Robustness of Natural Extension

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Abstract

How sensitive is the natural extension of an upper prevision against small perturbations in the assessments? We revise some basic results from the theory of systems of linear inequalities and equalities, and linear programming, and apply them to the theory of upper previsions. We find that stability is most easily characterized through a regularity condition on the constraints of the primal problem. We then study stability, and the existence of stable representations, in detail. We find necessary and sufficient conditions for the usual representations of natural extension to be stable, and necessary and sufficient conditions for natural extension to have a stable representation at all. We show that, by arbitrary small perturbation, we can force stability of the usual representations.

1 Introduction

Brevity pertains—see [8] for more about upper previsions.

Let Ω be any finite possibility space. A *gamble* is a real-valued function on Ω. The set of all such gambles is denoted by $\mathcal{L}$, so $\mathcal{L} = \mathbb{R}^{\Omega}$.

We are uncertain about the true value ω in Ω. A popular way of modeling our uncertainty about ω goes by means of an *upper prevision* $\overline{P}$. Specifically, assume that for each gamble g from a finite set $\mathcal{K} \subseteq \mathcal{L}$, we can specify an upper bound $\overline{P}(g)$ on its expectation. We limit ourselves to upper bounds, without loss of generality: a lower bound $\underline{P}(g)$ for g simply translates into an upper bound $\overline{P}(-g) = -\underline{P}(g)$ for $-g$.

A *probability mass function* x on Ω incurs a special kind of upper prevision, namely, one that fixes the expectation exactly, as $x(f) = -x(-f)$:[1]

$$x(f) = \sum_{\omega \in \Omega} x(\omega) f(\omega),$$

noting that, for convenience, we denote the expectation with respect to a probability mass function x also by x. We call x, as a function of gambles, a *linear prevision*. The set of all linear previsions on $\mathcal{L}$ is denoted by C, and it is a subset of the set P of all positive linear functionals (those x for which $x(\omega) \geq 0$ for all ω but not necessarily $x(1) = 1$) on $\mathcal{L}$:

$$C = \{x \in P \colon x(1) = 1\}.$$

For a general upper prevision $\overline{P}$, its *natural extension* $\overline{E}$ is of particular interest [8, §3.4.1]:

$$\overline{E}(f) = \max\{x(f) \colon x \in P,\, x(1) = 1,\, x \leq \overline{P}\} \qquad (1)$$

Here, $x \leq \overline{P}$ means that $x(g) \leq \overline{P}(g)$ for all $g \in \mathcal{K}$. Basically, $\overline{E}$ tells us how to accomplish inference from $\overline{P}$: given the bounds specified by $\overline{P}$, it gives us bounds for all other gambles.

The problem of natural extension in Eq. (1) is easily seen to be a linear programming problem. If it has a solution, then $\overline{P}$ is said to *avoid sure loss*. If $\overline{E}$ coincides with $\overline{P}$ on $\mathcal{K}$, then $\overline{P}$ is said to be *coherent*.

Its dual is (abusing notation for brevity) [8, §3.1.3(e)]:

$$\overline{E}(f) = \min\Big\{a + \textstyle\sum_{g \in \mathcal{K}} \lambda_g \overline{P}(g) \colon (a, \lambda_{\mathcal{K}}) \in Q^*, \\ a + \textstyle\sum_{g \in \mathcal{K}} \lambda_g g \geq f\Big\} \qquad (2)$$

where $Q^* = \{(a, \lambda_{\mathcal{K}}) \colon a \in \mathbb{R} \text{ and } \lambda_g \in \mathbb{R}^+\}$.[2]

[1] The notation 'x' for a probability mass function follows the usual convention in the linear programming literature, where x usually denotes the variable over which we optimize.

[2] Technically, $\lambda_{\mathcal{K}} \in (\mathbb{R}^+)^{\mathcal{K}}$, and we denote $\lambda_{\mathcal{K}}(g)$ by λ_g.

For the purpose of numerical analysis, but also for elicitation, it is important to know whether the solution is sensitive to perturbations in the assessments embodied by $\overline{P}$. The main purpose of this paper is to characterize those upper previsions that are insensitive to such perturbations. We investigate under what conditions a stable representation exists, and how to find this stable representation.

We extend, and to some extent, also simplify, earlier work by Hable, in particular, [2, pp. 118–125, Sec. 5.2] and [3, Sec. 2]. Doing so, we rely on well-known results about the stability of systems of linear inequalities and equalities.

The paper is structured as follows. Section 2 introduces and demonstrates the problem of instability of natural extension by means of a few simple examples. Section 3 reviews the theory of stability of systems of linear inequalities and equalities. Section 4 applies these results on the theory of lower previsions, and natural extension in particular. Section 5 concludes the paper.

2 Examples

Before we venture into the realm of the theory of systems of linear inequalities and equalities, we present some straightforward, yet insightful, examples. Although these examples present an oversimplified and naive view of the notion of stability of linear programs, they do capture the key aspects of the discussion that will follow.

2.1 Instability of Avoiding Sure Loss

We start with a special case of instability of natural extension, namely, when small perturbations cause the lower prevision to incur sure loss.

Consider $\Omega = \{\omega_1, \omega_2\}$, and the following assessments:[3]

$$\overline{P}(I_{\omega_2}) = 2/3 \qquad \overline{P}(I_{\omega_1}) = 1/3$$

By Eq. (1), it follows that we can calculate the natural extension $\overline{E}$ of for instance $I_{\omega_1} + 2I_{\omega_2}$ by the following linear program:

$$\text{maximize } \begin{bmatrix}1 & 2\end{bmatrix} \begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

subject to

$$x_1 \geq 0,\, x_2 \geq 0,\, x_1 + x_2 = 1 \tag{C}$$

$$\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix} \begin{bmatrix}x_1 \\ x_2\end{bmatrix} \leq \begin{bmatrix}2/3 \\ 1/3\end{bmatrix} \tag{S}$$

[3] By I_ω we denote the gamble which is 1 at ω and zero elsewhere.

Clearly, (C) + (S) have a non-empty feasible set: it includes the probability mass function x with $x(\omega_1) = 1/3$ and $x(\omega_2) = 2/3$ (in fact, this is the only element of the feasible set).

However, (C) + (S_ϵ), with

$$\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix} \begin{bmatrix}x_1 \\ x_2\end{bmatrix} \leq \begin{bmatrix}2/3 - \epsilon \\ 1/3\end{bmatrix} \tag{S_ϵ}$$

has an empty feasible set, for any $\epsilon > 0$. If a feasible system of constraints has no solution for some (but not necessarily all) arbitrary small perturbations, then we say that these *constraints are unstable*. Obviously, in such a case, the linear program is deemed unstable as well.

The above example shows that carelessly designed linear programming algorithms may fail to solve even this simple problem due to simple rounding errors.

In practice, implementations of linear programming get around this limitation by transforming to a so-called stable representation. Indeed, by identifying implicit linearities, the program becomes stable, at least in this case. Concretely, the modified system (C) + (S')

$$\begin{bmatrix}0 & 1\end{bmatrix} \begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}2/3\end{bmatrix} \tag{S'}$$

has the same feasible region as original problem. But, now, unlike the original system, all perturbations to the modified assessments:

$$\begin{bmatrix}0 \pm \epsilon & 1 \pm \delta\end{bmatrix} \begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}2/3 \pm \eta\end{bmatrix} \tag{$S'_{\epsilon,\delta,\eta}$}$$

have a solution for every sufficiently small ϵ, δ, η. In other words, the modified constraints are feasible for every sufficiently small perturbation, and so the modified system constraints is stable: we say that the original system has a *stable representation*. Moreover, the solution to the perturbed problem

$$x_2 = \frac{2/3 + \eta - \epsilon}{1 + \delta - \epsilon}$$

remains close to the original solution $x_2 = 2/3$. Whence, the linear program, under the stable representation, is stable too.

2.2 Instability of Natural Extension

The following example is adapted from an example given by Robinson [6, p. 443]. Consider $\Omega = \{a, b, c, d\}$, and the following assessments:

$$\overline{P}(I_a + 2I_b/3 + 2I_d) = 1/2 \qquad \overline{P}(I_b + 3I_c) = 3/2$$

By Eq. (1), it follows that we can calculate the natural extension $\overline{E}$ of for instance $2I_b + 2I_c$ by the following linear program:

$$\text{maximize } \begin{bmatrix} 0 & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix}$$

subject to

$$\begin{gathered} x_a \geq 0,\, x_b \geq 0,\, x_c \geq 0,\, x_d \geq 0 \\ x_a + x_b + x_c + x_d = 1 \end{gathered} \tag{C}$$

$$\begin{bmatrix} 1 & 2/3 & 0 & 2 \\ 0 & 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix} \leq \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} \tag{S}$$

Clearly, (C) + (S) have a non-empty feasible set: it consists of all the probability mass functions of the form (with $\alpha \in [0, 1]$)

$$\alpha \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 \\ 3/4 \\ 1/4 \\ 0 \end{bmatrix}$$

so $\overline{E}(2I_b + 2I_c) = 2$.

However, (C) + (S$_\epsilon$), with

$$\begin{bmatrix} 1 & 2/3-\epsilon & 0 & 2 \\ 0 & 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix} \leq \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} \tag{S$_\epsilon$}$$

has a very different feasible set, for any $\epsilon > 0$. Indeed, regardless of how small ϵ is chosen, the feasible set of the perturbed system contains only one probability mass function:

$$\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}$$

so, now, $\overline{E}(2I_b + 2I_c) = 1$. An arbitrary small perturbation can lead to an unproportionally large variation in the solution of the natural extension.

One can easily check that the system has perturbations that incur sure loss, for instance, by reducing the upper prevision of the first gamble to $1/2 - \epsilon$. We will prove that the natural extension is unstable if and only if there are perturbations which push the system into incurring sure loss (or equivalently, that the natural extension is stable if and only if all sufficiently small perturbations avoid sure loss).

Observe that the dual problem has an unbounded optimal solution:

$$\begin{aligned} \begin{bmatrix} 0 & 2 & 2 & 0 \end{bmatrix} \geq 2 &+ \lambda_1 \left(1/2 - \begin{bmatrix} 1 & 2/3 & 0 & 2 \end{bmatrix}\right) \\ &+ \lambda_2 \left(3/2 - \begin{bmatrix} 0 & 1 & 3 & 0 \end{bmatrix}\right) \end{aligned}$$

for all non-negative λ_1 and λ_2 such that $\lambda_1 = 3\lambda_2$. We will see that this is also tightly related to the instability of the primal problem.

Finally, it is unclear whether the system has a stable representation or not. Intuitively, it seems not; we will prove this later. For now, we present next a much simpler example which has clearly no stable representation.

2.3 Unrepairable Instability

As suggested already, not every upper prevision has a stable representation. Consider for instance the upper prevision defined on I_{ω_2} by

$$\overline{P}(I_{\omega_2}) = 0$$

To calculate its natural extension, we must consider the constraints (C) + (S2), with

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \end{bmatrix} \tag{S2}$$

The feasible region is non-empty: it contains the probability mass function x with $x(\omega_1) = 1$ and $x(\omega_2) = 0$ (in fact, here again, this is the only element of the feasible set). However, the perturbation

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} -\epsilon \end{bmatrix} \tag{S2$_\epsilon$}$$

has an empty feasible region, no matter how small $\epsilon > 0$. In fact, even after recognizing the implicit linearities, the system remains unstable under perturbations. In conclusion, it seems that there is no stable representation.

2.4 Main Issues

Assuming that we can generalize the above observations to arbitrary problems of natural extension, we are left with the following important questions:

1. For the stability of natural extension, does it matter whether we consider the primal or the dual representation?

2. In order to establish the stability of natural extension, is it sufficient to establish stability of the constraints of the primal linear program?

3. Under what conditions are the constraints of the usual representation of the primal linear program stable?

4. If this usual representation is not stable, under what conditions can it be transformed to a stable representation? In other words, when does a stable representation exist?

5. If a stable representation exists, how to find it?

3 Stability of Linear Programming

Robinson [5] characterizes stability of systems of linear inequalities and equalities, and [6] relates this characterization to the stability of natural extension. Here we quickly summarize his results. Also see [10] and [4].

3.1 Stability of Linear Systems of Inequalities and Equalities

Let X and Y be real Banach spaces, let Q be a non-empty convex cone in Y, let P be a non-empty convex set (usually, but not always, assumed to be a convex cone) in X, let b be a point in Y, and let A be a continuous linear operator from X into Y. For two points y_1 and y_2 in Y, we write $y_1 \leq_Q y_2$ if $y_2 - y_1 \in Q$. The cone is used to treat equalities and inequalities homogeneously. Distinguishing between them is crucial when studying stability.[4]

The solution set to

$$Ax \leq_Q b, \quad x \in P, \tag{*}$$

is denoted by F, and for the time being, we are interested in the stability of F with regard to perturbations in A and b.

3.1.1 Definition of Stability

Note that $x \in P$ is a solution of the above system of inequalities if and only if $b - Ax$ is in Q (this is immediate by the definition of $\leq_Q$). Hence, for any arbitrary $x \in P$, we can take the distance between $b - Ax$ and Q as a measure of how much x deviates from a solution of the system, or, if you like, as a measure of infeasibility with respect to the system.

$$\rho(x) = d(b - Ax, Q) = \inf_{q \in Q} \|b - Ax - q\|$$

The distance will be zero exactly when x satisfies the system.

[4]For example, $x = 0$ is obviously stable, but $\{x \geq 0, x \leq 0\}$ is obviously not (for instance, perturb the first inequality to $x \geq \epsilon$ for some $\epsilon > 0$).

Definition 1 (Robinson [5, p. 755])**.** *The system* (*) *is said to be* stable *if there is a positive number β, such that for each $x_0 \in F$ and for any continuous linear operator $A' \colon X \to Y$ and any $b' \in Y$, sufficiently close to A and b respectively, the distance from x_0 to the solution set of the perturbed system*

$$A'x \leq_Q b', \quad x \in P,$$

is not greater than $\beta\rho'(x_0)$, where

$$\rho'(x) = d(b' - A'x, Q) = \inf_{q \in Q} \|b' - A'x - q\|$$

is the distance between $b' - A'x$ and Q.

Note that stability implicitly demands that the original system is feasible, and that all (sufficiently small) perturbations of the original system are feasible.

In order to understand the reasoning behind Robinson's stability condition, let us rewrite the distance condition into something we can easily interpret:

$$\begin{aligned} d(x_0, F') &\leq \beta\rho'(x_0) \\ &= \beta \inf_{q \in Q} \|b' - A'x - q\| \\ &\leq \beta \inf_{q \in Q} \big(\|b' - b - (A'x_0 - Ax_0)\| \\ &\qquad + \|b - Ax_0 - q\|\big) \\ &= \beta\big(\|b' - b - (A'x_0 - Ax_0)\| \\ &\qquad + \inf_{q \in Q} \|b - Ax_0 - q\|\big) \\ &= \beta\|(b' - A'x_0) - (b - Ax_0)\| \end{aligned}$$

which we can further bound by

$$\begin{aligned} &= \beta\|b' - b - (A'x_0 - Ax_0)\| \\ &\leq \beta(\|b' - b\| + \|A'x_0 - Ax_0\|) \\ &\leq \beta(\|b' - b\| + \|A' - A\|\|x_0\|) \end{aligned}$$

Roughly speaking, the condition implies that any solution x_0 of the original system, is also a solution of the perturbed system up to an error that is proportional to the size of the perturbation and $\|x_0\|$.

3.1.2 Stability Criterion

Next, Robinson identifies a simple necessary and sufficient criterion for stability.

Definition 2 (Robinson [5, Def. 1])**.** *The system* (*) *is called* regular *if $b \in \operatorname{int}(AP + Q)$.*

Theorem 3 (Robinson [5])**.** *The system* (*) *is stable if and only if it is regular.*

Proof. As discussed in [5, p. 755, last paragraph], this follows immediately from [5, Thm. 1]. ☐

The following interesting result is an immediate consequence of [5, Thm. 1] (also see [6, Lem. 3]):

Theorem 4. *The system* (*) *is stable if and only if there is an* $\epsilon > 0$ *such that, for all* A' *and* b' *satisfying* $\max\{\|A - A'\|, \|b - b'\|\} < \epsilon$*, the system*

$$A'x \leq_Q b', \quad x \in P,$$

is feasible.

3.1.3 Stable Representation Criterion

In finite dimensions, we have the following result as well, where $\operatorname{ri} P$ denotes the topological interior of P relative to its affine span.

Theorem 5 (Robinson [5, Thm. 3])**.** *The system*

$$Gx \leq g,\ Hx = h, \quad x \in P \tag{3}$$

is representable as a regular system of inequalities and equalities over P *with the same solution set* F *if and only if* $F \cap \operatorname{ri} P \neq \emptyset$*. If the condition is satisfied, then the system can be made regular by changing certain inequalities to equalities and deleting certain redundant equalities.*

3.2 Stability of Linear Programming

Robinson's stability criterion for systems of linear inequalities and equalities does *not* say that the *Hausdorff distance* (see [7, Sec. 3] for a study of this metric in the context of credal sets) between the solution sets is small: it only says that the solution set of the perturbed system is contained, up to a small error, in the solution set of the original system. In fact, the solution set of the original system could be much larger (we hinted already at an example of this earlier, once realized that the dual constraints for natural extension are always stable).

Confusingly, when considering the primal constraints for natural extension, it turns out that stability of these constraints *do* imply that the Hausdorff distance between the credal sets of the original and perturbed systems is small. One of the underlying reasons for this is that the set C of probability mass functions is bounded.

The following result summarizes the relationship between stability of systems of linear inequalities and equalities and the stability of linear programs.

Note that we say that a linear program is *solvable* whenever it has an optimal solution, and that the dual $Q^* \subseteq \mathbb{R}^n$ of a cone $Q \subseteq \mathbb{R}^n$ is defined as

$$Q^* = \{z \in \mathbb{R}^n \colon (\forall x \in Q)(zx \geq 0)\}$$

where zx denotes the dot product of z and x.

Definition 6. *Consider a finite dimensional linear program (P) and its dual (D):*

$$\begin{array}{llll} \textit{maximize } cx & \textit{subject to} & Ax \leq_Q b & x \in P \\ \textit{minimize } ub & \textit{subject to} & uA \geq_{P^*} c & u \in Q^* \end{array}$$

where P *and* Q *are convex cones. The following conditions are equivalent. If any (and whence, all) of them are satisfied, then we say that the linear program (P) is* stable.

(A) The constraints of (P) and (D) are regular.

(B) The sets of optimal solutions of (P) and (D) are non-empty and bounded.

(C) For all sufficiently small perturbations (P')—with corresponding dual (D')—of the linear program (P), both (P') and (D') are solvable.

Proof of equivalence. See Robinson [6, Theorem 1]. □

Robinson [6, Theorem 1] also shows that, whenever a linear program is stable in the above sense, every optimal solution of (P') and (D') remains close to the the optimal solution set of (P) and (D). This obviously implies that the optimal value will not deviate much, which is exactly what we are after for the stability of natural extension. We refer to [6, Theorem 1] for a rigorous statement of what is meant by "sufficiently small" and "remains close" (we have omitted it here to keep the exposition as non-technical as possible).

3.3 Examples Revisited

Before we apply the above results to the specific problem of natural extension, we check stability and stable representability on the earlier examples.

For the first example, again look at Eq. (S), which we demonstrated to be unstable. The cone Q, in this case, is simply the set of non-negative gambles. Let us check that $b \notin \operatorname{int}(AC + Q)$, where $Ax \leq_Q b$ embodies the constraints $x \leq \overline{P}$ of Eq. (1), for $x \in C$.

Note that this turns out to be equivalent to checking that $b \notin \operatorname{int}(AP + Q)$, where $Ax \leq_Q b$ corresponds to the system *including* the constraint $x(1) = 1$, but $x \in P$ (see Theorem 7 further on).

A parametric representation of the set $AC + Q$ follows readily:

$$AC + Q = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\}$$

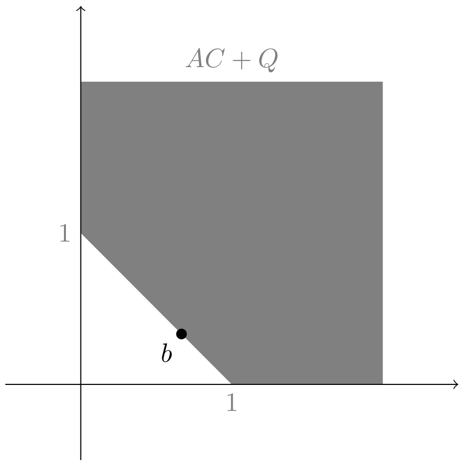

Figure 1: The region $AC{+}Q$ for (C) + (S). The vector b lies on the border, so the system is not stable.

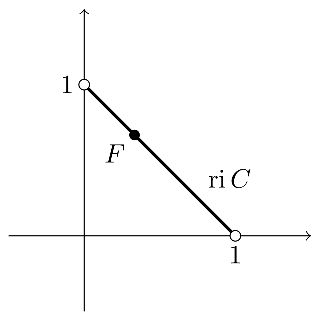

Figure 2: The relative interior of C, and solution set F, for (C) + (S). The solution set F has non-empty intersection with the relative interior of C, so the system has a stable representation.

over all $x_1, x_2, y_1, y_2 \geq 0$ such that $x_1 + x_2 = 1$. The vector

$$b = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

lies on the border of this set, but not in its interior (see Fig. 1). Whence, the system is not stable.

However, it has a stable representation: the solution set

$$F = \left\{ \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \right\}$$

intersects with the relative interior of the set C of all probability mass functions (see Fig. 2).

For the second example, one can similarly show that it does not satisfy the stability criterion. It is easy to show that it does not have a stable representation. Indeed, the feasible set lies on the edge of the set C of all probability mass functions, because $x_d = 0$ everywhere in the feasible region. So F does not intersect with the relative interior of C, and therefore there is no stable representation.

Figure 3: The region $AC + Q$ for (C) + (S2). The vector b lies on the border, so the system is not stable.

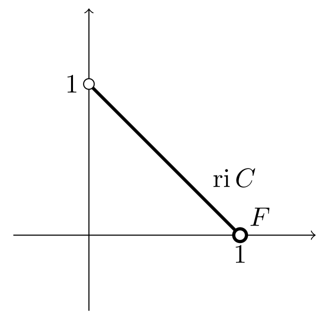

Figure 4: The relative interior of C, and solution set F, for (C) + (S2). The solution set F has empty intersection with the relative interior of C, so the system has no stable representation.

Let us now revisit the third example. Inspect Eq. (S2). A parametric representation of the region $AC + Q$ is

$$AC + Q = \left\{ \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + y_1 \right\}$$

over all $x_1, x_2, y_1 \geq 0$ such that $x_1 + x_2 = 1$, which reduces to

$$= \{y_1 \colon y_1 \geq 0\}$$

that is, the set of non-negative real numbers. The vector

$$b = \begin{bmatrix} 0 \end{bmatrix}$$

lies on the border of this set, but not in its interior (see Fig. 3). Whence, the system is not stable.

Moreover, we can now prove our earlier intuition that it has no stable representation: the solution set

$$F = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

does not intersect with the relative interior of the set C of all probability mass functions (see Fig. 4).

4 Stability of Natural Extension

4.1 Canonical Representations

We now rewrite the primal and dual forms of natural extension using the notation of the previous section on linear programming. The primal linear program, Eq. (1), is:

$$\text{maximize } c_f x \text{ subject to } A_{\overline{P}} x \leq_Q b_{\overline{P}},\ x \in P \quad (\mathbf{P})$$

with

$$c_f = \begin{bmatrix} f(\omega_1) & \dots & f(\omega_n) \end{bmatrix}$$

$$A_{\overline{P}} = \begin{bmatrix} 1 & \dots & 1 \\ g_1(\omega_1) & \dots & g_1(\omega_n) \\ \vdots & \ddots & \vdots \\ g_k(\omega_1) & \dots & g_k(\omega_n) \end{bmatrix} \quad b_{\overline{P}} = \begin{bmatrix} 1 \\ \overline{P}(g_1) \\ \vdots \\ \overline{P}(g_k) \end{bmatrix}$$

$$Q = \left\{ \begin{bmatrix} 0 \\ y_1 \\ \vdots \\ y_k \end{bmatrix} : y_1, \dots, y_k \in \mathbb{R}^+ \right\}$$

$$P = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1, \dots, x_n \in \mathbb{R}^+ \right\}$$

Note that the feasible region F is exactly the credal set of $\overline{P}$.

We call the linear program ($\mathbf{P}$) the *canonical representation* of the natural extension of $\overline{P}$.

If we omit the constraint $x(1) = 1$ from the system of inequalities, and consider the reduced optimization problem over $x \in C$ (as we did before in the examples), then we arrive at the *reduced canonical representation* of the natural extension of $\overline{P}$:

$$\text{maximize } c_f x \text{ subject to } A^-_{\overline{P}} x \leq b^-_{\overline{P}},\ x \in C \quad (\mathbf{P}^-)$$

where $A^-_{\overline{P}}$ is $A_{\overline{P}}$ without the first row, and $b^-_{\overline{P}}$ is $b_{\overline{P}}$ without the first element.

Studying the stability of this reduced system simply means that we do not consider perturbations in the normalization constraint $x(1) = 1$, which in fact seems a natural thing to do. However, the theory of stability of linear programs demands that the linear program has a dual, and ($\mathbf{P}^-$) does not have a dual, because the set C is not a cone. Fortunately, as we shall prove, stability properties are independent of whether we allow perturbations in $x(1) = 1$ or not.

Of course, ($\mathbf{P}$) does have a dual program, given earlier by Eq. (2):

$$\text{minimize } u b_{\overline{P}} \text{ subject to } u A_{\overline{P}} \geq_{P^*} c_f,\ u \in Q^* \quad (\mathbf{D})$$

with $P^* = \{z^T : z \in P\}$ and

$$Q^* = \left\{ \begin{bmatrix} a & \lambda_1 & \dots & \lambda_k \end{bmatrix} : a \in \mathbb{R}, \lambda_1, \dots, \lambda_k \in \mathbb{R}^+ \right\}$$

The linear program ($\mathbf{D}$) is the *canonical dual representation* of the natural extension of $\overline{P}$.

4.2 Stability of the Canonical Representation of Natural Extension

It will follow from our discussion in Section 3 that, to determine stability of natural extension in its canonical representation, it suffices to determine the regularity (or, stability) of the system of linear inequalities and equalities ($\mathbf{P}$) or equivalently, of ($\mathbf{P}^-$).

First, we need one more definition: a *linear-vacuous* mixture is any coherent upper prevision of the form

$$(1 - \alpha)x + \alpha \sup_{\omega \in \Omega}$$

for some $\alpha \in [0, 1]$ and $x \in C$. We say that this linear-vacuous mixture is non-linear whenever $\alpha > 0$.

Theorem 7. *Let $\overline{P}$ be any upper prevision. The following conditions are equivalent.*

(A) The linear program ($\mathbf{P}$) is stable.

(B) The linear program ($\mathbf{D}$) is stable.

(C) The system of linear inequalities and equalities of ($\mathbf{P}$) is regular.

(D) The system of linear inequalities and equalities of ($\mathbf{P}^-$) is regular.

(E) All sufficiently small perturbations of $\overline{P}$ avoid sure loss, that is, there is an $\epsilon > 0$ such that all $\overline{P}'$ on $\mathcal{K}$ satisfying $\overline{P}(g) - \epsilon \leq \overline{P}'(g) \leq \overline{P}(g)$ avoid sure loss.

(F) There is a linear prevision x such that $\overline{P}(g) > x(g)$ for all g in $\mathcal{K}$.

(G) $\overline{P}$ dominates a non-linear linear-vacuous mixture.

(H) $\overline{P}$ avoids sure loss and $\underline{E}(g) < \overline{E}(g)$ for all g in $\mathcal{K}$.

Proof. (A) and (B) are equivalent by Definition 6(A).

(A) and (C) are equivalent, again by Definition 6(A), once established that the system of linear inequalities and equalities of ($\mathbf{D}$) is *always* regular. Indeed, it suffices to show that

$$c_f \in \operatorname{int}(Q^* A - P^*)$$

This holds trivially because

$$Q^* A - P^* = \left\{ a + \textstyle\sum_{g \in \mathcal{K}} \lambda_g g - p^* : \dots \right\} = \mathbb{R}^n$$

as we vary over all $a \in \mathbb{R}$ and all $p^* \in P^*$.

(C) implies (D), by Theorem 4. [One can also quickly see that (F) implies (D) by [5, Theorem 2]—also see the discussion at [6, p. 444].]

Equivalence between (D) and (E) follows from Theorem 4, once noted that we only need to consider perturbations in $\overline{P}$ because probabilities sum one—whence every small perturbation in $A^-_{\overline{P}}$ and $b^-_{\overline{P}}$ can be bounded by a proportionally small perturbation in $b^-_{\overline{P}}$ only—and the usual properties of avoiding sure loss with respect to dominating upper previsions.

Equivalence between (E), (F), (G), and (H) follows trivially from the usual properties of lower previsions.

Finally, we establish equivalence between (C) and (F).

We rely on Robinson's regularity condition, $b_{\overline{P}} \in \operatorname{int}(A_{\overline{P}}P + Q)$. It is satisfied if and only if there is an $\epsilon > 0$ such that

$$b_{\overline{P}} + \epsilon B \subseteq A_{\overline{P}}P + Q$$

where B is the closed unit ball in $Y = \mathbb{R}^{\mathcal{K}}$, that is, the set $\{b \in Y\colon \sup|b| \leq 1\}$. Equivalently, now in matrix notation, we need that

$$\begin{bmatrix} 1 \\ \overline{P}(g_1) \\ \vdots \\ \overline{P}(g_k) \end{bmatrix} + \epsilon B \subseteq \left\{ \begin{bmatrix} x(1) \\ x(g_1) + y_1 \\ \vdots \\ x(g_k) + y_k \end{bmatrix} : x \in P,\, y \in Q \right\}.$$

Equivalently, there must be some $\epsilon > 0$ such that, for every $b \in B$ (that is, $b_i \in [-1, 1]$), there is an $x \in P$ and a $y \in Q$ such that

$$\begin{aligned} 1 + b_0\epsilon &= x(1) \\ \overline{P}(g_i) + b_i\epsilon &= x(g_i) + y_i \text{ for all } i \in \{1, \dots, k\}. \end{aligned}$$

If the above is satisfied, take $b_0 = 0$ and $b_1 = \cdots = b_n = 1$ to find that $\overline{P}(g_i) > x(g_i)$ for all i, and note that $x \in C$ because $b_0 = 0$.

Conversely, if there is some x' such that $\overline{P}(g_i) > x'(g_i)$ for all i, then the above is satisfied for sufficiently small ϵ. Indeed, fix any $0 < \epsilon < 1$, and let $x = (1 + b_0\epsilon)x'$—obviously $x \in P$, and the first equality is satisfied. The second equality can be satisfied as well, because

$$\overline{P}(g_i) - \epsilon \geq \max_{b'_0 \in \{-1,1\}} (1 + \epsilon b'_0)x'(g_i)$$

can always be achieved for small enough ϵ, because $\overline{P}(g_i) > x'(g_i)$, whence, for such ϵ,

$$\begin{aligned} \overline{P}(g_i) + b_i\epsilon &\geq \overline{P}(g_i) - \epsilon \\ &\geq \max_{b'_0 \in \{-1,1\}} (1 + \epsilon b'_0)x'(g_i) \\ &\geq (1 + \epsilon b_0)x'(g_i) = x(g_i) \end{aligned}$$

which concludes the proof. □

Informally, the canonical representation is stable if and only if $\overline{P}$ is inherently imprecise. This also means that we can always enforce stability by perturbation, for any upper prevision that avoids sure loss: simply mix $\overline{P}$ with a stable one, such as the vacuous upper prevision:

$$(1 - \alpha)\overline{P} + \alpha \sup_{\omega \in \Omega}$$

is *always* stable, for any $\alpha \in (0, 1]$. So, every upper prevision that avoids sure loss has arbitrarily close stable approximations.

Note that the natural extension of the above perturbation will not necessarily behave nicely as a function of α, particularly when $\overline{P}$ is unstable. For instance, in the perturbed example of Section 2.2, $\overline{E}(2I_b + 2I_c) = 1$ if $\alpha \ll \epsilon$ and $\overline{E}(2I_b + 2I_c) = 2$ if $\alpha \gg \epsilon$. In essence, one should pick α large enough to counter any (presumably unintended) implicit linearities, or near linearities.

If, for some reason, approximation is not an option, we have to find a stable representation. The conditions under which this is possible are uncovered in the next section.

4.3 Necessary and Sufficient Conditions for Stable Representations of Natural Extension

Definition 8. *A system of linear inequalities and equalities is said to be a* representation *of another system if it has the same feasible region F as that system.*

Definition 9. *A linear program is said to be a* representation *of another linear program if it has the same feasible region F and objective function as that linear program.*

Theorem 10. *Let $\overline{P}$ be any upper prevision. The following conditions are equivalent.*

(A) The linear program (**P**) *has a stable representation.*

(B) The linear program (**D**) *has a stable representation.*

(C) The system of linear inequalities and equalities of (**P**) *has a regular representation.*

(D) The system of linear inequalities and equalities of ($\mathbf{P}^-$) *has a regular representation.*

(E) There is a linear prevision x in the credal set F of $\overline{P}$ such that $x(\omega) > 0$ for all $\omega \in \Omega$.

(F) $\overline{P}$ avoids sure loss and $\overline{E}(I_\omega) > 0$ for all $\omega \in \Omega$.

Proof. The first part of the proof is similar to the proof of Theorem 7(A)&(B)&(C): again, the key observation is that the system of the dual is always regular. We also rely on the fact that the dual of a representation is a representation of the dual.

(C) $\iff$ (E). Such x belongs precisely to $F \cap \operatorname{ri} P$. Apply Theorem 5.

(D) $\iff$ (E). Such x belongs precisely to $F \cap \operatorname{ri} C$. Apply Theorem 5.

(E) $\implies$ (F). Immediate, because

$$\overline{E}(I_\omega) = \sup_{x' \in F} x'(\omega) \geq x(\omega) > 0.$$

(F) $\implies$ (E). Condition (F) implies that, for every ω, there is an x_ω in F such that $x_\omega(\omega) > 0$. Take any convex mixture x of x_ω with non-zero coefficients. Because F is convex, x belongs to F. Clearly, $x(\omega) > 0$ for all ω in F. □

The condition for having a stable representation is clearly much weaker than the one for stability: in essence, we only need to ensure that no singleton has zero upper probability. Again, it is obvious that this can be achieved by an arbitrary small perturbation, for any upper prevision that avoids sure loss: simply mix $\overline{P}$ with a linear prevision x that satisfies $x(\omega) > 0$ for all ω, such as the uniform one:

$$(1-\alpha)\overline{P} + \alpha \frac{1}{n} \sum_{\omega \in \Omega}$$

where n is the cardinality of Ω, *always* has a stable representation, for any $\alpha \in (0, 1]$. So, every upper prevision that avoids sure loss has arbitrarily close approximations that admit stable representations, and whose canonical representation is stable if and only if the canonical representation of $\overline{P}$ is stable (indeed, by Theorem 7!).

4.4 Finding the Stable Representation

Every reasonably advanced application for working with systems of linear inequalities and equalities has routines for finding all redundant constraints and all implicit linearities (see for instance [1]), effectively recovering the stable representation, when it exists.

5 Discussion and Conclusion

We have linked Robinson's stability criterion for systems of linear inequalities and equalities, and for linear programming, to the theory of upper previsions. We found a range of interesting necessary and sufficient conditions for the usual canonical representations of natural extension to be robust against perturbations, that is, to be stable. Thereby, we provided theoretical guarantees for small changes in the assessments not to have a large impact on any inferences made.

This is obviously rather useful in elicitation: if a subject makes assessments which violate stability, then the subject should at least be made aware of this. We provided a simple tool to fix unstable assessments, through perturbation with a vacuous model.

In case of instability of the canonical constraints, a subject could be unhappy to perturb with a vacuous model, for instance because she insists on certain assessments to be precise. We found that a stable representation may still exist after removal of redundant constraints and recognition of implicit linearities. Tools for doing so are readily available in the literature. Of course, it is *mandatory* to check that the subject actually agrees with the reduced system, and particularly that any linearities, or near linearities, are in agreement with her beliefs. When in doubt, we recommend the vacuous mixture.

In case the reduced system is still unstable, we found that it can be made stable via perturbation with for instance a uniform probability mass function—this may be preferred over vacuous perturbation in case the subject insists on particular assessments to remain precise.

In conclusion, we characterized the robustness of natural extension in a variety of ways, and we provided straightforward ways to work around instabilities by means of perturbation.

Many open problems remain, including the extension to non-finite spaces, and conditional lower previsions, which are typically solved by sequences of linear programs [9], and thus for which stability may be much harder to characterize.

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