Boundary linear utility and sensitivity of decisions with imprecise utility trade-off parameters

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July 2009

Background

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- Imprecise trade-offs between attributes.

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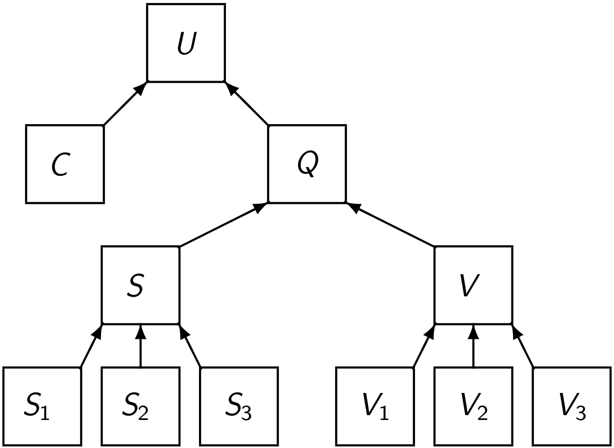
Imprecise utility tradeoffs : We suppose that DM can make preference statements over all outcomes of each individual attribute, and so may specify precise marginal utilities, but can only make preference statements for some, but not all, combinations of the various attributes. Each such preference statement imposes constraints on the tradeoff parameters which are used to combine the individual attributes into an imprecise multi-attribute utility.

Structure: Utility Hierarchy



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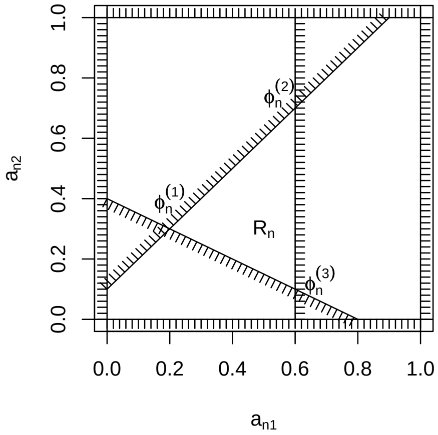
- At each node we have mutual utility independence over parents.
  - This allows a finite parameterisation of the combined utility function.
- All utilities are on a standard scale.
  - Worst outcome considered: $U = 0$.
  - Best outcome considered: $U = 1$.

  This allows us to interpret utilities and trade-offs at all nodes.

Elicitation and feasible set: Binary node



Combining all N nodes

$$
\begin{aligned}
R &= R_1 \times R_2 \times \cdots \times R_N \\
P_n &= \{\phi_n^{(1)}, \ldots, \phi_n^{(r_n)}\} \\
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  - Reduce the number of choices to be considered.
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- In this paper we explore the sensitivity of a choice, particularly using boundary linear utility.

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$$\bar{U}_\lambda = \sum_{i=1}^{r} \lambda_i U_i$$

where $\lambda = (\lambda_1, \ldots, \lambda_r)$ are non-negative constants such that $\sum_{i=1}^{r} \lambda_i = 1$ is a boundary linear utility.

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- For any such $\bar{U}_\lambda$, we may find d which maximises $\bar{U}_{d,\lambda} = \sum_{i=1}^{r} \lambda_i U_{d,i}$, where $U_{d,i}$ is the utility of alternative d with trade-off $\underline{\theta}^{(i)}$.

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- By varying λ we can emphasise or de-emphasize different parts of the feasible set.
- This gives us an alternative metric for the feasible set. The distance between any two vertices is now $\sqrt{2}$.

Exploring sensitivity

- Set of remaining alternatives : D.
- Proposed choice : d^*.
- "Almost equivalent" utility difference : ε.
- Original λ specification : λ_0.
- λ value at vertex v : λ_v.
- $\bar{U}_\lambda(d_j) - \bar{U}_\lambda(d_k) = \delta(d_j, d_k;\ \lambda)$.

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- Sensitivity in the θ-metric.

Volume sensitivity

- Compute the volume of λ-space, as a proportion of the total volume within which $\sum \lambda_j = 1$, over which the difference in utility between alternative d^* and each of the other retained alternatives is at least $-\varepsilon$.

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- Computation: Algorithm exploits structure of hierarchy and local computation properties.

Distance in λ-space

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- For each such v, find

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where

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- Large T_v suggests robustness of $d^\star$.

Sensitivity in the θ-metric

- Original central parameter value: $\underline{\theta}_0$.
- $P = \{\underline{\theta}^{(1)}, \ldots, \underline{\theta}^{(r)}\}$.
- $P_t = \{\underline{\theta}_t^{(i)} \ : \ \underline{\theta}_t^{(i)} = \underline{\theta}_0 + t(\underline{\theta}^{(i)} - \underline{\theta}_0)\}, \ (0 \leq t \leq 1)$.
- *Scaled range* R_t is the convex hull of P_t.
- Difference in expected utility: $\delta(d_j, d_k; \ \underline{\theta})$.
- Find $\max_{P_t}\{\delta(d_j, d^*; \ \underline{\theta}_t^{(i)})\}$.

Sensitivity in the θ-metric: Expansion plot