Stochastic PDEs with Random Set Coefficients

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Abstract
This contribution addresses stochastic PDEs with random set coefficients. A typical example is the elliptic PDE
\[-\text{div} \left( A(x) \text{grad} u(x) \right) = f(x)\]
where the excitation and the coefficient matrix are given by any of the following: (a) a random field (a stochastic
process with respect to the spatial variable); (b) a random set; (c) a random field whose parameters are random
sets; (d) a combination thereof. As soon as random sets and stochastic processes are involved, the solution
\(u\) is a set-valued process. The question arises in what sense it can be viewed as a random set.

For a stationary, Gaussian random field \(A\) it suffices to specify the expectation values \(\mu \equiv E(A(x))\) and the
autocovariance function \(C(\rho) = \text{COV}(A(x), A(y))\) which then depends only on the distance \(\rho = |x - y|\). As a
starting point, we consider a parametrized autocovariance function of the form \(C(\rho) = \sigma^2 \exp \left(- \frac{|\rho|}{L}\right)\)
with the field variance \(\sigma^2\) and the correlation length \(L\) as parameters. A useful feature of this type of random field
is that it can be obtained as solution to the Langevin equation, \(W_t\) denoting Wiener process,
\[
dX_t = -\frac{1}{L} X_t + \sqrt{\frac{2}{L}} \sigma \, dW_t, \quad X_0 \sim \mathcal{N}(0, \sigma^2).
\]

A random set is a map \(X\) which assigns to every \(\omega\) from a probability space \((\Omega, \Sigma, P)\) a subset \(X(\omega)\) of a target
space \(E\) such that the upper inverses \(X^{-}(B) = \{\omega \in \Omega : X(\omega) \cap B \neq \emptyset\}\) are measurable for every Borel subset
\(B\) of \(E\). An important tool is the fundamental measurability theorem that states (if \(E\) is a Polish space) the equivalence of the defining measurability property of \(X^{-}(B)\) for Borel, open, and closed subsets \(B\) as well as
the equivalence with the existence of a Castaing representation. A set-valued random variable such that \(X^{-}(B)\)
is measurable for every open set \(B\) is called Effros-measurable. Starting from a random field whose correlation
length, e.g., is an interval, the assignment
\[
\omega \mapsto \{A_L(x, \omega) : L \in [L_1, L_2]\},
\]
where \(x\) is a point in space and \(A_L(x, \omega)\) is a realization at point \(x\) of the field with correlation length \(L\),
defines a random set. It is the purpose of this contribution to present a proof of this fact. Thanks to the
representation (1), the continuity of the map \(L \to A_L(x, \omega)\) can be derived from the results of [1, 2]. From
there, a Castaing representation can be immediately obtained, which leads to the Effros measurability; the
fundamental measurability theorem completes the argument. The methods will be demonstrated at the hand
of a numerical example, employing polynomial chaos expansion as a computational device.

Keywords. Random fields, random sets, set-valued stochastic processes.

References
[2] B. Schmelzer. Set-valued assessments of solutions to stochastic differential equations with random set pa-