Efficient $L_1$-Based Probability Assessments Correction: Algorithms and Applications to Belief Merging and Revision

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A precise probability assessment is a quadruple $\pi = (V, U, p, C)$, where

- $V = \{X_1, \ldots, X_n\}$ is a finite set of propositional variables
- $U$ is a subset of $V$ that contains the effective events taken into consideration
- $p : U \to [0, 1]$ assigns a probability value to each variable in $U$
- $C$ is a finite set of logical constraints which lie among all the variables in $V$
A precise probability assessment is coherent if there exists a probability distribution $\mu : 2^V \rightarrow [0, 1]$ on the set of all truth-value assignment $2^V$ which satisfies the following properties:

1. For each $\alpha \in 2^V$, if there exists a constraint $c \in C$ such that $\alpha \not|= c$, then $\mu(\alpha) = 0$;
2. $\sum_{\alpha \in 2^V} \mu(\alpha) = 1$;
3. For each $X \in U$, $\sum_{\alpha \in 2^V, \alpha |= X} \mu(\alpha) = p(X)$. 

**Coherence of probability assessment**
INCOHERENCE

• What to do if the probability assessment is not coherent?
• A possible solution is to correct $p$ in $p'$ in a way that
  • $\pi' = (V, U, p', C)$ is coherent
  • $p'$ is as close as possible to $p$
• The correction is then a constrained minimization problem
• This approach follows the principle of minimum change of belief revision
• A distance between probability assessments is needed
L1 correction

• In this paper we use the L1 distance

\[ d_1(p, p') = \sum_{i=1}^{n} |p(X_i) - p'(X_i)| \]

• L1-distance minimization has a simple interpretation, since it implies a direct minimal modification of each single probability value

• Moreover, the related correction procedure has a much lower computational cost than other distances

• Note that the correction is not unique, i.e. there can be infinitely many corrections for an incoherent assessment

• Anyway, all the corrections form a convex set \( C(\pi) \)
It is possible to convert the problem of checking the coherence of a probability assessment into a mixed integer programming (MIP) problem [Cozman]

There exists fast procedures for solving MIP problems, even if this problem is NP-hard

We shortly describe the procedure **Correct**

The distance $\delta = d_1(p, p')$ between the original probability vector $p$ and any of its corrections $p'$ can be computed with a MIP program similar to the program for checking the coherence
**Procedure Correct**

- If $\delta = 0$, $p$ is already coherent and no correction is needed.
- Otherwise, we want to find the extremal points $q_1, \ldots, q_s$ of $C(\pi)$.
- Indeed $C(\pi) = Q \cap B_{\pi}(\delta)$ where
  - $Q$ is the convex set (polytope) of all vectors $q$ such that $(V, U, q, C)$ is coherent.
  - $B_{\pi}(\delta)$ is the ball of all vectors $q$ such that $d_1(p, q) \leq \delta$.
- Fast procedures for face-enumeration and vertex-enumeration can be used to compute the result.
We correct the following incoherent assessment with variables:

- $D \equiv \text{“the athlete uses banned performance-enhancing drugs” (i.e. ”doping”)}$
- $E \equiv \text{“the athlete is showing a performance-enhancing in the last period”}$
- $H \equiv \text{“the athlete is showing a significant change in his/her biological profile”}$

- probability values $p(D) = 0.9$, $p(E) = 0.8$ and $p(H) = 0.9$
- logical constraint $\mathcal{C} = \{ E \lor H, \neg D \lor E, \neg D \lor H \}$
EXAMPLE
Belief merging

- Given two coherent probability assessments $\pi_1 = (\mathcal{V}, U, p, \mathcal{C})$ and $\pi_2 = (\mathcal{V}, W, q, \mathcal{D})$, on the same propositional variables $\mathcal{V}$, we want to find a probability assessment $\pi_3$ as fusion of $\pi_1$ and $\pi_2$
- The basic procedure is
  - Join together $\pi_1$ and $\pi_2$ in a incoherent probability assessment $\pi'_3$
  - Correct $\pi'_3$
- We propose two approaches to perform the first operation
Belief merging I

• The first approach is to compute a “weighted average” of $\pi_1$ and $\pi_2$ with weights $\omega$ and $1 - \omega$.

• We define $\pi_1 + \omega \pi_2$ as the probability assessment $(V, U \cup W, r, C \cup D)$, where $r : U \cup W \rightarrow [0, 1]$ is now defined:

$$r(x) = \begin{cases} 
  p(x) & \text{if } x \in U \setminus W \\
  q(x) & \text{if } x \in W \setminus U \\
  \omega p(x) + (1 - \omega)q(x) & \text{if } x \in U \cap W 
\end{cases}$$

• The merging operator is defined as

$$\pi_1 \oplus_\omega \pi_2 = \text{Correct}(\pi_1 + \omega \pi_2)$$
**Example**

- Let $W = \{E, H, X_4 = (\neg D \land E \land H)\}$ and
- $\mathcal{D} \equiv \mathcal{C} \cup \{\neg D \lor \neg X_4, E \lor \neg X_4, H \lor \neg X_4\}$
- Let $\pi_1 = (V, W, \overline{p}, \mathcal{D})$ with
  \[\overline{p}(D) = 0.833, \overline{p}(E) = 0.867, \overline{p}(H) = 0.967 \text{ and } \overline{p}(X_4) = 0;\]
- Let $\pi_2 = (V, W, q, \mathcal{D})$ with
  \[q(E) = 0.867, q(H) = 0.967, q(X_4) = 0.01\]
- Choosing $\omega = \frac{1}{2}$, we have the starting weighted assessment $\pi_1 + \frac{1}{2} \pi_2$ with components $V, U \cup W = (D, E, H, X_4)$,
  \[r = (0.8333, 0.8667, 0.9667, 0.005)\]
Example

- $\pi_1 + \frac{1}{2} \pi_2$ is incoherent with an $L1$ minimal distance $\delta = 0.01$
- The correction $\pi_1 \oplus \frac{1}{2} \pi_2$ is the credal set with extremal values

\[
q_1 = (0.8333, 0.8742, 0.9667, 0.0075) \\
q_2 = (0.8308, 0.8642, 0.9667, 0.00) \\
q_3 = (0.8333, 0.8667, 0.9742, 0.0075) \\
q_4 = (0.8308, 0.8667, 0.9642, 0.00) \\
q_5 = (0.8358, 0.8692, 0.9667, 0.00) \\
q_6 = (0.8258, 0.8667, 0.9667, 0.0075)
\]
Belief merging II

- A different approach is to create a probability assessment which maintains both numerical values.
- The apparent contradiction is solved
  - by adding a new logical variable $X'_i$, for each event $X_i \in U \cap W$ such that $p(X_i) \neq q(X_i)$, and
  - by assigning the values $r(X_i) = p(X_i)$ and $r(X'_i) = q(X_i)$.
  - Moreover, the logical constraint $X_i = X'_i$ is added to $\mathcal{C} \cup \mathcal{D}$.
- $\pi_1 + \pi_2$ is obviously incoherent and the merging operation of $\pi_1$ and $\pi_2$ is computed as

$$\pi_1 \oplus_I \pi_2 = \text{Correct}(\pi_1 + \pi_2).$$
Example

- As in the previous example, but we add a new event $X'_4$
- We start with the assessment $\pi_1 + \pi_2$ with components $V$, $U' = (D, E, H, X_4, X'_4)$,
  $$r = (0.8333, 0.8667, 0.9667, 0.00, 0.01)$$
- The logical constraints have also $\neg X_4 \lor X'_4$, $X_4 \lor \neg X'_4$
- The correction leads now to a precise assessment with numerical values $(0.8333, 0.8667, 0.9667, 0.00, 0.00)$
**Comparison**

- The main difference between the two approaches is that $\oplus_I$ tries to automatically solve the contradiction, while the operator $\oplus_\omega$ needs an explicit way of solving it.
- The approach of $\oplus_\omega$ is in some sense a supervised one, because the user must explicitly provide a weight $\omega$,
- While $\oplus_I$ adopts an unsupervised approach, and these difference can leads to very different final results
- Thinking the probability assessments as belief states, the merging operators are a belief merging functions
Belief revision

• Suppose that $\pi_1 = (V, U, p, C)$ represents our current belief state and a new reliable information $\pi_2 = (V, W, q, D)$ arrives.

• We want to update our belief state with the new available information, with the idea that
  • we assume that the new information $\pi_2$ is correct
  • we allow to revise, as less as possible, $\pi_1$ in order to adapt it to the new information

• The revision can be performed as follows.
  • $\pi_1$ and $\pi_2$ are merged together with the operator $+_0$,
  • The resulting assessment is corrected by forbidding any change on the probabilities of the variables in $W$.

• The revision of $\pi_1$ with $\pi_2$ is then computed as

$$\pi_1 \ast \pi_2 = \text{Correct}2(\pi_1 +_0 \pi_2, W)$$
**Example**

- Suppose we want to consider $\pi_2$ as valid
- We start with an initial assessment $\pi_1 +_0 \pi_2$ with components $V, U \cup W = (D, E, H, X_4), W = (E, H, X_4)$, $r = (0.8333, 0.8667, 0.9667, 0.01)$ and logical constraints $\mathcal{D}$
- The only possibility to correct it is to reduce the numerical evaluation $r(D) = 0.8333$ to $r'(D) = 0.823$
- Hence the revision $\pi_1 \star \pi_2$ is the precise assessment with components $V, U \cup W = (D, E, H, X_4)$, $r' = (0.8233, 0.8667, 0.9667, 0.01)$ and the same logical constraints $\mathcal{D}$. 
Comparison with Jeffrey’s rule

• Revision operator $\star$ in general leads to an imprecise model

• It could be thought as an analogous of the famous Jeffrey’s rule of combination

• The main difference is that $\star$ minimizes the probability mass dislocation from the original assessment, maintaining as much as possible the magnitude of the values, hence working in an “additive” way

• While Jeffrey’s rule maintains as much as possible the odds ratios, hence working in a “multiplicative” way.

• Moreover the Jeffrey’s rule produces a final probability assessment which could be too different from $\pi$ since it inevitably alters all the values of $p$ on $U \setminus W$

• While $\star$ tries to modify $p$ as less as possible, in line with the belief revision methodology