A Logic with Upper and Lower Probability Operators

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ISIPTA 2015.
Example

P – a set of probability measures

P⋆(X) = sup {µ(X) | µ ∈ P},
P⋆(X) = inf {µ(X) | µ ∈ P}
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\begin{itemize}
\item \textbf{RED}
\item \textbf{OR}
\item \textbf{BLUE}
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\textbf{?}
$P$ – a set of probability measures

$P^*(X) = \sup\{\mu(X) \mid \mu \in P\}, \quad P_*(X) = \inf\{\mu(X) \mid \mu \in P\}$
Example

\[ L = 0 R, L = 0 B; \]
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\[ \begin{align*}
L & = 0 R, L = 0 B; \\
U & = 0.7 R, U = 0.7 B
\end{align*} \]
Example

$L = 0 R, L = 0 B; \quad U = 0.7 R, U = 0.7 B$

$((U \leq 0.3 G \land L \geq 0.3 G) \land U \leq 0.2 R) \Rightarrow L \geq 0.5 B.$
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Semantics

Definition (LUPP-structure)

Any tuple $M = \langle W, H, P, \nu \rangle$, where:

- $W$ is a nonempty set of worlds.
- $H$ is an algebra of subsets of $W$.
- $P$ is a set of finitely additive probability measures defined on $H$.
- $\nu: W \times L \rightarrow \{\text{true}, \text{false}\}$ evaluations of the primitive propositions.

Definition (Satisfiability relation)

$M \models \alpha$ iff $\nu(w)(\alpha) = \text{true}$, for all $w \in W$.

$M \models U \geq s \alpha$ iff $P^\star(\lbrack \alpha \rbrack) \geq s$.

Theorem (Decidability)

A satisfiability problem for LUPP-formulas is NP-complete.
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- $M \models \alpha$ iff $\nu(w)(\alpha) = \text{true}$, for all $w \in W$,
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Axiomatization issues

1) Non-compactness of *LU**P**P*-logic
   - consequence: there is no finitary axiomatization

2) Expressiveness of our propositional language
   - the representation theorem (Anger, Lembcke 1985)
Axiom schemes

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Axioms and Inference Rules

(1) all instances of the classical propositional tautologies

(2) $U \leq 1$ \quad $L \leq 1$

(3) $U \leq r \alpha \rightarrow U < s \alpha, s > r$

(4) $U < s \alpha \rightarrow U \leq s \alpha$

(5) $(U \leq r_1 \alpha_1 \land \cdots \land U \leq r_m \alpha_m) \rightarrow U \leq r \alpha, \text{if } \alpha \rightarrow \bigvee J \subseteq \{1, \ldots, m\}, |J| = k + n \land j \in J \alpha_j$ and $\bigvee J \subseteq \{1, \ldots, m\}, |J| = k \land j \in J \alpha_j$ are propositional tautologies, where $r = \sum_{i=1}^{m} r_i - k n, n \neq 0$

(6) $\neg (U \leq r_1 \alpha_1 \land \cdots \land U \leq r_m \alpha_m), \text{if } \bigvee J \subseteq \{1, \ldots, m\}, |J| = k \land j \in J \alpha_j$ is a propositional tautology and $\sum_{i=1}^{m} r_i < k$

(7) $L = 1 (\alpha \rightarrow \beta) \rightarrow (U \geq s \alpha \rightarrow U \geq s \beta)$
Axiom schemes

(1) all instances of the classical propositional tautologies

(2) \(U \leq_1 \alpha \land L \leq_1 \alpha\)

(3) \(U \leq_r \alpha \rightarrow U \leq_s \alpha, \ s > r\)

(4) \(U \leq_s \alpha \rightarrow U \leq_s \alpha\)

(5) \((U \leq_{r_1} \alpha_1 \land \cdots \land U \leq_{r_m} \alpha_m) \rightarrow U \leq_r \alpha, \) if \(\alpha \rightarrow \bigvee J \subseteq \{1, \ldots, m\}, |J|=k+n \land \bigwedge j \in J \alpha_j\) and \(\bigvee J \subseteq \{1, \ldots, m\}, |J|=k \land \bigwedge j \in J \alpha_j\) are propositional tautologies, where

\[r = \frac{\sum_{i=1}^{m} r_i - k}{n}, \ n \neq 0\]

(6) \(\neg(U \leq_{r_1} \alpha_1 \land \cdots \land U \leq_{r_m} \alpha_m),\) if \(\bigvee J \subseteq \{1, \ldots, m\}, |J|=k \land \bigwedge j \in J \alpha_j\) is a propositional tautology and \(\sum_{i=1}^{m} r_i < k\)

(7) \(L = 1(\alpha \rightarrow \beta) \rightarrow (U \geq_s \alpha \rightarrow U \geq_s \beta)\)
Inference Rules

(1) From $\rho$ and $\rho \rightarrow \sigma$ infer $\sigma$

(2) From $\alpha$ infer $L \geq 1 \alpha$

(3) From the set of premises $\{\phi \rightarrow U \geq s - k \alpha | k \geq 1 s\}$ infer $\phi \rightarrow U \geq s \alpha$

(4) From the set of premises $\{\phi \rightarrow L \geq s - k \alpha | k \geq 1 s\}$ infer $\phi \rightarrow L \geq s \alpha$. 
Inference Rules

(1) From $\rho$ and $\rho \rightarrow \sigma$ infer $\sigma$
(2) From $\alpha$ infer $L_{\geq 1} \alpha$
(3) From the set of premises

$$\{ \phi \rightarrow U_{\geq s - \frac{1}{k}} \alpha \mid k \geq \frac{1}{s} \}$$

infer $\phi \rightarrow U_{\geq s} \alpha$
(4) From the set of premises

$$\{ \phi \rightarrow L_{\geq s - \frac{1}{k}} \alpha \mid k \geq \frac{1}{s} \}$$

infer $\phi \rightarrow L_{\geq s} \alpha$.
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Axioms and Inference Rules

**Theorem (Strong completeness)**

A set of formulas $T$ is consistent iff it is LUPP $\text{Meas}$. 

**Sketch of the proof:**

1. Every consistent set $T$ can be extended to a maximal consistent set $T$. 
2. We use $T$ to construct a canonical model.
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Theorem (Strong completeness)

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Sketch of the proof:

1. Every consistent set $T$ can be extended to a maximal consistent set $T^*$.
2. We use $T^*$ to construct a canonical model.
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\[ LUPP_{Fr(n)} \]

Main differences:
1) All measures \( \mu \) have the finite range, i.e. for all \( \mu \in \mathcal{P} \):
   \[ H \rightarrow \{0, 1, \ldots, n, 1, \ldots, n, \ldots \} \]
2) The axiomatization is finite.
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