

- The α -junctions suffered from two main limitations :
 - **no interpretation** in the general case ;
 - **difficult to compute**.

→ These two issues tackled in the paper.

- Perspectives :
 - Demonstrate the usefulness of this family in applications ;
 - Usually, the reliability of the sources rather than their truthfulness is taken into account. This distinction matters !
 - Rules based on the α -junctive weight function ?