Optimal control of linear systems with quadratic cost and imprecise forward irrelevant input noise

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Linear systems

We consider a finite-state, discrete-time scalar linear system with a deterministic (known) current state \( x_t = x_t \). For all \( t \in \{k, \ldots, k_1\} \), the dynamics of the system is described by

\[
X_{t+1} = aX_t + bU_t + W_t.
\]

(DYN)

In this expression, \( a_t \) and \( b_t \) are real-valued parameters and the state \( X_t \) and noise \( W_t \), at time \( t \), are real-valued random variables. The control input \( u_t \) at time \( t \) is also real-valued.

State feedback Usually, the control input \( u_t \) is taken to be some real-valued function \( v_t \) of the previous states \( X_{t+1} \): \( u_t := (X_{t+1},X_{t+2},\ldots) \), called a feedback function. As the current state \( x_t \) is known, \( v_t \) is a constant. We call a tuple of feedback functions \( \psi_t := (v_t, v_{t+1}, \ldots, v_{k_1}) \) a control policy. We use \( \Psi_{k_1} \) to denote the set of all control policies \( \psi_{k_1} \).

LQ cost functional We measure the performance of a control policy \( \psi_{k_1} \) by means of the associated cost. For all \( t \in \{0, \ldots, k_1\} \), all \( \psi_t \in \Psi_{k_1} \), and all \( x_t \in \mathbb{R} \) we define the linear-quadratic (LQ) cost functional \( \eta \) as

\[
\eta(\psi_{k_1},x_t) := \sum_{t=1}^{k_1} r_t \psi_t(X_t)^2 + q_t X_t^2,
\]

where \( r_t > 0 \) and \( q_t > 0 \) are real coefficients.

Precise noise model \( \mathcal{P} \)

In order to model the noise \( W_{k_1} := (W_1, W_2, \ldots, W_{k_1}) \), we consider an initial time \( k_0 \), let \( k_0 \leq k \leq k_1 \), and focus on modelling \( W_{k_1} \).

Precise noise model We model our beliefs about \( W_{k_1} \) using conditional probability density functions: for all \( k \in \{k_0, \ldots, k_1\} \) and all \( w_{k-1} \in \mathbb{R}^{k-1} \), we are given a conditional probability density function \( \tilde{P}_k(w_{k_1-1}) \), and we use \( \tilde{P}_k \) to denote the corresponding conditional linear prevision operator (expectation operator). It then follows from the law of iterated expectation that for any gambles \( \mathcal{g} \) on \( \mathbb{R}^{k-1} \):

\[
\tilde{P}_k(\mathcal{g}(w_{k-1})) = \tilde{P}_k(\tilde{P}_k(\cdots(\tilde{P}_k(\mathcal{g}(w_{k-2})))\cdots(w_{k-1},W_1))(w_{k-1}),
\]

where we assume that our conditional probability density functions are sufficiently well-behaved for the previsions in this expression to exist. We denote the set of all such precise noise models \( \mathcal{P} \) by \( \mathcal{P} \).

White noise model In the literature, it is often assumed that the noise is independent. This means that all the conditional probability density functions (and associated linear previsions) are equal to marginal ones.

Imprecise noise model \( \mathcal{D} \)

Imprecise noise model Our beliefs about \( W_{k_1} \) are modelled by a set \( \mathcal{D} \subseteq \mathcal{P} \) of precise noise models. This definition allows us to use the results obtained in the precise LQ problem.

Forward irrelevant noise model \( \mathcal{D} \) is said to be a forward irrelevant product if there are sets of marginal probability density functions \( \tilde{P}_k, k \in \{k_0, \ldots, k_1\} \), such that \( \mathcal{D} \) is the largest subset of \( \mathcal{P} \) for which it holds that

\[
\tilde{P}_k(w_{k-1}) \in \mathcal{D}
\]

for all precise models \( \mathcal{P} \), all \( k \in \{k_0, \ldots, k_1\} \) and all \( w_{k_1-1} \in \mathbb{R}^{k_1-1} \).

Simulations

How do we choose which element of \( \mathcal{D} \) to apply? We propose two possible options:

1. use the control policy that corresponds to a white noise model
2. lazily choose the \( \bar{k} \in [k_0,\bar{k}] \) that minimises \( |\bar{k}| \).

We ran two simulations to compare their performance

Small difference in cost, but the lazy control has more zero inputs more research is definitely necessary

The precise LQ problem

Local optimality A control policy \( \psi_{k_1} \) is locally optimal for \( x_t \in \mathbb{R} \) and \( w_{k_1-1} \in \mathbb{R}^{k_1-1} \) if

\[
\Psi_{k_1}(\psi_{k_1}|x_t,w_{k_1-1}) = \arg\min_{\psi \in \Psi_{k_1}} \eta(\psi_{k_1},x_t)\] w_{k_1-1}.

Optimality A control policy \( \psi_{k_1} \) is optimal for \( x_t \in \mathbb{R} \) and \( w_{k_1-1} \in \mathbb{R}^{k_1-1} \) if, for all \( t \in \{k_0, \ldots, k_1\} \) and all \( x_{k_1}, w_{k_1-1} \in \mathbb{R}^{k_1-1} \):

\[
\Psi_{k_1}(\psi_{k_1}|x_t,w_{k_1-1}) \in \text{loc-opt}\psi_{k_1}(\forall_{k_1}|x_t,w_{k_1-1}),
\]

where \( \forall_{k_1} \) is derived from (DYN) and \( x_{k_1} \). The set of all such optimal control policies is denoted by \( \text{opt}_{k_1}(\forall_{k_1}|x_t,w_{k_1-1}) \).

Precise noise solution For any current state \( x_t \in \mathbb{R} \) and noise history \( w_{k_1-1} \in \mathbb{R}^{k_1-1} \), the set \( \text{opt}_{k_1}(\forall_{k_1}|x_t,w_{k_1-1}) \) consists of a single optimal control policy. For any \( t \in \{k_0, \ldots, k_1\} \) and \( x_{k_1}, w_{k_1-1} \in \mathbb{R}^{k_1-1} \), it is given by

\[
\Psi_{k_1}(\psi_{k_1}|x_t,w_{k_1-1}) = \arg\min_{\psi \in \Psi_{k_1}} \eta(\psi_{k_1},x_t)\] w_{k_1-1}.

(OCP)

The parameters \( m_{k+1} \) and \( r_{k+1} \) are obtained from the initial condition \( m_{k+1} := q_1 + \ell w_{k+1} - r_{k+1} w_{k+1}^2 \), and the recursive Riccati equation \( m_t := q_t + \ell w_{t+1} - r_{t+1} w_{t+1}^2 \), with \( r_{t+1} = (r_t + \ell w_{t+1})^2 \). The noise feedforward \( h_{k+1} \) is obtained from the initial condition \( h_{k+1} := q_1 + \ell w_{k+1} \).

Calculating this feedforward is intractable!

White noise solution For white noise, the recursive feedforward relation simplifies to

\[
h_t := m_{k+1} P_t(W) + r_{k+1} w_{k+1}^2.
\]

with initial condition \( h_{k+1} = 0 \).

The imprecise LQ problem

E-admissibility A control policy \( \hat{\psi}_{k_1} \) is \( E \)-admissible for \( x_t \in \mathbb{R} \) and \( w_{k_1-1} \in \mathbb{R}^{k_1-1} \) if

\[
\hat{\psi}_{k_1} \in \text{opt}_{E}(\forall_{k_1}|x_t,w_{k_1-1}) := \bigcup_{\hat{\mathcal{P}}} \text{opt}_{E}(\forall_{k_1}|x_t,w_{k_1-1})\]

Imprecise noise solution Every \( \hat{\mathcal{P}} \in \mathcal{D} \) corresponds to a single \( E \)-admissible control policy \( \hat{\psi}_{k_1} \)—see Equation (OCP)—that is a combination of the same state feedback and a possibly different noise feedforward. Calculating all possible feedforwards is intractable!

Forward irrelevant noise solution If \( \mathcal{D} \) is a forward irrelevant product, then for all \( t \in \{k_0, \ldots, k_1\} \) and all \( w_{k_1-1} \in \mathbb{R}^{k_1-1} \):

\[
h_{k_1-t} \in \mathcal{D} \subset \mathcal{P}
\]

for all \( k \in \{k_0, \ldots, k_1\} \) and all \( w_{k_1-1} \in \mathbb{R}^{k_1-1} \).

Convergence For stationary linear systems (constant \( a_t, b_t, c_t, d_t \) and \( \mathcal{D} \)), and large \( k_1 - k \), the parameters \( m_k, r_k, \tilde{P}_k, \tilde{P}_k(W) \) and \( \tilde{P}_k(W) \) converge to easily calculable limit values.