Rationalizability under Uncertainty

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Abstract

The game-theoretic solution concept called rationalizability ([1], [4]) captures the idea of rational behavior constrained only by the common knowledge that each player maximizes expected utility with respect to a single personal probability distribution representing uncertainty. Here I generalize the concept of rationalizability by using sets of probabilities to model uncertainty in games, and examine how game theory can be informed by introducing imprecise probability when it is common knowledge among players that each player maximizes the minimum expectation (known as Γ-maximin, see [2]).

Consider a finite normal form game \( G = \{ I, \{ S_i \}, \{ u_i \} \}_{i \in I} \), where \( I \) denotes a finite set of players, \( S_i \) denotes the finite set of actions of player \( i \), and \( u_i : S \rightarrow \mathbb{R} \) denotes player \( i \)'s payoff function (where \( S = \prod_{i \in I} S_i \)). And let \( \Delta_i \) denote the set of player \( i \)'s mixed strategies, which can be regarded as probability measures on \( S_i \).

Rationalizability requires that each player maximizes her own expected payoff against her belief about the opponents’ strategy choices. A belief of player \( i \) about the other players’ strategy choices in a game \( G \) is a probability distribution over the set of actions \( S_{-i} = \prod_{j \neq i} S_j \). Note that this formulation of beliefs allows a player to hold a belief that the other players’ actions are correlated. We say that a strategy \( \delta_i \in \Delta_i \) is rational if there exists a belief \( \delta_{-i} \in \Delta_{-i} \) such that \( \delta_i \) maximizes player \( i \)'s expected payoff. In this case, \( \delta_i \) is called a best response to the belief \( \delta_{-i} \). We then formulate the concept of rationalizability as follows ([3]).

Definition 1 (Rationalizability) In a game \( G \), an action \( s_i \) of player \( i \) is rationalizable if for each player \( j \in I \) there exists a set \( Z_j \) of actions such that (i) \( s_i \in Z_i \), and (ii) for each player \( j \in I \), every action \( s_j \in Z_j \) is a best response to a belief \( \delta_{-j} \) of player \( j \) that assigns positive probability only to those actions in \( Z_{-j} \).

In analogy with rationalizability, the new solution concept we call Γ-maximin rationalizability captures the idea that each player believes that her opponents maximizes their own minimum expected payoff with respect to their conjectures about the other players’ strategies. A conjecture \( C_{-i} \) of player \( i \) about her opponents’ strategy choices is a nonempty, closed, and convex set of probability measures on \( S_{-i} \). And a strategy \( \delta_i \in \Delta_i \) is called rational under uncertainty if there exists a conjecture \( C_{-i} \) such that \( \delta_i \) maximizes player \( i \)'s minimum expected payoff with respect to \( C_{-i} \). In this case, we say that \( \delta_i \) is Γ-maximin admissible relative to \( C_{-i} \). We then define:

Definition 2 (Γ-maximin Rationalizability) In a game \( G \), an action \( s_i \) of player \( i \) is Γ-maximin rationalizable if for each player \( j \in I \) there exists a set \( A_j \) of actions such that (i) \( s_i \in A_i \), and (ii) for each player \( j \in I \), every action \( s_j \in A_j \) is Γ-maximin admissible relative to a conjecture \( C_{-j} \) of player \( j \) such that each probability measure in \( C_{-j} \) assigns positive probability only to those actions in \( A_{-j} \).

Clearly, Γ-maximin rationalizability has rationalizability as a special case when all players’ conjectures are comprised by a single probability measure. In order to illustrate the difference between these two solution concepts, consider the \( 3 \times 2 \) game shown to the left. Assume that each player’s feasible options are pure strategies only, that is, explicit randomization is excluded; no non-trivial mixed strategy is feasible for each player.

Note that row player’s action \( M \) is never a best response to any precise conjecture over \( \{ L, R \} \). Thus, the only rationalizable actions for both players are \( D \) and \( R \) respectively. However, all actions in this game are Γ-maximin rationalizable. The crucial part of this claim is to argue that row player’s action \( M \) is Γ-maximin rationalizable. This can be shown by considering the following case: let \( A_1 = \{ U, M \} \) and \( A_2 = \{ L, R \} \) be the sets of actions for row and column player respectively. Assume that row and column player’s conjecture is depicted respectively by \( C_{-1} = \{ P_1(\cdot) : 0 \leq P_1(R) \leq 0.6 \} \) and \( C_{-2} = \{ P_2(\cdot) : P_2(D) = 0, 0 \leq P_2(U) \leq 1 \} \). Then, the action \( M \) is Γ-maximin rationalizable. Based on this, it is easy to verify that all the other actions of both players are Γ-maximin rationalizable.

Keywords. Uncertainty, sets of probabilities, game theory, rationalizability, Γ-maximin rationalizability.

References