Efficient Computing of a Least Favorable Pair for Two Hypothesis of Probability Intervals

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Problem (Maximintest):

We pose $\Omega = \{\omega_i \mid 1 \le i \le n\}$ as finite sample space and **H_0 the feasible probability interval** with the intervals $[L_0(E_i)\,,\,U_0(E_i)]$ for $E_i = \{\omega_i\}$ and $1 \le i \le n$ and $\mathcal{M}_0$ the set of probabilities which lie in all these intervals.
Let **H_1 the (feasible) probability interval** with the intervals $[L_1(E_i)\,,\,U_1(E_i)]$ for $E_i = \{\omega_i\}$ and $1 \le i \le n$ and $\mathcal{M}_1$ the set of probabilities which lie in all these last introduced intervals.
If $\mathcal{M}_0 \cap \mathcal{M}_1 = \emptyset$, **a least favorable pair of probabilities** $(q_0\,,\,q_1)$ with $q_0 \in \mathcal{M}_0$ and $q_1 \in \mathcal{M}_1$ exists. **Determine $(\mathbf{q_0}\,,\,\mathbf{q_1})$** !

Solution:

The risk-function of the Neyman-Pearson-Test of q_0 vs. q_1 is at the same time the risk-function (β) of the Niveau α-Maximintest H_0 vs. H_1 and at the same time the **lower convex envelope** (called $\mathbf{KUR(X,.)}$) of the pointset

$$X = \{(0,1),(1,0)\} \cup \{(U_0(A), U_1(\neg A)) | \emptyset \neq A \subseteq \Omega\}$$

where $U_0(A) = \max_{p \in \mathcal{M}_0} p(A)$ and $U_1(A) = \max_{p \in \mathcal{M}_1} p(A)$.
To find q_0 and q_1 for two hypothesis H_0 and H_1 of disjoint probability intervals, one may construct the lower convex envelope of the points X ($|X| \le 2^n$) to have the risk-function q_0 vs. q_1.
In order to avoid to determine many points for large n a supplementary proceeding can be choosen:
Take Z as indexing variable of "possible formular cases"

$$Z \in \{(I,I),(I,II),(II,I),(II,II)\} \text{ and}$$

regard the values $(val_x(A), val_y(A)) = (U_0(A), U_1(\neg A))$ lying in a plane.
For the singletons E and for $A \neq \emptyset$ set

$$val_{x\,Z}(A) = \begin{cases} \sum_{E \subseteq A} U_0(E), & Z = (I,I),(II,I) \\ 1 - \sum_{E \subseteq \neg A} L_0(E), & Z = (I,II),(II,II) \end{cases}$$

$$val_{y\,Z}(A) = \begin{cases} 1 - \sum_{E \subseteq A} L_1(E), & Z = (I,I),(I,II) \\ \sum_{E \subseteq \neg A} U_1(E), & Z = (II,I),(II,II) \end{cases}$$

$Z = I, II$ f.e. means that the first formular is taken for val_y and the second formular for val_x.
We build for $Z \in \{(I,I),(I,II),(II,I),(II,II)\}$ the four sets

$$X_Z = \{(0,1),(1,0)\} \cup \{(val_{x\,Z}(A), val_{y\,Z}(A)) | \emptyset \neq A \subseteq \Omega\}.$$

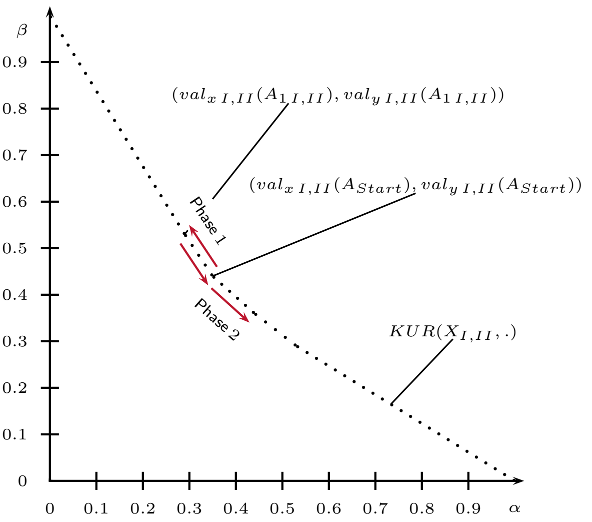
For all Z the **lower convex envelope** of the sets $\mathbf{X_Z}$ (called $KUR(X_Z,.)$) has **no point below** the **lower convex envelope** of the set $\mathbf{X}$, since
$U_0(A) = \min_Z val_{xZ}(A)$ and $U_1(\neg A) = \min_Z val_{yZ}(A)$,
see [4] p. 402.
Each of the lower convex envelope of $X_{I,I}, X_{I,II}, X_{II,I}, X_{II,II}$ is (via an unified algorithm) constructed by at most n points.

Example:

In [3] there is a numeric example of the solution of the given problem with a space Ω containing 12 elements. At first the unique algorithm is illustrated (only) for $Z = I, II$ to construct $KUR(X_{I,II},.)$ (next figure).
The point $(val_{x\,I,II}(A_{1\,I,II}), val_{y\,I,II}(A_{1\,I,II}))$ must be found to get the steepest descent from the point $(0,1)$. This is done in **Phase 1** of the algorithm. Parting from the point $(val_{x\,I,II}(A_{Start}), val_{y\,I,II}(A_{Start}))$ the descent becomes stepwise steeper by taking away appropriate subsets from A_{Start}. In the example $A_{1\,I,II} = \{\omega_2, \omega_3, \omega_4, \omega_5\}$.

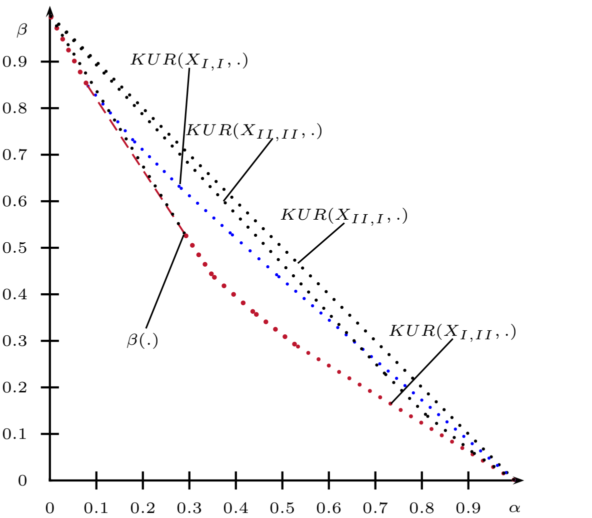
Then in **Phase 2** the curve will be completed piecewise linearly in descending order of the $\pi_{I,II} = \frac{L_1(\cdot)}{L_0(\cdot)}$.

Construction of $\mathbf{KUR(X_{I,II},.)}$



The convex lower envelope of X can be constructed (efficiently) by building the lower convex envelope of all these at most four times n resulting points of the construction for each X_Z.

Construction of $\mathbf{KUR(X,.)}$



Now by the risk-funktion $\beta(.)$ the least favorable pair of probabilities can be calculated.

References:

[1] Augustin, T. (1998):
Optimale Tests bei Intervallwahrscheinlichkeit. Vandenhoeck und Ruprecht, Göttingen

[2] Campos, L.M. de; Huete, J.F.; Moral, S. (1993):
Probability Intervals: A Tool for Uncertain Reasoning. Journal of Uncertainty and Knowledge Based Systems, Vol. 2, p. 167 ..

[3] Gümbel, M. (2009):
Über die effiziente Anwendung von F-PRI - Ein Beitrag zur Statistik im Rahmen eines allgemeineren Wahrscheinlichkeitsbegriffs. Pinusdruck, Christiane u. Karl Jürgen Mühlberger, Augsburg see www.martin-guembel.de

[4] Weichselberger, K. (2001):
Elementare Grundbegriffe einer allgemeineren Wahrscheinlichkeitsrechnung I. Physica, Heidelberg

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