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Decision Theory under Imprecise Probabilities

Tutorial

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Imprecise Probabilities

Coherent Lower/Upper Previsions (Walley, 1991)

Lower prevision:

$$\underline{P} : \ \mathcal{L}_\infty(\Omega) \ \to \ \mathbb{R}\,, \qquad f \ \mapsto \ \underline{P}[f]$$

Upper prevision:

$$\overline{P} : \ \mathcal{L}_\infty(\Omega) \ \to \ \mathbb{R}\,, \qquad f \ \mapsto \ \overline{P}[f]$$

Credal set

$$\mathcal{M} \ = \ \Big\{ P \text{ precise prevision} \ \Big| \ \underline{P}[f] \leq P[f] \leq \overline{P}[f] \ \ \forall\, f \in \mathcal{L}_\infty(\Omega) \Big\}$$

Decision Theory

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- "if this is necessary" $\longrightarrow$ decision problem
- "decisions" $\longrightarrow$ What can be done?
- "good decisions" $\longrightarrow$ Optimality criterion?

Decision Theory

Decision theory = Theory of "How to make good decisions, if this is necessary"

- "if this is necessary" → decision problem
- "decisions" → What can be done?
- "good decisions" → Optimality criterion?

Examples:

- Making an investment in a certain company
- Statistical decision theory:
  - Hypothesis testing: Reject the null hypothesis H_0 or not?
  - Estimation of a parameter

In the following: mathematical formalization of such things

Decisions and States of Nature

Decisions

$$\mathbb{D} \;=\; \text{set of all possible decisions (actions) } t$$

States of nature

- State of nature θ = description of the actual situation
- Whole set of possible states of nature

$$\theta \in \Theta$$

Which $\theta \in \Theta$ is true? $\longrightarrow$ This is unknown!

Example: Decisions and States of Nature

Making an Investment in a Certain Company

- States of nature θ: success of the company
- Decisions t: whether to make an investment

	investment	no investment
Company will be very successful.	excellent	slightly bad
Company will do reasonably well.	okay	okay
Company will collapse.	disastrous	slightly good

Example: Decisions and States of Nature

Making an Investment in a Certain Company

- States of nature θ: success of the company
- Decisions t: whether to make an investment

	investment	no investment
Company will be very successful.	100	-10
Company will do reasonably well.	0	0
Company will collapse.	-200	10

Utility Function

What are the consequences of my decision $t \in \mathbb{D}$?

Utility function:

$$u : \ \Theta \times \mathbb{D} \ \to \ \mathbb{R}\,, \quad (\theta, t) \ \mapsto \ u_\theta(t)$$

Every decision $t \in \mathbb{D}$ leads to a certain utility

$$u_\theta(t) \ \in \ \mathbb{R}$$

depending on the (unknown) state of nature $\theta \in \Theta$

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- high utility: good
- small utility: bad
- negative utility = positive loss

Alternative: Loss Function

What are the consequences of my decision $t \in \mathbb{D}$?

Loss function instead of utility function

$$W: \quad \Theta \times \mathbb{D} \;\rightarrow\; \mathbb{R}\,, \qquad (\theta, t) \;\mapsto\; W_\theta(t)$$

Every decision $t \in \mathbb{D}$ leads to a certain loss

$$W_\theta(t) \;\in\; \mathbb{R}$$

depending on the (unknown) state of nature $\theta \in \Theta$

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$$W_\theta(t) \ \in \ \mathbb{R}$$

depending on the (unknown) state of nature $\theta \in \Theta$

Equivalence of loss function and utility function:

$$W_\theta(t) \ = \ -u_\theta(t) \qquad \forall\, t \in \mathbb{D} \quad \forall\, \theta \in \Theta$$

Loss function is used in the following!

What is a good decision?

	investment	no investment
Company will be very successful.	excellent	slightly bad
Company will do reasonably well.	okay	okay
Company will collapse.	disastrous	slightly good

$\longrightarrow$ **Rating depends on the unknown state of nature θ .**

What is a good decision?

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Company will be very successful.	excellent	slightly bad
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$\longrightarrow$ **Rating depends on the unknown state of nature θ.**

What can be done?

- If we have a prior distribution (prevision) π for θ:
  $\rightarrow$ Deciding according to the expected loss / Bayes risk

What is a good decision?

Precise prior distribution

- Find a decision $\tilde{t}$ which minimizes the "Bayes risk"

$$\pi[W_{\cdot}(t)] \;=\; \int_{\Theta} W_{\theta}(t)\,\pi(d\theta) \;\in\; \mathbb{R}$$

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Imprecise prior distribution

- leads to a whole interval of "Bayes risks"

$$\left[\underline{\pi}[W_{\cdot}(t)]\,,\,\overline{\pi}[W_{\cdot}(t)]\right] \;\subset\; \mathbb{R}$$

$\longrightarrow$ How to compare overlapping intervals?

$\longrightarrow$ Different possibilities (i.e. different optimality criteria)

Γ-minimax, Γ-minimin and a mixture

Γ-minimax (Γ-maximin)

- Find a decision $\tilde{t}_1$ which minimizes the upper bound

$$\overline{\pi}[W.(t)] \;=\; \text{upper Bayes risk}$$

$\rightarrow \tilde{t}_1$ is optimal in the worst case (pessimistic)!

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Popular in robust Bayesian statistics.

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Γ-minimin (Γ-maximax)

- Find a decision $\tilde{t}_2$ which minimizes the lower bound

$$\underline{\pi}[W_{\cdot}(t)] \;=\; \text{lower Bayes risk}$$

$\rightarrow$ $\tilde{t}_2$ is optimal in the best case (optimistic)!

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and a mixture

- Find a decision $\tilde{t}_3$ which minimizes a convex combination of the lower and the upper bound

$$\alpha\overline{\pi}[W_{\cdot}(t)] + (1-\alpha)\underline{\pi}[W_{\cdot}(t)]$$

$\rightarrow$ compromise between pessimism and optimism

E-admissibility and Maximality

Let

$$\mathcal{P} \;=\; \left\{ \pi \text{ precise prior} \;\middle|\; \underline{\pi}[h] \le \pi[h] \le \overline{\pi}[h] \;\; \forall\, h \in \mathcal{L}_\infty(\Theta) \right\}$$

be the credal set of the imprecise prevision.

E-admissibility

- Find a decision $\tilde{t}_4$ which minimizes

$$\pi[W_{\cdot}(t)] \;=\; \int_\Theta W_\theta(t)\,\pi(d\theta)$$

  for at least one $\pi \in \mathcal{P}$.

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for at least one $\pi \in \mathcal{P}$.

Minimality (Maximality)

- Find a decision $\tilde{t}_5$ such that: For every $t \in \mathbb{D}$, there is at least one $\pi_t \in \mathcal{P}$ such that

$$\pi_t[W_\cdot(\tilde{t}_5)] \;\leq\; \pi_t[W_\cdot(t)]$$

Interval dominance

- A decision $\tilde{t}_6$ is optimal with respect to **interval dominance** if

  there is no decision $t \in \mathbb{D}$ such that

$$\overline{\pi}\big[W_{\cdot}(t)\big] \;\leq\; \underline{\pi}\big[W_{\cdot}(\tilde{t}_6)\big]$$

Which is the right one?

The following implications hold:

$$\begin{tikzcd}
\Gamma\text{-minimin} \arrow[d] \arrow[dr] & \Gamma\text{-minimax} \arrow[d] \\
\text{E-admissibility} \arrow[r] \arrow[dr] & \text{Minimality} \arrow[d] \\
 & \text{Interval Dominance}
\end{tikzcd}$$

Confer Troffaes (2007).

How to calculate an optimal decision?

- In case of a finite $\Theta = \{\theta_1, \ldots, \theta_m\}$:
  - Calculations by linear programming
  - See Utkin & Augustin (2005) and Kikuti et al. (2005).
- In case of an infinite Θ:
  - Θ can be discretized
  - Decision problems can be approximately solved by linear programming
  - See Troffaes (2008).

Data-Based Decision Theory

Often: Decision making on base of observations is possible

$$\text{Observation} \,/\, \text{data } x \qquad \longrightarrow \qquad \text{decision } t = \delta(x)$$

Example: Statistical decision theory is always data-based

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So, we have

- a **sample space** $(\mathcal{X}, \mathcal{A})$
- an **observation** $x \in \mathcal{X}$
- a **distribution of the observation**

$$x \sim P_\theta \qquad \text{(depending on the state of nature } \theta\text{)}$$

We have to **chose** not only one decision t but **a decision function**

$$\delta : \mathcal{X} \to \mathbb{D}\,, \qquad x \mapsto \delta(x)$$

Data-Based Decision Theory

The Bayes risk of a decision function $\delta : x \mapsto \delta(x)$

- **Precise previsions**:

$$\int_{\Theta} \int_{\mathcal{X}} W_\theta(\delta(x))\, P_\theta(dx)\, \pi(d\theta)$$

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- **Precise previsions**:

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- **Imprecise previsions**:

$$\left\{ \int_\Theta \int_\mathcal{X} W_\theta(\delta(x))\, P_\theta(dx)\, \pi(d\theta) \;\middle|\; \pi \in \mathcal{P}\,,\ \ P_\theta \in \mathcal{M}_\theta \ \ \forall\, \theta \in \Theta \right\}$$

  - $\mathcal{P}$ credal set of an imprecise prevision of θ
  - $\mathcal{M}_\theta$ credal set of an imprecise prevision of the observation x

  $\rightarrow$ imprecise model $\big(\mathcal{X}, \mathcal{A}, (\mathcal{M}_\theta)_{\theta \in \Theta}\big)$

Example: Estimating (Statistical Decision Theory)

Data $x_1, \ldots, x_n$ are modeled by a vector of random variables

$$X = (X_1, \ldots, X_n) \quad \sim \quad P_0$$

P_0 is the true (precise) distribution.

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P_0 is the true (precise) distribution.

Assumptions: precise case

- $\{ P_\theta \mid \theta \in \Theta \}$ a known parametric set of precise distributions
- There is a true parameter $\theta_0 \in \Theta$ such that $P_0 = P_{\theta_0}$
- There might be a precise prior π on Θ .

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Decision theoretic formalization:

- States of nature: the possible parameters $\theta \in \Theta$
- Decision function: an estimator S

$$\delta = S \ : \ \mathcal{X} \ \rightarrow \ \mathbb{D} = \Theta \, , \quad x \ \mapsto \ S(x)$$

- Loss function: e.g. the least squares loss

$$W_\theta\big(S(x)\big) \ = \ \big(\theta - S(x)\big)^2 \qquad \forall\, \theta \in \Theta$$

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- $\left\{\mathcal{M}_\theta \,\middle|\, \theta \in \Theta\right\}$ a known parametric set of imprecise distributions
- There is a true parameter $\theta_0 \in \Theta$ such that $P_0 \in \mathcal{M}_{\theta_0}$
- There is a imprecise prior $\mathcal{P}$ on Θ .

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- States of nature: the possible parameters $\theta \in \Theta$
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Example: Estimating (Statistical Decision Theory)

The Bayes risk of an estimator $S : x \mapsto S(x)$

- **Precise previsions**:

$$\int_\Theta \int_\mathcal{X} \big(\theta - S(x)\big)^2 \, P_\theta(dx)\, \pi(d\theta)$$

- **Imprecise previsions**:

$$\left\{ \int_\Theta \int_\mathcal{X} \big(\theta - S(x)\big)^2 \, P_\theta(dx)\, \pi(d\theta) \;\middle|\; \pi \in \mathcal{P},\; P_\theta \in \mathcal{M}_\theta \;\; \forall\, \theta \in \Theta \right\}$$

  - $\mathcal{P}$ credal set of an imprecise prevision of θ
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  $\rightarrow$ imprecise model $\big(\mathcal{X}, \mathcal{A}, (\mathcal{M}_\theta)_{\theta \in \Theta}\big)$

How to solve data-based decision problems?

In case of precise previsions:

$$\int_\Theta \int_\mathcal{X} W_\theta(\delta(x))\, P_\theta(dx)\, \pi(d\theta)$$

- **By updating:** Find a decision $\delta(x) \in \mathbb{D}$ which minimizes

$$\int_\Theta W_\theta(\delta(x))\, \pi(d\theta|x) \quad = \quad \text{posterior Bayes risk}$$

- **Without updating:** Find a decision function $\delta : \mathcal{X} \to \mathbb{D}$ which minimizes

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Both ways are equivalent!

How to solve data-based decision problems?

In case of imprecise previsions:

$$\left\{ \int_\Theta \int_\mathcal{X} W_\theta(\delta(x))\, P_\theta(dx)\, \pi(d\theta) \;\middle|\; \pi \in \mathcal{P},\; P_\theta \in \mathcal{M}_\theta \;\; \forall\, \theta \in \Theta \right\}$$

$\longrightarrow$ Again, we have a whole set of (possible) Bayes risks.

$\longrightarrow$ Again, different optimality criteria can be used.

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$\longrightarrow$ Again, different optimality criteria can be used.

The Γ-**minimax** criterion is used in the following.

How to solve data-based decision problems?

In case of imprecise previsions:

- **By updating:** Find a decision $\delta(x) \in \mathbb{D}$ which minimizes

$$\sup_{\pi \in \mathcal{P}_x} \int_\Theta W_\theta(\delta(x))\, \pi(d\theta|x) \quad = \quad \text{upper posterior Bayes risk}$$

where $\mathcal{P}_x = \left\{ \pi(\cdot|x) \,\middle|\, \pi \in \mathcal{P},\ P_\theta \in \mathcal{M}_\theta\ \forall\, \theta \in \Theta \right\}$

- **Without updating:** Find a decision function $\delta : \mathcal{X} \to \mathbb{D}$ which minimizes

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Both ways are <u>not</u> equivalent! (See e.g. Augustin (2003).)

Calculations in data-based decision problems?

In case of imprecise previsions:

- **Solution by updating:**
  - In general, it is hard to compute the set of all (possible) posteriors

$$\mathcal{P}_x = \left\{ \pi(\cdot|x) \,\middle|\, \pi \in \mathcal{P},\ P_\theta \in \mathcal{M}_\theta \ \forall\, \theta \in \Theta \right\}$$

  - Once $\mathcal{P}_x = \left\{ \pi(\cdot|x) \,\middle|\, \pi \in \mathcal{P},\ P_\theta \in \mathcal{M}_\theta \ \forall\, \theta \in \Theta \right\}$ is calculated, problem can be treated as a data-free problem.

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- **Solution without updating:**
  - If Θ and $\mathcal{X}$ are finite, calculations by linear programming
  - Computationally costly and, quite often, intractable so far

Other/Further Generalizations of Decision Theory

- **Imprecise loss functions**
- **Fuzzy sets**
- **Dempster-Shafer theory**
- ...

Some References

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