Introduction

- Conformity of a marginal and a conditional lower prevision means that they can be derived from some joint by means of natural extension.
- We study this notion, also together with assumptions of epistemic irrelevance and independence between the variables.
- The connection with independent products, and in particular with the strong product, is also investigated.
- Settings: two variables $X_1$, $X_2$ taking values in possibility spaces $X_1$, $X_2$ (not necessarily finite).

BASIC NOTIONS:

Let $L(X_1 \times X_2) := \{ f : X_1 \times X_2 \rightarrow \mathbb{R} \text{ bounded} \}$.

$P : L(X_1 \times X_2) \rightarrow \mathbb{R}$ is a coherent lower prevision when it is the lower envelope of a family of expectations with respect to finitely additive probabilities.

Its restrictions to $X_1, X_2$—measureable gambles is its marginals $P_{X_1}, P_{X_2}$.

Similarly, given $x_1 \in X_1$, $P(\cdot|x_1) : L(X_1 \times X_2) \rightarrow \mathbb{R}$ is a conditional coherent lower prevision when it is the lower envelope of a family of conditional expectations with respect to finitely additive probabilities.

In that case, $P(\cdot|X_1) := \sum_{x_2 \in X_2} I_{x_2} \cdot P(x_1,x_2)$ is a separately coherent conditional lower prevision.

Results

Preliminary concepts

Conditional natural extension: given $P$ on $L(X_1 \times X_2)$ and $x_2 \in X_2$, $E(f(x_2))$ is given by

$$\sup_{\mu} \{ \mu : P(I_{x_2}(f - \mu)) \geq 0 \}$$

if $P(x_1) > 0$ otherwise.

$$E(f|X_1), E(f|X_2)$$ satisfy irrelevance w.r.t $P_{X_1}, P_{X_2}$ when

$$E(f(x_1)) = P_{X_1}(f(x_1, \cdot))$$

for all $f \in L(X_1 \times X_2), x_1 \in X_1$.

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for all $f \in L(X_1 \times X_2), x_1 \in X_1, x_2 \in X_2$.

$P$ is an independent product of the marginals $P_{X_1}, P_{X_2}$ when $P_{X_1}, P(\cdot|X_1)$, $P(\cdot|X_2)$ are coherent, where $P(\cdot|X_1)$, $P(\cdot|X_2)$ are defined by epistemic irrelevance. The smallest one is the independent natural extension $P_{X_1} \otimes P_{X_2}$.

An independent product is an independent envelope of its marginals when it is a lower envelope of factorising previsions. The smallest is the strong product $P_{X_1} \otimes P_{X_2}$, given by

$$\min\{P_1 \times P_2 : P_1 \geq P_{X_1}, P_2 \geq P_{X_2}.\}$$

Definitions

$P_{X_1}, P_{X_2}$ are conforming when there exists $P$ on $L(X_1 \times X_2)$ with marginal $P_{X_1}$ and conditional natural extension $P(\cdot|X_1)$.

$P_{X_1}, P_{X_2}$ are conforming with $X_1$-$X_2$ irrelevance when there exists $P$ with marginals $P_{X_1}, P_{X_2}$ and whose conditional natural extension $E(\cdot|X_1)$ satisfies irrelevance w.r.t $P_{X_2}$.

We say that $P_{X_1}, P_{X_2}$ are conforming with $X_1$-$X_2$ independence when there exists $P$ with marginals $P_{X_1}, P_{X_2}$ whose conditional natural extensions $E(\cdot|X_1), E(\cdot|X_2)$ satisfy irrelevance w.r.t $P_{X_1}, P_{X_2}$.

Consider $P$ with marginals $P_{X_1}, P_{X_2}$, and the conditions:

$$E(f) \leq E(P_{X_1}(f|X_2))$$ \hspace{1cm} (1)

and

$$P(g - f) \geq \min_{x_1 \in X_1} P(g - f^{x_1})$$ \hspace{1cm} (2)

for all $f, g \in L(X_1 \times X_2), P_{X_1} \geq P_{X_1}$, where

$$f^{x_1} : X_1 \times X_2 \rightarrow \mathbb{R}$$

$$(x_1, x_2) \mapsto f(x_1, x_2).$$

Results

- $P_{X_1}, P(\cdot|X_1)$ are conforming $\iff$ $P(\cdot|x_1)$ is vacuous whenever $P_{X_1}(x_1) = 0$.
- If $X_1$ is finite and there is some $P$ conforming with $P_{X_1}, P(\cdot|X_1)$, the smallest one is $P_{X_1}(P(\cdot|X_1))$.

Let $\mathcal{P}_{X_1}(P_{X_1})$ be the set of these compatible joints.

$\mathcal{P}_{X_1}(P_{X_1}) \neq \emptyset$ $\iff$ (a) either $P_{X_1}(x_1) > 0$ for every $x_1$ or $P_{X_1}$ is vacuous.

$\mathcal{P}_{X_1}(P_{X_1})$ is closed under lower envelopes.

If $X_1$ is finite and $\mathcal{P}_{X_1}(P_{X_1}) \neq \emptyset$, the smallest model in this set is $P_{X_1}(P(\cdot|X_1))$, where $P(\cdot|X_1)$ is derived from $P_{X_1}$ by irrelevance.

Thus, conforming natural extension = irrelevant natural extension when $X_1$ is finite (but not in general).

Let $\mathcal{F}_{X_1}(P_{X_1})$ be the set of these compatible previsions.

$\mathcal{F}_{X_1}(P_{X_1}) \neq \emptyset$ $\iff$ (a) either $P_{X_1}(x_1) > 0$ for every $x_1$ or $P_{X_1}$ is vacuous; and (b) either $P_{X_1}(x_1) > 0$ for every $x_1$ or $P_{X_1}$ is vacuous.

If $\mathcal{F}_{X_1}(P_{X_1}) \neq \emptyset$, then any independent product of $P_{X_1}, P_{X_2}$ belongs to $\mathcal{F}_{X_1}(P_{X_1})$.

Essential references


At a glance

Results:
- Conformity clashes with the existence of zero lower probabilities.
- In the finite case the conforming natural extension becomes the irrelevant/independent natural extension.
- We have a behavioural characterisation of the strong product.

Open problems:
- Study this problem with other updating rules, like regular extension.
- Extension to more than two models.
- Connection with sets of gambles.