Abstract

We review de Finetti’s two coherence criteria for determinate probabilities: coherence1 defined in terms of previsions for a set of random variables that are undominated by the status quo – previsions immune to a sure-loss – and coherence2 defined in terms of forecasts for events undominated in Brier score by a rival forecast. We propose a criterion of IP-coherence based on a generalization of Brier score for IP-forecasts that uses 1-sided, lower and upper, probability forecasts. However, whereas Brier score is a strictly proper scoring rule for forecasts assessed by Brier score, the bookie’s previsions are incoherent, i.e., they are undominated by any rival set of forecasts. In his later presentations de Finetti favored coherence2 over coherence1 because, in addition to providing an equivalent criterion for coherence, also proper scores provide a method for incentive compatible elicitation, unlike the situation with coherence, and the prevision game, as we call it. In section 2, we make precise and explain these claims.

De Finetti’s theory of coherent previsions, coherence1, serves as the basis for numerous IP generalizations – see [7, 18, 19] for examples. However, we know of no parallel development of IP theory based on proper scoring rules. It is our purpose in this essay to report basic findings about scoring-rule based IP theory. In section 3 we explain one approach to an IP version of coherence2. In section 4 we present an impossibility result for a real-valued proper IP scoring rule. By contrast, we illustrate a strictly proper, lexicographic (vector-valued) IP version of Brier score. In section 5 we conclude with remarks about the approach begun here.

2. De Finetti’s two criteria for coherence

2.1 Coherence1 and coherence2. The prevision game, is formulated for a class of bounded variables, \( \mathcal{X} = \{ X_i : i \in I \} \) each of which is measurable with respect to a space \( (\Omega, \mathcal{F}) \), where \( I \) serves an index set.

One player, the bookie, posts a fair, or 2-sided prevision \( P(X) \) for each \( X_i \in \mathcal{X} \). The bookie’s opponent, the gambler, may choose finitely many non-zero real numbers \( \{ \alpha_i \} \) where, when the state \( \omega \in \Omega \) obtains, the bookie’s payoff is \( \sum_i \alpha_i (X_i(\omega) - P(X_i)) \), and the gambler’s payoff is the negative, \( -\sum_i \alpha_i (X_i(\omega) - P(X_i)) \). That is, the bookie is obliged either to buy (if \( \alpha > 0 \)) or to sell (if \( \alpha < 0 \)) \( |\alpha| \)-many units of \( X \) at the price, \( P(X) \).

Hence, the previsions are described as being 2-sided or fair buy/sell prices.

The bookie’s previsions are incoherent, if the gambler has a strategy that insures a uniformly negative payoff for the bookie, i.e., if there exist a finite set \( \{ \alpha_i \} \) and \( \varepsilon > 0 \) such that, for each \( \omega \in \Omega \), \( \sum_i \alpha_i (X_i(\omega) - P(X_i)) < -\varepsilon \). Otherwise, the bookie’s previsions are coherent.

De Finetti’s Fundamental Theorem of Previsions: The bookie’s previsions \( \{ P(X) : X \in \mathcal{X} \} \) are coherent if and only if there is a finitely additive probability \( P \) whose expected value for \( X \), \( E_P[X] \), is the bookie’s prevision:

- Coherence1 if and only if \( E_P[X] = P(X) \).

This result extends to include coherence2, for conditional expectations given non-null events, using the device of called-off previsions. Let \( F \) be an event with \( F(\omega) \) its indicator function. The bookie’s called-off prevision,
When the conditioning event example, when the bookie (believes he/she) knows the we focus on forecasting events, represented by their probabilistic forecasting subject to Brier score. Hereafter to mitigate strategic aspects of the prevision game, de Finetti's formulation has been discussed many times in the literature, and with a variety of different proposals to remedy the situation. For three different corrections to this defect in coherence; see [8, 10, and 20]. However, the problem with conditioning on null events does not arise for the questions addressed in this essay. So we use de Finetti’s version of coherence.

De Finetti [3] noted that strategic aspects of betting may affect elicitation of a bookie’s fair previsions. For example, when the bookie (believes he/she) knows the gambler’s betting odds, then announcing a prevision is subject to strategic play in the game and may fail to reveal the bookie’s fair prevision.

**Example 1:** Suppose the bookie’s fair (2-sided) prevision for an event $G$ is .50. But suppose the bookie is confident the gambler’s fair prevision for $G$ is .75. So the bookie announces $P(G) = .70$, anticipating that the gambler will find it profitable to buy units of $G$ at the inflated price. Elicitation using the prevision game fails to identify the bookie’s fair price for $G$.

**Aside:** There are other issues concerning elicitation in the prevision game. Among these is the challenge of state-dependent utilities [13], which we mention in Section 5.

To mitigate strategic aspects of the prevision game, de Finetti turned to a different coherence criterion: probabilistic forecasting subject to Brier score. Hereafter we focus on forecasting events, represented by their indicator functions. $E(\omega) = 1$ if $\omega \in E$ and $E(\omega) = 0$ if $\omega \notin E$.

The bookie’s previsions serve as probabilistic forecasts subject to Brier score: squared-error loss. The penalty for the forecast $P(E)$ when $\omega \in \Omega$ is given by two functions $\{g_1, g_0\}$ depending upon the state:

$$
g_1(P(E), \omega) = (1 - P(E))^2 \quad \text{if event } \omega \in E \text{ obtains;}
$$

$$
g_0(P(E), \omega) = (0 - P(E))^2 \quad \text{if event } \omega \in E^c \text{ obtains},
$$

which is summarized by the squared-error penalty score $E(\omega) - P(E)^2$.

For the conditional (called-off) forecast $P_1(X)$, on condition that event $F$ obtains, the score is $F(\omega)(E(\omega) - P(E))^2$. And just as in the prevision game, the score for a finite set of forecasts is the sum of the separate scores.

**Definition:** A forecast set $\{P(X): X \in \mathcal{X}\}$ is coherent; if, for each finite subset of $\mathcal{X}$, there is no rival forecast set $\{P'(X): X \in \mathcal{X}\}$ whose scores uniformly dominates in $\Omega$.

The two senses of coherence are equivalent, as de Finetti established.

**Proposition 1:** A set of previsions is coherent, in the prevision-game if and only if those same previsions are a coherent set of forecasts under Brier score.

**Proof:** Here is a geometric version of de Finetti’s projection argument for establishing that coherence is coherence with unconditional previsions/forecasts. We use these ideas in Section 3 to extend coherence to an IP setting.

Let $\mathcal{X} = \{X_1, X_2\}$ where $X_1$ is the indicator for an event $A$ and $X_2$ is the indicator for the complementary event $A^c$. In Figure 1, below, a pair of forecasts, $\{Q(A), Q(A^c)\}$ with $0 \leq Q(A), Q(A^c) \leq 1$, is depicted by the point $(Q(A), Q(A^c))$ in the unit square. Note: If either forecast is outside the unit interval, then it is outside the range for the variable being forecasted. And then it is trivial to dominate that forecast with a rival forecast chosen to be closer to the nearest endpoint of the range of the variable in question.

The coherent forecasts lie along the reverse diagonal, the simplex on two states, where $Q(A) + Q(A^c) = 1$. No such point is dominated by any other coherent forecast, since moving along this line segment increases the distance, and hence increases the squared error relative to one endpoint or the other.

**Example 2:** Consider, the incoherent, previsions: $P(A) = .6$ and $P(A^c) = .7$. A Book is achieved against these previsions with the gambler’s strategy $\alpha_1 = \alpha_2 = 1$. Then the net payoff to the bookie is -0.3 regardless which state $\omega$ obtains. In order to see that these are also incoherent forecasts, review Figure 1.

If the forecast previsions are not coherent, they lie outside the probability simplex. Project these incoherent forecasts into the simplex. As in Example, $(.6, .70)$ projects onto the coherent, previsions depicted by the point $(.45, .55)$. By elementary properties of Euclidean projection, the resulting coherent forecasts are closer to each endpoint of the simplex. Thus, the projected forecasts have a dominating Brier score regardless which state obtains. This establishes that the initial forecasts are incoherent. Since no coherent forecast set can be so dominated, we have coherence of the previsions if and only coherence of the corresponding forecasts.
Just as coherence fails to regulate called-off previsions given a null event, coherence does not regulate called-off forecasts given a null event. See [5] for a parallel revision to coherence.

2.2 Incentive Compatible Scoring

Brier score is just one of an infinite class of (strictly) proper scoring rules: A coherent forecaster (uniquely) minimizes expected score by announcing previsions. Thus, forecasting with a (strictly) proper scoring rule avoids the problem of strategic behavior present in the prevision game: there is no opponent. Even allowing different proper scoring rules for different forecasts, by taking the combined score for a finite set of forecasts as the sum of the individual scores, the result is again (strictly) proper. Savage [11] and Schervish [12] characterize the \((g_0, g_t)\) pairs for proper scoring rules. In [14] we establish that all (proper) scoring rules produce the same distinction between coherent; and incoherent; forecasts as with Brier score, both for unconditional forecasts and for conditional forecasts given a non-null event.

Proposition 2 [14]:

2.1 When the scoring rule is proper, finite, and continuous, each incoherent forecast set is dominated by some coherent; forecast set.

2.2 When the scoring rule is proper, finite, but not continuous, each incoherent forecast set is dominated, but not necessarily by a coherent; forecast set.

Note: Result 2.1 can be established by a generalization of de Finetti’s geometric argument, where the projection depends upon the scoring rule. See [9]. The demonstration in [14] uses game-theoretic reasoning.

3. Coherence with a Brier IP scoring rule.

Recall C.A.B. Smith’s [17] modification of de Finetti’s prevision game that provides a criterion of IP-coherence for (closed, convex) IP sets. Rather than requiring a 2-sided, fair price, permit the bookie to fix a pair of 1-sided previsions for each \(X \in \mathcal{X}\):

- The bookie announces one rate \(P(X)\) as a buying price for use when \(\alpha > 0\), and a possibly different selling price \(\overline{P}(X)\) for use when \(\alpha < 0\).

The result is a generalized Book argument. See [19, chapter 2] for some history and basic results.

Proposition 3:

(3.1) A bookie’s 1-sided previsions avoid sure loss if and only if there is a maximal, non-empty (closed, convex) set of finitely additive probabilities \(\mathcal{P}\) where

\[
P(X) \leq \inf_{\mathcal{P}(X)} E_r[X]
\]

And

\[
\overline{P}(X) \geq \sup_{\mathcal{P}(X)} E_r[X].
\]

When these inequalities are equalities, the 1-sided previsions are said to be IP-coherent.

(3.2) By requiring lower and upper previsions for sufficiently many variables (from the linear span of \(\mathcal{I}\)), the 1-sided previsions avoid sure loss if and only if they are also IP-coherent. See Theorem 1.ii of [15].

We offer a parallel version for defining IP-coherence based on Brier score for 1-sided forecasts, as follows:

Use a **lower forecast** to assess a penalty score when the event forecasted fails:

Use an **upper forecast** to assess a penalty score when the event forecasted obtains.

Let \(\{E_i: i = 1, \ldots, m\}\) be \(m\) events defined over a finite partition \(\Omega = \{\omega_j: j = 1, \ldots, n\}\). The forecaster gives lower and upper probability forecasts \(\{p_i, q_i\}\) for each event \(E_i\).

Scoring forecasts with a Brier-styled IP scoring rule:

Fix a state \(\omega \in \Omega\).

If \(\omega \in E_i\) the score for the forecast of \(E_i\) is

\[
(1-q_i)^2 = g(q_i, \omega)
\]

If \(\omega \notin E_i\) the score for the forecast of \(E_i\) is

\[
p_i = g(p_i, \omega)
\]

That is, use the most favorable forecast value from the pair \(\{p_i, q_i\}\) for determining the score. Just as with the other coherence criteria discussed here, the score for a set of forecasts is the sum of the individual forecast scores.

**Dominance**: A forecast set \(\mathcal{F}\) (strictly) dominates another \(\mathcal{G}\) if, for each \(\omega \in \Omega\), the score for \(\mathcal{G}\) is (strictly) less than the score for \(\mathcal{F}\).

But, since the vacuous \(\{0 = p_i, q_i = 1\}\) forecast dominates each rival \(0 < p_i, q_i < 1\), we require an additional restriction on the class of competing forecasts in order to avoid triviality of the resulting theory of IP-coherence. Aside: This is analogous to a problem that is usually ignored within traditional IP theory. With 1-sided previsions, it remains coherent to be strategic: announce a lower buying (and/or a higher selling) price than one is prepared to accept. That is, knowing who is the *Gambler*...
in the 1-sided Prevision Game, the Bookie may play strategically and mimic having a less determinate IP-coherent set of previsions in order to secure strictly favorable gambles.

We propose that \( IP\)-coherence, takes into account both a rival model class \( M \) of coherent, forecasts and the relative imprecision in a forecast set. Stated informally, a set of 1-sided forecasts \( \mathcal{F} \) are incoherent when:

(i) there exists a dominating set of forecast \( \mathcal{G} \) that are
(ii) at least as precise/deterministic as \( \mathcal{F} \) and
(iii) where \( \mathcal{G} \) belongs to the IP-coherent model class \( M \).

We illustrate this idea by filling in the details of the two concepts: the rival model class \( M \) and relative informativeness between forecast sets.

**Example 3:** \( M \) is the \( \epsilon \)-contamination class. Let \( P \) be a particular probability distribution over \( \Omega = \{ \omega_1, \ldots, \omega_h \} \). Fix \( 0 \leq \epsilon \leq 0.5 \). Let \( \mathcal{Q} \) be the simplex of all probability distributions on \( \Omega \). The \( \epsilon \)-contamination model with focus \( P, \mathcal{Q} \), is the set of probability distributions on \( \Omega \) defined by \( \mathcal{R}_\epsilon = \{ (1-\epsilon)P + \epsilon Q : Q \in \mathcal{Q} \} \). For our purposes, it is useful to know that this class is characterized by specifying (IP-coherent) lower probabilities for atomic events, and using the largest closed convex set of distributions satisfying those bounds.

In what follows we illustrate one index of relative indeterminacy associated with our Brier-style IP-scoring rule.

**IP-forecasts over a finite partition for Brier-styled, \( \epsilon \)-contamination coherence:**

Let \( \mathcal{F} = \{ \{p_i, q_i^*\} : i = 1, \ldots, n \} \) be forecasts for each state \( \omega_i \in \Omega = \{ \omega_1, \ldots, \omega_n \} \).

Define \( \mathcal{F} \)'s score set \( \mathcal{S} \) by an ordered \( n \)-tuple of \( n \)-dimensional points:

\[
\mathcal{S} = \{ (q_1, p_1, \ldots, q_n), (q_1, p_2, \ldots, q_n), \ldots, (q_1, p_2, \ldots, q_n) \}.
\]

Thus, \( \mathcal{S} \) contains at most \( n \)-many distinct points. Each point in \( \mathcal{S} \) has \( n \)-many coordinates.

Observe that the IP-Brier-style score for \( \mathcal{F} \) evaluated at state \( \omega_i \) is the square of the Euclidean distance from the \( i^{th} \) point of \( \mathcal{S} \) to the \( i^{th} \) corner of the probability simplex on \( \Omega \). Clearly, the IP-score for a forecast set can be improved merely by moving a lower forecast closer to 0, or by moving an upper forecast closer to 1. So, consider dominating forecast sets only when the dominating forecast has a score that is less indeterminate than the score set for the dominated forecast. Here is a candidate for relative indeterminacy which, when combined with our Brier-style IP-score, allows a characterization of \( \epsilon \)-contamination IP-coherence.

**Definition:** Forecast set \( \mathcal{F} \) is at least as indeterminate as forecast set \( \mathcal{F}_i \) (or \( \mathcal{F} \) is at least as determinate as \( \mathcal{F}_i \)) if the convex hull of score set \( \mathcal{S}_i, H(\mathcal{S}_i) \), is isomorphic under rigid movements (where both shape and sized are held fixed) to a subset of the convex hull of score set \( \mathcal{S}_i, H(\mathcal{S}_i) \).

Note that this relation of relative imprecision, or relative indeterminacy, is merely a partial order. We opt for such a concept so that relative indeterminacy may be extended to a variety of different real-valued indices of imprecision, e.g., by using generalized volume of the score set to quantify indeterminacy.

We use these notions to define IP-coherence generally, and then continue with our illustration of IP-coherence with respect to the \( \epsilon \)-contamination model.

**Definition:** Given an IP-scoring rule, a set \( \mathcal{F} \) of IP-forecasts is \( IP\)-incoherent with respect to the IP-model \( M \) provided that there is a dominating set of rival forecasts \( \mathcal{G} \) from the model \( M \) where the set \( \mathcal{G} \) is at least as determinate than the set \( \mathcal{F} \). Say that \( \mathcal{F} \) is \( IP\)-coherent with respect to \( M \) if it is not \( IP\)-incoherent with respect to \( M \). For convenience we will write these as \( M \)-coherent \( \mathcal{F}_i \) and \( M \)-incoherent \( \mathcal{F}_i \).

Observe that IP-incoherence reduces to de Finetti’s incoherence when all forecasts in \( \mathcal{F} \) are determinate, i.e., when \( p_i = q_i \) for each forecasted event \( E_i \), \((i \in I)\), and when \( M \) is the class of determinate, coherent forecasts. To see this, assume that \( |\Omega| = k \). Then the score set \( \mathcal{S} \) is the ordered set with \( k \)-many repetitions of the same \( |I| \)-dimensional point. Since the lower and upper \( \mathcal{F} \) forecasts for an event are identical, the \( k \)-many points in \( \mathcal{S} \) do not vary with \( \omega \). So a dominating rival forecast set \( \mathcal{G} = \{p_i^*, q_i^*\} \) must also assign the same lower and upper values to each event \( E_i \) (that is, for each \( i \in I, p_i^* = q_i^* \) ), in order for \( \mathcal{G} \) to be at least as determinate as \( \mathcal{F} \). By Proposition 2.1, then if \( \mathcal{G} \) dominates \( \mathcal{F} \) the rival forecast set \( \{p_i^*\} \) establishes that \( \mathcal{F} \) is incoherent, then incoherent.

Next, we provide two basic results for IP-coherence with respect to the \( \epsilon \)-contamination model.

**Proposition 4:** Let \( 0 \leq p_i \leq q_i \leq 1 \), with \( n \)-many forecasts \( \mathcal{F} \) solely for atoms in a finite algebra \( \Omega = \{ \omega_1, \ldots, \omega_h \} \).

\((4.1)\) The score set \( \mathcal{S} \) for \( \mathcal{F} \) lies entirely within the probability simplex on \( \Omega \) if and only if the lower and upper forecasts \( \mathcal{F} \) match an \( \epsilon \)-contamination model. And then \( \mathcal{F} \) cannot be dominated by rival forecasts from a more determinate \( \epsilon \)-contamination model.

\((4.2)\) If all the elements of a score set \( \mathcal{S} \), associated with forecast set \( \mathcal{F} \), lie outside the probability simplex on \( \Omega \), there is a dominating \( \epsilon \)-contamination forecast model \( \mathcal{F}^* \) greater determinacy than \( \mathcal{F} \). \( \mathcal{F} \) is \( IP\)-incoherent against rivals from the \( \epsilon \)-contamination model.
Proof:

(4.1) is established by elementary calculations. If and only if each point of the score set $S$ belongs to the probability simplex then, when state $\omega_0$ obtains, corresponding to the $j^{th}$ point of $S$, $1 = q_j + \sum_{i \neq j} p_i$, and this equality obtains for each $j = 1, \ldots, n$. Then there exists an $\varepsilon \geq 0$ such that for each $i = 1, \ldots, n$, $q_i = p_i + \varepsilon$, which defines an $\varepsilon$-contamination model. In the opposite direction, if forecasts are based on an $\varepsilon$-contamination model, for $i = 1, \ldots, n$, $q_i = p_i + \varepsilon$, and then $1 = q_i + \sum_{i \neq j} p_i$ so that all of the score set $S$ lies in the probability simplex.

Last, if $S$ belongs to the probability simplex and a rival $\varepsilon$-contamination model $\mathcal{F}$ (with corresponding score set $S'$) dominates, then $H(S)$ is a proper subset of $H(S')$ because for each $j = 1, \ldots, n$, the $j^{th}$ point of $S'$ is closer to the $j^{th}$ extreme point of the probability simplex than is the $j^{th}$ point of $S$. So, $\mathcal{F}$ is less determinate than $\mathcal{F}$. Thus $\mathcal{F}$ is IP-coherent$_2$ with respect to the $\varepsilon$-contamination model.

(4.2) follows by the Brouwer Fixed-Point Theorem. Begin with a forecast set $\mathcal{F} = \mathcal{F}_0$, whose score set $\mathcal{S}_0$ has each of its $n$-many ordered points outside the simplex of coherent forecasts. Recursively create rival forecast sets as follows. Apply the (de Finetti) projection to each of these $n$-many ordered points of $\mathcal{S}_0$, taking them into the probability simplex of coherent forecasts. This creates (at most) $n$-points $T_1 = \{t_1, \ldots, t_n\}$ where each $t_i \in T_1$ is a probability distribution $P(\bullet)$ over $\Omega$. Form the new forecast set $\mathcal{F}_1 = \{\{p_{ij}, q_{ij}^1; i = 1, \ldots, n\}$ where $p_{ij} = \min_{\omega \in \Omega} \{P(\omega)\}$ and $q_{ij} = \max_{\omega \in \Omega} \{P(\omega)\}$. This determines a new score set $\mathcal{S}_1$. Since none of the points in $\mathcal{S}_1$ belongs to the probability simplex by the same reasoning used in de Finetti’s analysis for Proposition 1, $\mathcal{F}_1$ dominates $\mathcal{F}_0$.

Just in case $\mathcal{S}_1$ lies in the simplex, when result (4.1) applies, the recursive procedure halts. Otherwise forecast set $\mathcal{F}_2$ is created from a projection of score set $\mathcal{S}_1$ into the probability simplex, etc. (See Appendix 2 for an illustration.)

Since Euclidean projections are continuous functions and the probability simplex is compact, the recursive process with forecast sets $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$ has a fixed point $\mathcal{F}^*$ in the class of $\varepsilon$-contamination models. By a simple adaptation of de Finetti’s argument for Proposition 1, the forecast set $\mathcal{F}_{i+1}$ (weakly) dominates the forecast set $\mathcal{F}_i$ unless $\mathcal{F}_i$ is a fixed point of the process.

Note: It may be that $\mathcal{F}_{i+1}$ merely weakly dominates $\mathcal{F}_i$ for $i \geq 1$, since some but not all the points in $\mathcal{S}_i$ may lie in the probability simplex. However, since all the points of $\mathcal{S}_i$ lie outside the probability simplex, $\mathcal{F}_i$ dominates $\mathcal{F}_0$.

Last, the projection of a closed, convex set, e.g., the projection of $H(S)$ into the probability simplex, is isomorphic to a subset of $H(S)$. Thus, assuming that the each of the points of $\mathcal{S}_0$ is outside the probability simplex on $\Omega$, the fixed point $\mathcal{F}^*$ of the process $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$, which belongs to the $\varepsilon$-contamination model class, strictly dominates $\mathcal{F}_0$, and is at least as determinate as $\mathcal{F}^*$. Hence, $\mathcal{F}_0$ is IP-incoherent$_2$ with respect to the $\varepsilon$-contamination class.

Example: Here is an illustration of Proposition 4, IP-coherence, with respect to the $\varepsilon$-contamination model, using 5 different forecast sets. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Forecasts are for the three atoms only. The five forecast sets $\mathcal{F}(i = 1, \ldots, 5)$ are given in the form $\{\{p_1, q_1\} \times \{p_2, q_2\} \times \{p_3, q_3\}\}$. The respective score sets have three points with coordinates $\{(q_1, p_2, p_3), (p_1, q_2, p_3), (p_1, p_2, q_3)\}$, as described above. Figure 2 diagrams the convex hull of each score set and shows the shaded 2-dimensional, triangular simplex of probability functions on $\Omega$.

Example: Score set \( S' \) dominates the other four score sets and is at least as determinate as each of them.

Figure 2 (for Example 4)

The convex hull of the five score sets are color coded. The simplex of probability distributions is shaded. Each score set projects onto \( S' \), the score set for forecast set \( \mathcal{F}' \), corresponding to an \( \varepsilon \)-contamination model.

\[
\mathcal{S}_1 = \{0.55, 0.80\}, \{0.55, 0.80\}, \{0.55, 0.80\}
\]

\[
\mathcal{S}_2 = \{0.80, 0.55\}, \{0.55, 0.80\}, \{0.55, 0.80\}
\]

\[
\mathcal{S}_3 = \{0.25, 0.50\}, \{0.25, 0.50\}, \{0.25, 0.50\}
\]

\[
\mathcal{S}_4 = \{0.50, 0.25\}, \{0.25, 0.50\}, \{0.25, 0.50\}
\]

\[
\mathcal{S}_5 = \{0.20, 0.45\}, \{0.20, 0.45\}, \{0.20, 0.45\}
\]

\[
\mathcal{S}_6 = \{0.45, 0.20, 0.20\}, \{0.20, 0.45, 0.20\}, \{0.20, 0.45\}
\]

\[
\mathcal{S}_7 = \{0.10, 0.35\}, \{0.10, 0.35\}, \{0.10, 0.35\}
\]

\[
\mathcal{S}_8 = \{0.35, 0.10, 0.10\}, \{0.10, 0.35, 0.10\}, \{0.10, 0.10, 0.35\}
\]

\[
\mathcal{S}_9 = \{0.05, 0.30\}, \{0.05, 0.30\}, \{0.05, 0.30\}
\]

\[
\mathcal{S}_{10} = \{0.30, 0.05\}, \{0.05, 0.30\}, \{0.05, 0.30\}
\]
The two forecast sets $\mathcal{F}^1$ and $\mathcal{F}^2$ are IP-incoherent, in accord with Proposition 3. Their 1-sided prevalences lead to sure losses as, respectively, their lower (upper) forecasts are too great (too small). There is no determinate probability distribution agreeing with either set’s lower and upper forecasts.

Forecast set $\mathcal{F}^1$ corresponds to an $\varepsilon$-contamination model with focus the uniform probability $P = (1/3, 1/3, 1/3)$ and $\varepsilon = 1/6$. The convex hull of the score set $S^1$ lies in the probability simplex, as per Proposition (4.1). It is IP-coherent, and IP-coherent, with respect to the $\varepsilon$-contamination model class.

Forecast set $\mathcal{F}^1$ is IP-coherent, as it has lower and upper forecasts agreeing with a closed convex set of probabilities. Those values agree with an ALUP model, but not with an $\varepsilon$-contamination model. That is, $\mathcal{F}^1$ is IP-coherent, with respect to an IP-model class defined by

$$P = (1/3, 1/3, 1/3) \text{ and IP-model class determined solely by atomic lower and upper probabilities [ALUP], but not so with respect to the $\varepsilon$-contamination class, which is an IP-model class determined solely by atomic lower probabilities.}$$

(See Appendix 1 for details.)

Forecast set $\mathcal{F}^1$ has lower and upper forecasts that do not match those from a closed convex set of probabilities. Its intervals are too wide. However, the uniform probability agrees with these forecasts, i.e., the probability values $(1/3, 1/3, 1/3)$ fall inside the forecast intervals from $\mathcal{F}^1$. Thus, in accord with Proposition 3, the forecasts from $\mathcal{F}^1$ do not suffer a sure-loss in the 1-sided prevision game; however, $\mathcal{F}^1$ is IP-incoherent, and IP-incoherent, with respect to the $\varepsilon$-contamination model class.

As indicated by Figure 2, each of the other four convex hulls projects to $H(S^2)$. That is, the process described in the proof of Proposition (4.2) has $\mathcal{F}$ as its fixed point for each of the five forecast sets, and the process terminates after (at most) one projection.

See Appendix 2 for an illustration of Proposition (4.2) where the fixed point is merely a limit of the process.

4. Incentive compatible IP-elicitation

Recall that de Finetti favored coherence over coherence because, in addition to serving as an equivalent criterion of coherence, Brier score provides a strictly proper score. It provides incentive compatible elicitation for determinate probabilities. For a forecaster whose degrees of belief about events are represented by a single probability function $P(\bullet)$ and who maximizes expected utility, she/he has a unique strategy for announcing forecasts (and called-off forecasts) that minimize expected Brier score. Announce the probability $P(E)$ for the forecast of event $E$. If $H$ is not-null, then announce the conditional probability $P(E | H)$ for the called-off forecast of event $E$, on condition that $H$ obtains. Recall that when $H$ is null, coherence places no restrictions on the called-off forecasts given $H$. There is no difference to the expected score contributed by any conditional forecast of $E$, called-off if $H$ fails, regardless whether that forecast is or is not coherent. See [5] for an improved version of coherence.

What can be done to extend Brier score to an incentive compatible IP-scoring rule? The question is ill-formed without a decision rule that extends maximizing expected utility to IP contexts. We consider only decision rules that reduce to the rule of maximizing expected utility when those IP sets collapse onto the special case of a singleton set, where upper and lower probabilities are identical and a single probability distribution represents uncertainty. Also, we require that decision rules respect the following weak form admissibility. Let $S(\mathcal{F}, \omega)$ be a real-valued IP-scoring rule for forecast set $\mathcal{F}$ in state $\omega$.

Recall that scores are given in the form of a loss so that smaller is better.

**Admissibility Principle:** If for each $\omega \in \Omega$ $S(\mathcal{F}, \omega) \leq S(\mathcal{F}', \omega)$, then $\mathcal{F}$ is admissible in a pairwise choice between rival forecasts $\mathcal{F}$ and $\mathcal{F}'$. Moreover, if for each $\omega$ this inequality is strict then $\mathcal{F}$ is inadmissible whenever $\mathcal{F}$ is an option.

In this section we report two results about eliciting upper and lower probabilities for events when the forecaster’s opinion is represented by a closed, convex sets of probabilities on a finite state space.

**Proposition 5:** There is no real-valued (strictly) proper IP continuous scoring rule.

By contrast, however,

**Proposition 6:** Under either the $\Gamma$-Maximin decision rule, or using one of Levi’s [8] lexicographic decision rules – $E$-admissibility followed by $\Gamma$-Maximin security – there is a strictly proper lexicographic IP-Brier scoring rule.

The IP-decision rules we investigate in Proposition 6 are summarized as follows, with details given in Section 4.2: $\Gamma$-Maximin: The admissible options in $D$ are those that maximize their lower expected value.

$E$-admissibility: An option $X \in D$ is $E$-admissible if for some $P \in P$ and each $Y \in D$, $E_P[X] \geq E_P[Y]$.

$E$-admissibility-followed-by-$\Gamma$-Maximin: Apply $\Gamma$-Maximin to the set of $E$-admissible options in $D$.

Next, we establish and explain these findings.

4.1 Proof of Proposition 5 The impossibility reported in this result is made evident by considering the demands on a real-valued strictly proper IP-scoring rule $S(\mathcal{F}, \omega)$, for forecasting one event, $E$.
Let the interval \([p, q]\), \(0 \leq p \leq q \leq 1\), represent the forecaster’s uncertainty for \(E\). In general, the IP-scoring rule may be written
\[
g_{\omega}(p, q) = g(0) + \frac{1}{2}(1-q)\lambda(dx)\]
if \(\omega \in E\) obtains, and
\[
g_{\omega}(p, q) = g(0) + \frac{1}{2}q\mu(dx)\]
if \(\omega \in E^c\) obtains.

When \(p = q\), in order to be strictly proper and real-valued, the scoring rule must satisfy Theorem 4.2 of Schervish [12]. Specifically, with \(0 \leq x \leq 1\), the loss for the point forecast \(S(x, x, \omega), x\) satisfies
\[
g_{\omega}(x) = g(1) + \frac{1}{2}(1-q)\lambda(dx)\]
if \(\omega \in E\) obtains;
\[
g_{\omega}(x) = g(0) + \frac{1}{2}q\mu(dx)\]
if \(\omega \in E^c\) obtains,
where \(g(1)\) and \(g(0)\) are finite, and \(\lambda(dx)\) is a measure on \([0, 1]\) that gives positive measure to every non-degenerate interval. Continuity of the scoring rule results from a continuous measure \(\lambda\) with no point masses. For example, Brier score results by letting \(\lambda\) have the constant density 2 on the unit interval.

When \(p < q\), the impossibility of a strictly proper IP-scoring rule is a consequence of the fact that, since \(\lambda\) is positive on non-degenerate sub-intervals of the unit interval \([0,1]\) and continuous, there will be rival interval forecasts \([p, q]\) and \([p', q']\) with
\[
g_{\omega}(p, q) - g_{\omega}(p', q') \geq 0,
\]
and
\[
g_{\omega}(p, q) - g_{\omega}(p', q') \geq 0.
\]
Then the interval forecast \([p', q']\) is admissible against the rival interval forecast \([p, q]\). When the interval \([p, q]\) is the forecaster’s IP-uncertainty for event \(E\), she/he will not have reason to announce that as her/his forecast rather than the rival forecast \([p', q']\) and the IP-scoring rule is not strictly proper. If for each \(\omega\) the inequality is strict, then the IP-scoring rule is not proper.

Example 5. We illustrate Proposition 5 using the ideas about IP-coherence, presented in section 4. Consider Brier score adapted to a forecast interval \([p, q]\). That is, let \(b([p, q], \omega) = g([p, q], \omega) = (1-q)^2\) if \(\omega \in E\), and \(b([p, q], \omega) = g([p, q], \omega) = p^2\) if \(\omega \in E^c\). Introduce a real-valued index of indeterminacy for a forecast set \(\mathcal{C}, I[\mathcal{C}]\), where \(I\) agrees with the partial order of relative imprecision used to define IP-coherence. For instance, let \(I(p, q) = q^2\). For real values \(x, y\), let \(H(x, y)\) be a real-valued function increasing in each of its arguments, e.g., \(H(x, y) = x + y\). Define an IP-Brier score for forecast set \(\mathcal{C}\) by \(b(\mathcal{C}, \omega) = H(b(\mathcal{C}, \omega), I[\mathcal{C}])\). Then by Proposition 5, \(B\) is an improper-IP scoring rule. To complete the example, consider event \(E\) and compare the two interval forecasts \([.25, .75]\) and \([.50, .50]\). Then \(B([.25, .75], \omega) = 1/16 + 1/2 = 9/16\) and \(B([.50, .50], \omega) = 1/4 + 0 = 1/4\). Hence, the interval forecast \([.25, .75]\) is inadmissible under this IP-Brier scoring rule \(B\).

4.2 Proof of Proposition 6

First we review the two decision rules mentioned in the text. Let \(\mathcal{P}\) be a closed, convex set of probabilities \(P\) on the space \(\{\Omega, \mathcal{F}\}\). Let \(\chi\) be the class of bounded random variables, \(X\), each measurable with respect to this space. For each \(X\), write \(X\) for the infemum over \(\mathcal{P}\) of the expected value of \(X\),
\[
X = \inf_{\mathcal{P}} E[\mathcal{P} X],
\]
which identifies the lower expected value for \(X\) with respect to \(\mathcal{P}\). Identify a decision problem, \(D\), with a closed subset of \(\chi\). That is, the options in a decision problem form a closed set of bounded variables.

The two IP-decision rules we investigate in Proposition 6 are defined as follows:

\[\text{\textit{\textbf{Gamma-Maximin}}:}\]

The admissible options in \(D\) are those that maximize their lower expected value.

\[\text{\textit{\textbf{Note}}:}\]

By making both \(\mathcal{P}\) and \(D\) closed sets, this max-min operation is well defined.

\[\text{\textit{\textbf{E-admissibility}}:}\]

An option \(X \in D\) is E-admissible if for some \(P \in \mathcal{P}\) and each \(Y \in D\), \(E[\mathcal{P} X] \geq E[\mathcal{P} Y]\).

\[\text{\textit{\textbf{E-admissibility-followed-by-Gamma-Maximin}}:}\]

Apply Gamma-Maximin to the set of E-admissible options in \(D\).

In general, these decision rules have very different axiomatic characterizations. Gamma-Maximin is represented by a real-valued ordering of \(\chi\) using \(X\)-values to index each option. But that ordering violates the independence axiom for preferences. E-admissibility is not represented by an ordering. In fact, it does not even reduce to pairwise comparisons. (See [16] for related discussion.) Nonetheless, next we construct a lexicographic IP-Brier score that is strictly proper under either of the two decision rules mentioned in Proposition 6.

Proposition 5 precludes a proper IP-scoring rule that elicits both endpoint of the interval forecast \([p, q]\) for event \(E\). However, we may elicit either endpoint alone. Define the lower-Brier scoring rule, \(b([x, y], \omega) = b(x, \omega)\) as:
\[
b_{\omega}(x) = (1-x)^2\]
if \(\omega \in E\),
\[
b_{\omega}(x) = 1 + x^2\]
if \(\omega \in E^c\).
and the upper-Brier scoring rule, \(\tilde{b}((x,y), \omega) = \tilde{b}(y, \omega)\) as:
\[
\tilde{b}_{\omega}(y) = (1+y)^2 + 1\]
if \(\omega \in E\),
\[
\tilde{b}_{\omega}(y) = y^2\]
if \(\omega \in E^c\).

Each of these is a strictly proper scoring rule for eliciting determinate forecasts. This follows immediately from Schervish’s representation (above,) where \(g(1) = \tilde{g}_0(0) = 0\), \(g(0) = \tilde{g}_1(1) = 1\), and \(\lambda = 2\) is the uniform (Brier) score density for both rules.

Lemma 1: Under the Gamma-Maximin decision rule, respectively, the lower- (upper-) Brier score is strictly proper for the lower (upper) endpoint of the IP-forecast \([p, q]\) of event \(E\).

Proof of Lemma 1: We give the argument for the lower-Brier score. The reasoning for the upper-Brier score is
similar. Let $p = \min_{\omega \in \mathcal{P}} P(E)$ and $q = \max_{\omega \in \mathcal{P}} P(E)$, so that
\[ \forall \omega \in \mathcal{P} \quad p \leq P(E) \leq q, \] and these bounds are tight. The lower-Brier score of the forecast $[r, s]$ for $E$ depends solely on $r$. The $P$-Expected score for forecast $[r, s]$ is:
\[
E_{[r]} = P(E)(1-r) + (1-P(E))(1+r) = (1-r)^2 + 2r(1-P(E)).
\]
By simple dominance, $0 \leq r \leq 1$. For a given forecast $r$, this expected penalty score is greatest at $P(E) = p$, when the expected score is $(1-r)^2 + 2r(1-p)$. But since lower-Brier score is strictly proper, this worst value is best, i.e., the worst of these expected scores is smallest uniquely for a forecast with $r = p$.  \[\text{Lemma 1}\]

Lemma 2: Under the $E$-admissibility-followed-by-$\Gamma$-Maximin decision rule, respectively, the lower- (upper-) Brier score is strictly proper for the lower (upper) endpoint of the IP-forecast $[p,q]$ of event $E$.

Proof of Lemma 2: Again, we give the argument only for the lower-Brier score. Since lower-Brier score is a strictly proper scoring rule for determinate forecasts, the $E$-admissible forecasts are those of the form $[r, s]$ where $p \leq r \leq q$. Then, by Lemma 1, the $\Gamma$-Maximin solution from this set is uniquely solved at $r = p$.  \[\text{Lemma 2}\]

By Proposition 5, unfortunately, the real-valued composite score obtained by adding together these two scores, $\mathbf{b}([r,s]) = \mathbf{b}([r,s]) + \mathbf{b}([r,s])$, is not IP-proper, which we illustrate with the following example.

Example 6: We illustrate the inappropriety of the real-valued IP-score, $\mathbf{b}([r,s])$, in accord with Proposition 5. Consider an extreme case where the forecaster is maximally uncertain of event $E$, so that the vacuous probability interval $[0, 1]$ represents her/his uncertainty. The forecast $[.5, .5]$ has constant $\mathbf{b}$-score, i.e.,
\[ \mathbf{b}([.5, .5], \omega) = 1 + \frac{1}{4} + \frac{1}{4} = 1.5, \]
independent of $\omega$.
The straightforward forecast $[0, 1]$ has the constant score $\mathbf{b}([0, 1], \omega) = 1 + 1 = 2$, independent of $\omega$. So forecast $[.5 , .5]$ strictly dominates forecast $[0, 1]$ under the $\mathbf{b}$-scoring rule.  

Therefore, we use a 2-tier lexicographical composite scoring to combine these two rules in a manner that create a strictly proper IP-Brier score.

Definition: The two-tier, lexicographical IP-Brier score for the interval forecast $[p,q]$ of event $E$, which we write as $\mathbf{b}_{\text{IP}}([r,s])$, is the 2-tier lexicographic loss function
\[
\mathbf{b}_{\text{IP}}([r,s], \omega) = \mathbf{b}([r,s], \omega), \mathbf{b}([r,s], \omega), \mathbf{b}([r,s], \omega). \]
That is, lexicographically, first apply the loss function $\mathbf{b}([r,s], \omega)$, and among those forecasts have equal $\mathbf{b}$-value, then apply the $\mathbf{b}([r,s])$ loss function. By the preceding two lemmas, under the two decision rules named in Proposition 6, only the interval $[p,q]$ is $\mathbf{b}_{\text{IP}}$-optimal for forecasting event $E$ when the forecaster’s uncertainty for that event is the IP-interval $[p,q]$.

Aside: It is evident that the order of the components is irrelevant in this 2-tiered, lexicographical IP-Brier score.

To elicit an IP-forecast set $\mathcal{P} = \{[p_i, q_i] : i = 1, \ldots, n\}$ for the events $\{E_1, E_2, \ldots, E_n\}$ use, e.g., the $2n$ tiered lexicographic IP-Brier score
\[ < \mathbf{b}([r_1,s_1]), \mathbf{b}([r_1,s_1]), \ldots, \mathbf{b}([r_n,s_n]), \mathbf{b}([r_n,s_n]) >. \] Then the following is immediate from Proposition 6.

Corollary. The $2n$-tiered, lexicographical IP-Brier score is strictly proper under either the $\Gamma$-Maximin or $E$-admissibility-followed-by-$\Gamma$-Maximin decision rules. As above, the order of the $2n$-terms is irrelevant.

5. Summary

When coherence, of 2-sided previsions is not enough, and elicitation also matters, then Brier score offers an incentive compatible scoring rule with an equivalent coherence criterion: coherence, – avoid dominated forecasts. This is de Finetti’s analysis, Proposition 1.

We extend Brier scoring to IP-coherence, of interval-valued forecasts, analogous to the familiar use of 1-sided (lower and upper) previsions for defining IP-coherence. Subject to an IP-scoring rule for forecasting events, the coherent forecaster gives lower and upper probabilistic forecasts for a particular set of events that characterize elements of an IP-model class $M$ – e.g., the $\epsilon$-contamination class is characterized by IP-forecasts for the atoms of the measure space – Proposition 4. Coherence, of the set of IP-forecasts requires that these lower and upper forecasts are not dominated by any more determinate IP model within the model class $M$, subject to the same IP scoring rule.

However, a distinguishing feature between coherence, and coherence, namely that Brier score is incentive compatible for elicitation of 2-sided (real-valued) forecasts for events, does not extend to 1-sided forecasts. That is, according to Proposition 5, there is no strictly proper, real-valued IP-scoring rule for events. However, by relaxing the conditions on scoring rules to permit lexicographical utility, subject to either of two IP-decision rules, there do exist strictly proper IP-scoring rules for eliciting closed, interval-valued probability forecasts.

There are numerous open questions relating to the preliminary work reported in this paper. We list three topics on which we are currently at work.

1) A different challenge to elicitation, even when probability is determinate, is the problem posed by state-dependent utilities. This arises in the choice of the
numéraire that is to be used, either with outcomes of
previsions for coherence1, or in scoring forecasts for
coherence2. (See [13] for discussion of the problem in
the setting of coherence.)

Does forecasting afford any advantage over betting
in this context and is there a difference also with IP-
elicititation?

2) As noted in Section 2, neither coherence1 nor
coherence2 constrains, respectively, a called-off prevision
for an event or a called-off forecast for an event, given a
null event. However, lexicographic expected utility [8] is
one approach among several others available [5, 10, 20]
for improving the treatment of 2-sided conditional
probability with called-off previsions given a null event.
(See [1] for a review of some of the open issues.)

Proposition 6 relies on a lexicographic scoring rule to
establish propriety with respect to interval valued
forecasts.

Can we use lexicographic scoring rules also to elicit
called-off forecasts given a null event?

3) De Finetti’s theory of coherence is designed to
accommodate all finitely additive probabilities. That is,
countable additivity is not a requirement of coherence1 or
coherence2. This is achieved by insisting that
incoherence, i.e., a failure of simple dominance, is
achieved using only finitely many previsions or only
finitely many forecasts at one time. In other words, a
coherent set of previsions or forecasts may be dominated
when more than finitely many are combined at once,
even though they cannot be dominated when only finitely
many are combined. It is interesting, we find, that even
with determinate probabilities, coherence, and
coherence2 are not equivalent in this regard. There are
settings where countably many coherent, forecasts may
be combined and remain undominated by all rival
forecasts, though these same previsions may result in a
sure-loss when countably many are combined into a
single option [17].

In order to accommodate all finitely additive
probabilities, when does IP-coherence2 depend upon the
restriction that violations of dominance matter
only when finitely many forecasts are scored at the
same time?

Acknowledgements

Earlier versions of these results were presented at the
University of Warwick’s Subjective Bayes Workshop, the
Purdue Wimer Memorial Lectures workshop, and
CMU’s Games and Decisions discussion group, and we
thank the participants at these meetings for their helpful
comments. In particular we appreciate suggestions from
Timos Athanasiou, Luca Rigotti, and Kevin Zollman.

Appendix 1

The Atomic Lower-Upper Probability [ALUP] class.
This IP-class consists of closed, convex sets of
probabilities defined by lower and upper probabilities for
atomic events. That is an ALUP model is the largest
(closed) convex set of distributions that satisfy such
bounds, where the bounds are achieved by the lower and
upper probability values given for the atoms of the space.
See [6] for discussion about this IP-class of models.

IP-coherence2, where rival forecasts are taken from the
ALUP class, arises when the forecaster is called upon to
give lower-and-upper forecasts for each atom, \( \omega \), and for
the complement to each atom, \( \omega' \), in the space. That is,
in order to duplicate Proposition 4 for the ALUP class
the forecaster is called upon to give 2n-many forecasts
when \( \Omega = \{ \omega_1, \ldots, \omega_n \} \). Example 7 illustrates this.

Example 7 (a continuation of Example 4): An illustration
of ALUP-coherence2. We provide 3 forecast sets for the
atoms, and their the their complements in a space defined by
\( \Omega = \{ \omega_1, \omega_2, \omega_3 \} \). That is, each forecast set includes IP-
forecasts for 6 events. Forecast sets \( \mathcal{F} (j = 2, 3, 4) \) are
given as 6 pairs: \( [p_1, p_2] \) for \( \omega_1, \omega_2 \), \( i = 1, 2, 3 \). Each of
the corresponding 3 score sets is comprised by 3 points,
corresponding to the 3 states in \( \Omega \). Each point in a score
set has 6 coordinates, corresponding to the scores for
forecasts of \( (\omega_1, \omega_2, \omega_3, \omega_1, \omega_2, \omega_3) \).

\[
\mathcal{F}^1 = \begin{bmatrix}
0.25, 0.50 \\
0.50, 0.25 \\
0.10, 0.90 \\
0.10, 0.90 \\
0.50, 0.25 \\
0.90, 0.10 \\
\end{bmatrix}
\]

\[
\mathcal{F}^2 = \begin{bmatrix}
0.25, 0.50 \\
0.50, 0.25 \\
0.10, 0.90 \\
0.10, 0.90 \\
0.50, 0.25 \\
0.90, 0.10 \\
\end{bmatrix}
\]

\[
\mathcal{F}^3 = \begin{bmatrix}
0.25, 0.50 \\
0.50, 0.25 \\
0.10, 0.90 \\
0.10, 0.90 \\
0.50, 0.25 \\
0.90, 0.10 \\
\end{bmatrix}
\]

Forecast sets \( \mathcal{F}^1 \) and \( \mathcal{F}^2 \) are ALUP-coherent. There do not exist
more precise forecast sets from the ALUP-model that dominate
either of these sets of forecasts. Their score sets lie in the
probability simplex for these 6 events.

Forecast set \( \mathcal{F}^3 \) is ALUP-incoherent. A de Finetti projection of
\( \mathcal{F}^3 \) produces a more determinate rival ALUP forecast with
dominating IP-Brier score. In fact, the projection produces a
more informative \( e\)-contamination model that dominates.
The respective IP-Brier scores for \( \mathcal{F}^1 \) and for \( \mathcal{F}^2 \) are independent of
For \( \mathcal{S} \) the score is a constant penalty of 0.885. For \( \mathcal{F} \) it is a constant penalty of 0.750.

**Appendix 2**

**Example 8** – This construction provides a more complicated illustration of Proposition 4 where the fixed point \( \mathcal{F} \) of the process is a limit of the recursive procedure given in the proof of (4.2). Let \( \Omega = \{ \omega_1, \omega_2, \omega_3 \} \). Forecast sets \( \mathcal{F} \) are of the form \( \{ p_i, q_i \} : i = 1, 2, 3 \),

\[
\begin{align*}
\mathcal{F} &= \mathcal{F}_0 = \{ (0.25, 0.60), (0.20, 0.50), (0.10, 0.40) \} \\
\mathcal{S} &= \mathcal{S}_0 = \{ (0.60, 0.20, 0.10), (0.25, 0.50, 0.10), (0.25, 0.20, 0.40) \}
\end{align*}
\]

(Step 1) Project score set \( \mathcal{S}_0 \) to form set

\[
\begin{align*}
\mathcal{T}_1 &= \{ (0.6, 0.2, 0.3), (0.3, 0.5, 0.1), (0.3, 0.25, 0.45) \}
\end{align*}
\]

Form the new forecast and score sets \( \mathcal{F}_1, \mathcal{S}_1 \) based on the probabilities in set \( \mathcal{T}_1 \)

\[
\begin{align*}
\mathcal{F}_1 &= \{ (0.30, 0.63, 0.13), (0.23, 0.55, 0.45) \} \\
\mathcal{S}_1 &= \{ (0.6, 0.2, 0.3), (0.30, 0.55, 0.13), (0.30, 0.2, 0.45) \}
\end{align*}
\]

(Step 2) Project set \( \mathcal{T}_1 \) to form set

\[
\begin{align*}
\mathcal{T}_2 &= \{ (0.30, 0.740, 0.13, 0.740) \}
\end{align*}
\]

Form the new forecast and score sets \( \mathcal{F}_2, \mathcal{S}_2 \) based on the probabilities in set \( \mathcal{T}_2 \)

\[
\begin{align*}
\mathcal{F}_2 &= \{ (0.30, 0.67, 0.14) \} \\
\mathcal{S}_2 &= \{ (0.6, 0.2, 0.3), (0.2, 0.55, 0.45) \}
\end{align*}
\]

(Step 3) Project \( \mathcal{S}_2 \) to form set

\[
\begin{align*}
\mathcal{T}_3 &= \{ (0.30, 0.55, 0.13, 0.55) \}
\end{align*}
\]

Form the new forecast and score sets \( \mathcal{F}_3, \mathcal{S}_3 \) based on the probabilities in set \( \mathcal{T}_3 \)

\[
\begin{align*}
\mathcal{F}_3 &= \{ (0.30, 0.55, 0.13) \} \\
\mathcal{S}_3 &= \{ (0.6, 0.2, 0.3), (0.2, 0.55, 0.45) \}
\end{align*}
\]

(Step 4) Project \( \mathcal{F}_3 \) to form set

\[
\begin{align*}
\mathcal{T}_4 &= \{ (0.30, 0.55, 0.13, 0.55) \}
\end{align*}
\]

Form the new forecast and score sets \( \mathcal{F}_4, \mathcal{S}_4 \) based on the probabilities in set \( \mathcal{T}_4 \)

\[
\begin{align*}
\mathcal{F}_4 &= \{ (0.30, 0.55, 0.13) \} \\
\mathcal{S}_4 &= \{ (0.6, 0.2, 0.3), (0.2, 0.55, 0.45) \}
\end{align*}
\]

Iterate the process which converges to forecast set

\[
\mathcal{F}^* = \{ (0.30, 0.6, 0.13) \}
\]

and score set

\[
\mathcal{S}^* = \{ (0.6, 0.2, 0.3), (0.30, 0.55, 0.13) \}
\]

\( \mathcal{F}^* \) is an e-contamination model whose IP-Brier score dominates \( \mathcal{S} \)’s score. \( \mathcal{F}^* \) has greater informativeness (greater determinacy) than forecast \( \mathcal{S} \) as the hull \( H(\mathcal{S}) \) is isomorphic to a proper subset of the hull \( H(\mathcal{S}^*) \).

**References**


