Nonparametric Predictive System Reliability with Redundancy Allocation.

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Coolen [1] presented lower and upper probabilities for prediction of Bernoulli random quantities, which have strong internal consistency properties within the theory of interval probability. These lower and upper probabilities followed from an assumed underlying latent variable model, with future outcomes of random quantities related to data by Hill's assumption $A_{(n)}$ [4], and they are part of a wider statistical methodology called 'Nonparametric Predictive Inference' (NPI).

1 NPI for k-out-of-m systems

k-out-of-m systems, also called 'voting systems', consist of m exchangeable components, such that the system functions if and only if at least k of its m components function, with series systems ($k = m$) and parallel systems ($k = 1$) as special cases.

Coolen-Schrijner, Coolen and MacPhee (2008) considered NPI for system reliability, and in particular for series systems with subsystem i a k_i-out-of-m_i system. Such systems are common in practice, and can offer the important advantage of building in redundancy by increasing some m_i, to increase the system reliability. For the k-out-of-m system, let Y_1^n and Y_{n+1}^{n+m} denote the random number of successes in trials 1 to n and $n+1$ to $n+m$ respectively. When considering such a system , the event $Y_{n+1}^{n+m} \geq k$ is of interest as this corresponds to successful functioning of a k-out-of-m system, following n tests of components that are exchangeable with the m components in the system considered. Given data consisting of s successes from n components tested, the NPI lower and upper probabilities for the event $Y_{n+1}^{n+m} \geq k$ are

$$\begin{aligned}\overline{P}(m:k|\,n,s) &= \overline{P}(Y_{n+1}^{n+m} \geq k | Y_1^n = s) \\ &= \binom{n+m}{n}^{-1} \left[\binom{s+k}{s}\binom{n-s+m-k}{n-s} \right. \\ &\qquad \left. + \sum_{l=k+1}^{m} \binom{s+l-1}{s-1}\binom{n-s+m-l}{n-s} \right]\end{aligned}$$

and, via the conjugacy property,

$$\begin{aligned}\underline{P}(m:k|\,n,s) &= \underline{P}(Y_{n+1}^{n+m} \geq k | Y_1^n = s) \\ &= 1 - \binom{n+m}{n}^{-1} \left[\sum_{l=0}^{k-1} \binom{s+l-1}{s-1}\binom{n-s+m-l}{n-s} \right]\end{aligned}$$

For systems consisting of a series of $N \geq 2$ independent subsystems, with subsystem i ($i = 1, \ldots, N$) a k_i-out-of-m_i system consisting of exchangeable components, the NPI upper and lower probabilities for such a series system to function are

$$\begin{aligned}\overline{P}(\mathbf{m}:\mathbf{k}|\,\mathbf{n},\mathbf{s}) &= \prod_{i=1}^{N} \overline{P}(m_i : k_i|\, n_i, s_i) \\ \underline{P}(\mathbf{m}:\mathbf{k}|\,\mathbf{n},\mathbf{s}) &= \prod_{i=1}^{N} \underline{P}(m_i : k_i|\, n_i, s_i)\end{aligned}$$

2 Path counting for upper and lower probabilities

We explain briefly the idea of how to work with the lower probability of successful functioning of the system considered. The n past and m future observations can be represented with latent variables on the real line. Let $\nu(n,m)$ denote the number of equally likely orderings of those variables for which the data $Y_1^n = s$ must be followed by $Y_{n+1}^{n+m} \geq k$. Then $\underline{P}(Y_{n+1}^{n+m} \geq k | Y_1^n = s) = \nu(n,m)/\binom{n+m}{n}$. Calculating ν is possible by representing each ordering of the variables as a path from $(0,0)$ to (n,m) through pairs $(x,y) \in \mathbf{Z}^2$ with every step either rightwards (increasing x by 1) or upwards (increasing y by 1) as explained in [3]. The count $\nu(n,m)$ includes only those paths which have $y \geq k$ when they first attain x-value s. The advantage of this is that we can use the path counting identity $\nu(n,m+1) = \nu(n,m) + \nu(n-1,m+1)$ to establish results by induction.

3 Redundancy allocation after component testing

With reliability measured by the NPI lower probability for system functioning, optimal allocation of (any number of) additional components, to enhance the system reliability, can be achieved by adding the components sequentially according to the following algorithm (given in pseudo-code), in which, for $i = 1, \ldots, N$ and $j_i \geq 0$, where j_i are additional components added to subsystem i

$$\rho(i,j_i) = \frac{\underline{P}(m_i + j_i + 1 : k_i|\, n_i, s_i)}{\underline{P}(m_i + j_i : k_i|\, n_i, s_i)}$$

Optimal allocation algorithm

1. Set $j_i = 0$ and calculate $\rho(i,j_i) = \rho(i,0)$ for all $i = 1, \ldots, N$;
2. Determine i_m such that
$$\rho(i_m, j_{i_m}) = \max_{1 \leq i \leq N} \rho(i,j_i)$$
If this i_m is not a unique value, then pick any one of these values;
3. Add an extra component to subsystem i_m: set $j_{i_m} := j_{i_m} + 1$ and calculate $\rho(i_m, j_{i_m})$;
4. Return to Step 2, using the same values $\rho(i,j_i)$ as in the previous step for $i \neq i_m$, together with the new value $\rho(i_m, j_{i_m})$ for subsystem i_m, as just calculated in Steps 2 and 3.

This algorithm can be stopped at any time, whatever stop-criterion is defined, and will always give optimal allocation of extra components. After stopping the algorithm, the vector $\mathbf{j} = (j_1, \ldots, j_N)$ gives the number of extra components that is added to each subsystem, and the NPI lower probability for successful functioning of the system after adding these extra components is equal to

$$\underline{P}(\mathbf{m}+\mathbf{j}:\mathbf{k}|\,\mathbf{n},\mathbf{s}) = \underline{P}(\mathbf{m}:\mathbf{k}|\,\mathbf{n},\mathbf{s}) \times \prod_{i=1}^{N}\prod_{l_i=0}^{j_i-1} \rho(i,l_i)$$

4 Example

Consider a basic system consisting initially of four independent k_i-out-of-m_i subsystems in series configuration, with the values k_i and m_i as given in Table 1. Several scenarios of allocation of additional components, to increase redundancy optimally, will be illustrated for this system, with different numbers of successes in the tests of different components in Table 2. Throughout this example, we assume that 5 components of each type were tested, so $n_i = 5$ for $i = 1, \ldots, 4$.

Subsystem i: k_i-out-of-m_i

i	1	2	3	4
k_i	1	2	3	1
m_i	2	3	5	4

Optimal allocation sequences of 5 components

(s_1, s_2, s_3, s_4)	allocation sequence	initial reliability	final reliability
$(5,5,5,5)$	2-3-1-2-3	0.7703	0.9259
$(4,5,5,5)$	1-2-3-1-2	0.6960	0.8877
$(5,4,5,5)$	2-2-3-2-1	0.6186	0.8677
$(5,5,4,5)$	3-2-3-3-1	0.6227	0.8479
$(5,5,5,4)$	2-3-1-2-3	0.7485	0.8963

For the zero-failure case, so with $s_i = 5$ for all $i = 1, \ldots, 4$, this example was also presented by Coolen-Schrijner, *et al* [2], who showed that, due to the fact that subsystem 4, a 1-out-of-4 (parallel) system, has the largest built-in redundancy, the first extra component added to this subsystem is actually only the 12th in the optimal allocation sequence. This sequence of the first 12 extra components is presented again in Table 3, together with corresponding sequences for situations with one or more components of type 4 failing in the test, while no other components failed. This clearly illustrates that, for increasing number of failed components of a particular type in the test, one allocates extra components to the corresponding subsystem earlier in the optimal sequence. For the last case, with only 1 out of 5 tested components of type 4 functioning successfully in the test, one clearly adds a large number of extra components to subsystem 4, but the effect of reduced component reliability still causes the final reliability to be substantially smaller than for the other test results and optimal allocation sequences reported.

Optimal allocation sequences of 12 components

(s_1, s_2, s_3, s_4)	allocation sequence	initial reliability	final reliability
$(5,5,5,5)$	2-3-1-2-3-2-3-1-3-2-1-4	0.7733	0.9742
$(5,5,5,4)$	2-3-1-2-3-4-2-3-1-4-3-2	0.7485	0.9595
$(5,5,5,3)$	2-3-4-1-2-4-3-4-2-4-3-1	0.6867	0.9295
$(5,5,5,2)$	4-2-4-3-4-1-4-2-4-3-4-4	0.5630	0.8556
$(5,5,5,1)$	4-4-4-2-4-4-3-4-1-4-4-2	0.3464	0.6460

5 Future work

- Inclusion of different costs per component of the different types, with the aim of increasing system reliability to a required level at minimum costs, or to obtain optimal system reliability with a fixed budget.
- Consideration of more general system structures,

References

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[4] Hill, B.M. (1968). Posterior distribution of percentiles: Bayes' theorem for sampling from a population. *Journal of the American Statistical Association* **63**, 677-691.