Convergence of Continuous-Time Imprecise Markov Chains

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We provide necessary and sufficient conditions for the unique convergence of a continuous-time imprecise Markov chain to a stationary distribution.

Problem Statement. Consider the set of all the continuous-time non-stationary Markov chains with finite state space $\mathcal{X}$ of which the transition rate matrix $Q_t$ is a function of time such that $Q_t \in \mathcal{Q}$, where $\mathcal{Q}$ is a closed convex set of transition rate matrices that has separately specified rows, meaning that

$$Q \in \mathcal{Q} \iff (\forall x \in \mathcal{X}) \ Q(x, \cdot) \in \mathcal{Q}_x$$

where, for all $x \in \mathcal{X}$, $\mathcal{Q}_x$ is a closed convex set of row vectors. We call such a set of Markov chains a continuous-time imprecise Markov chain.

Fix any $t > 0$. Then for all $f \in \mathbb{R}^{\mathcal{X}}$ and $x \in \mathcal{X}$, the expected value $E_t(f|X_0 = x)$ of $f$ at time $t$, conditional on $X_0 = x$, ranges over a closed interval of which we will denote the lower bound by $T_x(f|x)$. For all $x \in \mathcal{X}$, $T_x(\cdot|x)$ is a coherent lower prevision on $\mathbb{R}^{\mathcal{X}}$. The corresponding lower transition operator $T_x : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$ is defined by

$$T_x f(x) := T_x(f|x) \text{ for all } x \in \mathcal{X}.$$ 

By a recent result of Skulj [1], $\int_0^t := T_t f$ is the solution to the differential equation

$$\frac{d}{dt} f_t = Q f_t$$

with initial condition $f_0 = f$, where for all $h \in \mathbb{R}^{\mathcal{X}}$:

$$Q h(x) := \min_{Q \in \mathcal{Q}} \sum_{y \in \mathcal{X}} Q(x, y) h(y)$$

for all $x \in \mathcal{X}$.

We study the limit behaviour of $T_x$. In particular, we provide necessary and sufficient conditions for $\mathcal{Q}$ to be Perron-Frobenius-like (PF), meaning that there is some $P_\infty : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}$ such that, for all $x \in \mathcal{X}$:

$$\lim_{t \to +\infty} T_x f(x) = P_\infty f \text{ for all } f \in \mathbb{R}^{\mathcal{X}},$$

or, equivalently, for $T_x(\cdot|x)$ to converge to a stationary distribution $P_\infty$ that does not depend on $x$.

Results. Our main result is that the following four conditions are equivalent:

1. $\mathcal{Q}$ is PF,
2. $T_x$ is PF for some $t > 0$,
3. $T_x$ is PF for all $t > 0$,
4. $\mathcal{Q}$ is regularly absorbing,

where (i) for any $t > 0$, we say that $T_x$ is PF if the discrete-time imprecise Markov chain that has $T_x$ as its lower transition operator is PF, in the sense that, for all $f \in \mathbb{R}^{\mathcal{X}}$, $\lim_{n \to \infty} T_n f$ exists and is constant and (ii) "regularly absorbing" is a qualitative property of $\mathcal{Q}$ that is fully determined by the signs of the upper transition rates to singletons $Q(x, y) := \max_{Q \in \mathcal{Q}} Q(x, y)$ and the lower transition rates to sets $\overline{Q}(x, A) := \min_{Q \in \mathcal{Q}} \sum_{y \in \mathcal{X}} Q(x, y)$, for $x, y \in \mathcal{X}$, $x \neq y$ and $A \subset \mathcal{X} \setminus \{x\}$. See the poster for more details.

As future work, we would like to develop coefficients of ergodicity that characterise whether $\mathcal{Q}$ is PF and that provide—tight—bounds on the rate of convergence.

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References