IP theory Based on Proper Scoring Rules – Preliminary Findings

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Abstract

De Finetti's [1974] theory of coherent 2-sided previsions serves as the basis for numerous IP generalizations. His 2-person, zero-sum prevision-game uses a class of bounded random variables $\mathcal{X}$ measurable with respect to some common space $\{\Omega, \mathcal{B}\}$. One player, the *bookie*, is required to post a "fair" 2-sided prevision $\boldsymbol{P}[X]$ for each $X \in \mathcal{X}$. The bookie's opponent, the *gambler*, may choose finitely many non-zero real numbers $\{\alpha_i\}$ where, when the state $\omega \in \Omega$ obtains, the payoff to the bookie is $\Sigma_i\ \alpha_i(\ X_i(\omega) - \boldsymbol{P}[X_i]\ )$, and the opposite payoff, $-\Sigma_i\ \alpha_i(\ X_i(\omega) - \boldsymbol{P}[X_i]\ )$, is for the gambler. That is, the bookie is obliged either to buy (when $\alpha > 0$), or to sell (when $\alpha < 0$) $|\alpha|$-many units of X at the price, $\boldsymbol{P}(X)$. Hence, the previsions are *2-sided* or *fair*. The bookie's previsions are *incoherent* if the gambler has a strategy that insures a uniformly negative payoff for the bookie. Otherwise, the bookie's previsions are *coherent*. De Finetti's *Fundamental Theorem of Previsions* insures that a bookie's previsions are coherent if and only if there exists a finitely additive probability P that determines the expected values for each $X \in \mathcal{X}$, and these expected values are the bookie's previsions: $\mathbf{E}_P[X] = \mathbf{P}[X]$. This result extends to include conditional expectations using the device of called-off previsions.

In order to create an IP-theory, modify de Finetti's game to allow the bookie to fix a pair of 1-sided previsions for each $X \in \mathcal{X}$: For each X, the bookie announces one rate $\underline{\boldsymbol{P}}[X]$ as a buying price, and a (possibly) different rate $\overline{\boldsymbol{P}}\ [X]$ as a selling price. Likewise, one defines a pair of 1-sided called-off previsions for X given event F. A modified Fundamental Theorem obtains, so that a bookie's 1-sided previsions are coherent if and only if there is a maximal, non-empty (convex) set of finitely additive probabilities $\mathcal{P}$ where $\underline{\boldsymbol{P}}(X) = \text{infemum}_{P \in \mathcal{P}}\ \mathbf{E}_P[X]$ and $\overline{\boldsymbol{P}}\ [X] = \text{supremum}_{P \in \mathcal{P}}\ \mathbf{E}_P[X]$. This idea has been developed by many researchers, e.g., C.A.B.Smith [1961].

De Finetti was concerned with strategic aspects of his game. The bookie might take advantage of knowledge about the gambler to **announce** previsions different from her/his own degrees of belief. In order to mitigate such strategic aspects, de Finetti used probabilistic forecasting subject to Brier score as an alternative framework for a second but equivalent criterion of coherence. We focus on probabilistic forecasting of events. Thus, each $X \in \mathcal{X}$ is an indicator function. The bookie's previsions now serve as probabilistic forecasts subject to Brier score: squared-error loss. The penalty associated with the forecast $\mathrm{P}[\boldsymbol{G}]$ when the state $\omega \in \Omega$ obtains is $(\boldsymbol{G}(\omega) - \mathrm{P}[\boldsymbol{G}])^2$, and the score for the conditional forecast $\boldsymbol{P}_F[\boldsymbol{G}]$, called-off if event F fails, is $\boldsymbol{F}(\omega)(\boldsymbol{G}(\omega) - \mathrm{P}[\boldsymbol{G}])^2$. The Brier penalty score from a finite set of forecasts is the sum of the separate forecasts. The set of forecasts, $\{\mathrm{P}[X]: X \in \mathcal{X}\}$, is coherent in the second sense if and only if there is no rival set of forecasts $\{\mathrm{P}'[X]: X \in \mathcal{X}\}$ where, for some finite subset of $\mathcal{X}$, the score from the rival P′-forecasts uniformly dominates the score from the corresponding finite set of P-forecasts. De Finetti established the equivalence between these two senses of coherence. The second sense of coherence, however, does not involve strategic forecasting, in contrast with the opportunity for strategic action in the 2-person previsions-game.

In ongoing work, we investigate a modification of de Finetti's second criterion of coherence so that it may serve as a basis for IP theory. Analogous to the use of 1-sided previsions in the prevision-game, we distinguish the *lower forecast*, used to assess the penalty score when the event fails, from the *upper forecast*, used to assess the penalty score when the event obtains. The issues we address with this generalization go beyond de Finetti's motivation to use the second criterion of coherence in order to avoid the threat of strategic behavior.

Keywords: Coherence, scoring rules, Brier score.

References

[1] De Finetti, B. *Theory of Probability* (2 vols). Wiley, 1974.
[2] Smith, C.A.B. Consistency in Statistical Inference and Decision. *J.Royal Stat. Soc*. **B**, 23: 1-25, 1961