On Conditional Independence in Evidence Theory

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ISIPTA'09

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- ISIPTA'29?? — multidimensional models for desirable gambles

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- Therefore, operator of composition for basic assignment will not work in possibility theory.
- It seemed that this problem could be avoided if we took into account the fact that both evidence and possibility theories could be considered as special kinds of imprecise probabilities.
- Strong independence implies possibilistic independence based on product t-norm.
- Unfortunately, application of strong independence to two general bodies of evidence (neither Bayesian nor consonant) leads to models beyond the framework of evidence theory.

Independence

Let m be a basic assignment on $\mathbf{X}_N$ and $K, L \subset N$ be disjoint. We say that groups of variables X_K and X_L are *independent with respect to basic assignment m* (and denote it by $K \perp\!\!\!\perp L\ [m]$) if

$$m^{\downarrow K \cup L}(A) = m^{\downarrow K}(A^{\downarrow K}) \cdot m^{\downarrow L}(A^{\downarrow L})$$

for all $A \subseteq \mathbf{X}_{K \cup L}$ for which $A = A^{\downarrow K} \times A^{\downarrow L}$, and $m(A) = 0$ otherwise.

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commonality function

$$Q(A) = \sum_{B \supseteq A} m(B)$$

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Lemma

Let K, L be disjoint, then $K \perp\!\!\!\perp L\ [m]$ if and only if

$$Q^{\downarrow K \cup L}(A) = Q^{\downarrow K}(A^{\downarrow K}) \cdot Q^{\downarrow L}(A^{\downarrow L})$$

for all $A \subseteq \mathbf{X}_{K \cup L}$.

Conditional non-interactivity

Let m be a basic assignment on $\mathbf{X}_N$ and $K, L, M \subset N$ be disjoint, $K \neq \emptyset \neq L$. Groups of variables X_K and X_L are *conditionally non-interactive given X_M with respect to m* (Ben Yaghlane et al.) (and denote it by $K \perp\!\!\!\perp L|M\ [Q]$) if and only if the equality

$$Q^{\downarrow K\cup L\cup M}(A) \cdot Q^{\downarrow M}(A^{\downarrow M}) = Q^{\downarrow K\cup M}(A^{\downarrow K\cup M}) \cdot Q^{\downarrow L\cup M}(A^{\downarrow L\cup M})$$

holds for any $A \subseteq \mathbf{X}_{K\cup L\cup M}$.

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holds for any $A \subseteq \mathbf{X}_{K\cup L\cup M}$.

conditional independence (Shenoy, Studený)

Conditional non-interactivity — formal properties

Theorem

(B. Ben Yaghlane, Ph. Smets and K. Mellouli)
Conditional non-interactivity satisfies:

(A1) $K \perp\!\!\!\perp L|M\ [Q] \ \Rightarrow\ L \perp\!\!\!\perp K|M\ [Q]$,

(A2) $K \perp\!\!\!\perp L \cup M|I\ [Q] \ \Rightarrow\ K \perp\!\!\!\perp M|I\ [Q]$,

(A3) $K \perp\!\!\!\perp L \cup M|I\ [Q] \ \Rightarrow\ K \perp\!\!\!\perp L|M \cup I\ [Q]$,

(A4) $K \perp\!\!\!\perp L|M \cup I\ [Q] \ \wedge\ K \perp\!\!\!\perp M|I\ [Q] \ \Rightarrow\ K \perp\!\!\!\perp L \cup M|I\ [Q]$,

(A5) $K \perp\!\!\!\perp L|M \cup I\ [Q] \ \wedge\ K \perp\!\!\!\perp M|L \cup I\ [Q] \ \Rightarrow\ K \perp\!\!\!\perp L \cup M|I\ [Q]$.

Example

Let X, Y and Z be three binary variables with values in $\mathbf{X} = \{x, \bar{x}\}, \mathbf{Y} = \{y, \bar{y}\}, \mathbf{Z} = \{z, \bar{z}\}$ and m_{XZ} and m_{YZ} two basic assignments:

$$m_{XZ}(\{(x, \bar{z}), (\bar{x}, \bar{z})\}) = m_{XZ}(\{(x, \bar{z}), (\bar{x}, z)\}) = .5,$$
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Since they are projective

$$m_{XZ}^{\downarrow Z}(\{\bar{z}\}) = m_{YZ}^{\downarrow Z}(\{\bar{z}\}) = .5,$$
$$m_{XZ}^{\downarrow Z}(\{z, \bar{z}\}) = m_{YZ}^{\downarrow Z}(\{z, \bar{z}\}) = .5,$$

there exists an extension of both of them.

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$$m_{NI}(\mathbf{X} \times \mathbf{Y} \times \{\bar{z}\}) = .25,$$
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$$m_{NI}(\{(x,y,\bar{z}),(\bar{x},\bar{y},z)\}) = .5,$$
$$m_{NI}(\{(x,y,\bar{z})\}) = -.25.$$

Independence

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$$m^{\downarrow K \cup L}(A) = m^{\downarrow K}(A^{\downarrow K}) \cdot m^{\downarrow L}(A^{\downarrow L})$$

for all $A \subseteq \mathbf{X}_{K \cup L}$ for which $A = A^{\downarrow K} \times A^{\downarrow L}$, and $m(A) = 0$ otherwise.

Lemma

Let K, L be disjoint, then $K \perp\!\!\!\perp L\ [m]$ if and only if

$$Q^{\downarrow K \cup L}(A) = Q^{\downarrow K}(A^{\downarrow K}) \cdot Q^{\downarrow L}(A^{\downarrow L})$$

for all $A \subseteq \mathbf{X}_{K \cup L}$.

Conditional independence

Let m be a basic assignment on $\mathbf{X}_N$ and $K, L, M \subset N$ be disjoint, $K \neq \emptyset \neq L$. We say that groups of variables X_K and X_L are *conditionally independent given X_M with respect to m* (and denote it by $K \perp\!\!\!\perp L | M \; [m]$), if and only if the equality

$$m^{\downarrow K \cup L \cup M}(A) \cdot m^{\downarrow M}(A^{\downarrow M}) = m^{\downarrow K \cup M}(A^{\downarrow K \cup M}) \cdot m^{\downarrow L \cup M}(A^{\downarrow L \cup M})$$

holds for any $A \subseteq \mathbf{X}_{K \cup L \cup M}$ such that $A = A^{\downarrow K \cup M} \otimes A^{\downarrow L \cup M}$, and $m(A) = 0$ otherwise.

An *extension (join)* of two sets $A \subseteq \mathbf{X}_K$ and $B \subseteq \mathbf{X}_L$ is the set

$$A \otimes B = \{x \in \mathbf{X}_{K \cup L} : x^{\downarrow K} \in A \;\; \& \;\; x^{\downarrow L} \in B\}.$$

Example — continued

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then

$$m(\mathbf{X} \times \mathbf{Y} \times \{\bar{z}\}) = .5,$$
$$m(\{(x, y, \bar{z}), (\bar{x}, \bar{y}, z)\}) = .5,$$

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iff $X \perp\!\!\!\perp Y | Z[m]$.

Conditional independence

Theorem

Let m_1 and m_2 be projective basic assignments on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively. Let us define a basic assignment m on $\mathbf{X}_{K\cup L}$ by the formula

$$m(A) = \frac{m_1(A^{\downarrow K}) \cdot m_2(A^{\downarrow L})}{m_2^{\downarrow K\cap L}(A^{\downarrow K\cap L})}$$

for $A = A^{\downarrow K} \otimes A^{\downarrow L}$ such that $m_2^{\downarrow K\cap L}(A^{\downarrow K\cap L}) > 0$ and $m(A) = 0$ otherwise. Then

$$\begin{aligned} m^{\downarrow K}(B) &= m_1(B), \\ m^{\downarrow L}(C) &= m_2(C) \end{aligned}$$

for any $B \in \mathbf{X}_K$ and $C \in \mathbf{X}_L$, respectively, and $(K \setminus L) \perp\!\!\!\perp (L \setminus K)|(K \cap L)\ [m]$. Furthermore, m is the only basic assignment possessing these properties.

Conditional independence — formal properties

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Conditional independence satisfies

(A1) $K \perp\!\!\!\perp L|M\ [m] \ \Rightarrow\ L \perp\!\!\!\perp K|M\ [m]$,

(A2) $K \perp\!\!\!\perp L \cup M|I\ [m] \ \Rightarrow\ K \perp\!\!\!\perp M|I\ [m]$,

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Theorem

Let m be a basic assignment on $\mathbf{X}_N$ such that $m(A) > 0$ if and only if $A = \times_{i \in N} A_i$, where A_i is a focal element on $\mathbf{X}_i$. Then (A5) is satisfied.