Nonlinear Filtering of Convex Sets of Probability Distributions

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Abstract

A solution is provided to the problem of computing a convex set of conditional probability distributions that characterize the state of a nonlinear dynamic system as it evolves in time. The estimator uses the Galerkin approximation to solve Kolmogorov's equation for the diffusion of a continuous-time nonlinear system with discrete-time measurement updates. Filtering of the state is accomplished for a convex set of distributions simultaneously, and closed-form representations of the resulting sets of means and covariances are generated.

Keywords. nonlinear filtering theory, convex sets of probability distributions, set-valued estimation

1 Introduction

The classical filtering problem of estimation theory is to compute real-time the conditional distribution of the state of a dynamic system given the observations. For linear Gaussian systems with a unique probability model, this problem is solved by computing the conditional expectation and covariance of the state, resulting in the well-known Kalman filter. This approach is appropriate for linear systems when there exists a unique *a priori* distribution for the initial system state. To deal with situations where the *a priori* distribution is not unique, Morrell and Stirling [10] developed the *set-valued Kalman filter*. Under this approach, the *a priori* state is represented by a convex set of distributions, and the problem is to compute the corresponding convex set of conditional expectations.

For nonlinear systems, the filtering problem becomes difficult because the entire conditional density function, rather than just the first two moments, must be computed at each time. The solution to this problem is represented by the so-called Kolmogorov's equation, a partial differential equation that describes the evolution of the conditional density of the state of a dynamic system. Although a general closed-form solution does not exist, recently a new approach, using the Galerkin approximation [1], has been shown to be an effective means of approximating the solution to Kolmogorov's equation. The resulting *nonlinear projection filter* thus represents an alternative to traditional approaches to nonlinear estimation, such as the extended Kalman filter, which relies on linearization.

In this paper we extend the nonlinear projection filter to deal with non-unique *a priori* distributions. This extension is accomplished by constructing a convex set of *a priori* density functions represented by a convex set of Euclidean r- dimensional vectors. The coordinates of these vectors are the coefficients of expansion of these density functions in terms of a set of basis functions. Each such vector represents the Galerkin approximation to one of the densities in the prior set and every such density is represented in this way. A functional is defined to represent this set with a finite number of parameters. A recursive representation of the time and measurement updating of the whole set of conditional densities is found by exploiting a certain property of the nonlinear measurement update equations. Based on this set of parameterized density functions a description of the sets of associated means and covariances is derived.

2 The Nonlinear Projection Filter

In this section we summarize the development of the approximate point-valued filter proposed by Beard et al. [1]. The system is represented by the nonlinear stochastic vector differential equation

$$d\mathbf{x}_t = \mathbf{f}(\mathbf{x}_t, t)dt + \mathbf{G}(\mathbf{x}_t, t)d\boldsymbol{\beta}_t, \quad t \geq t_0 \qquad (1)$$

where $\mathbf{x}_t \in \mathbb{R}^n$ is the state of the system at time t and $\{\boldsymbol{\beta}_t, t \geq t_0\}$ is a p-dimensional Brownian motion with covariance matrix $\mathbf{Q}(t)dt$. Let m-dimensional noisy measurements be made at discrete times t_k

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_{t_k}, t_k) + \mathbf{v}_k, \quad k = 1, 2, \cdots, \tag{2}$$

where $\{\mathbf{v}_k, k \geq 1\}$ is an m-dimensional white Gaussian sequence independent of $d\boldsymbol{\beta}_t$ with covariance matrix $\mathbf{R}_k$. Define the collection of measurements taken up to and including time t as $\mathbf{Y}_t = \{\mathbf{y}_k : t_k \leq t\}$. It is desired to determine the statistical properties of the stochastic process described by the system (1) and (2) given the a priori distribution $\mathrm{p}(\mathbf{x}, t_0)$.

The solution to this system is the conditional probability density function $\mathrm{p}_{t|t_k}(\mathbf{x}_t|\mathbf{Y}_{t_k})$ which contains all of the statistical information about the stochastic process $\mathbf{x}_t$ for every value of the parameter t. It is a well established fact that $\mathrm{p}(\mathbf{x}, t|\mathbf{Y}_{t_k})$ evolves in time between observations according to Kolmogorov's forward equation [5, page 130]. That is, between observations at times t_k and t_{k+1}, $\mathrm{p} \equiv \mathrm{p}(\mathbf{x}, t|\mathbf{Y}_{t_k})$ evolves according to

Prediction Equation:

$$\frac{\partial \mathrm{p}}{\partial t} = -\sum_{i=1}^{n} \frac{\partial(\mathrm{p} f_i)}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2[\mathrm{p}(\mathbf{GQG}^T)_{ij}]}{\partial x_i \partial x_j} \tag{3}$$

where the initial condition to (3) is $\mathrm{p}(\mathbf{x}, t_k|\mathbf{Y}_{t_k})$, the measurement updated density at time t_k, and $(\mathbf{GQG}^T)_{ij}$ is the $(i,j)^{th}$ element of the matrix $\mathbf{GQG}^T$. The information obtained by taking a measurement at time t_{k+1} may be computed according to Bayes rule, yielding

Filter Equation:

$$\mathrm{p}(\mathbf{x}, t_{k+1}|\mathbf{Y}_{t_{k+1}}) = \frac{\mathrm{p}(\mathbf{y}_{k+1}|\mathbf{x})\mathrm{p}(\mathbf{x}, t_{k+1}^-|\mathbf{Y}_{t_k})}{\int \mathrm{p}(\mathbf{y}_{k+1}|\boldsymbol{\xi})\mathrm{p}(\boldsymbol{\xi}, t_{k+1}^-|\mathbf{Y}_{t_k})d\boldsymbol{\xi}} \tag{4}$$

where

$$\mathrm{p}(\mathbf{y}_k|\mathbf{x}) = (2\pi)^{-m/2}(det\mathbf{R}_k)^{-1/2} \times exp\left(-\frac{1}{2}[\mathbf{y}_k - \mathbf{h}(\mathbf{x}, t_k)]^T \mathbf{R}_k^{-1}[\mathbf{y}_k - \mathbf{h}(\mathbf{x}, t_k)]\right) \tag{5}$$

and $\mathrm{p}(\mathbf{x}, t_{k+1}^-|\mathbf{Y}_{t_k})$ is the state conditional density obtained from (3) immediately prior to the measurement at time t_{k+1}.

The optimal nonlinear state estimator for the system (1) and (2) is given by (3) and (4) for the case of continuous dynamics and discrete measurements. Between measurements the statistical properties of the state are predicted by the evolution of its density function according to (3). At the time of an observation the new information from the measurement is incorporated into the state estimate through the difference equation (4). The problem with this optimal filter is that closed form solutions for (3) exist for only a few special cases.

Galerkin's method [2, 6, 8] is a general method for the approximate solution of partial differential equations, in which the exact solution is considered to be an element of a Hilbert space with compact support, Ω, and an approximate solution is chosen from a finite dimensional subspace, S_N, such that it solves the equation projected onto that subspace. Let $\{\phi_\ell\}_{\ell=0}^{\infty}$ be a complete orthonormal basis for the Hilbert space, $\mathcal{H}$. The inner product associated with $\mathcal{H}$ is given by

$$\langle f, g \rangle = \int_\Omega g(\boldsymbol{\xi}) f^*(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{6}$$

The approximate solution to the filtering problem using Galerkin's method is obtained by solving for the expansion coefficients of the approximating density function given by

$$\hat{\mathrm{p}}(\mathbf{x}, t|\mathbf{Y}_t) = \sum_{\ell=0}^{N-1} c_\ell(t)\phi_\ell(\mathbf{x}), \tag{7}$$

where $\phi_0, \ldots, \phi_{N-1}$ are the basis functions that span the approximation subspace S_N. Between measurements the approximate solution to Kolmogorov's equation becomes the solution to the system of ordinary differential equations

$$\dot{\mathbf{c}}(t) = \mathbf{A}(t)\mathbf{c}(t), \tag{8}$$

where

$$\mathbf{c}(t) = [c_0(t), \ldots, c_{N-1}(t)]^T$$

and

$$\mathbf{A}_{q,\ell}(t) = \sum_{i=1}^{n} \left\langle -\frac{\partial[\phi_\ell f_i]}{\partial x_i} + \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2[\phi_\ell(\mathbf{GQG}^T)_{ij}]}{\partial x_i \partial x_j}, \phi_q \right\rangle. \tag{9}$$

The solution to this equation may be characterized by a state-transition matrix $\boldsymbol{\Phi}(\cdot;\tau)$, where

$$\mathbf{c}(t) = \boldsymbol{\Phi}(t; t_k)\mathbf{c}(t_k), \quad t \in [t_k, t_{k+1}). \tag{10}$$

Note that if $\mathbf{f}$, $\mathbf{G}$, and $\mathbf{Q}$, are not time dependent, then $\mathbf{A}(t) = \mathbf{A} =$ constant matrix and $\boldsymbol{\Phi}(t; t_k) = e^{\mathbf{A}(t-t_k)}$.

The application of measurement information according to Bayes law is accomplished by the difference equation (4) which becomes

$$\mathbf{c}(t_{k+1}) = \frac{\boldsymbol{\Upsilon}_{k+1}\mathbf{c}(t_{k+1}^-)}{\boldsymbol{\upsilon}_{k+1}^T\mathbf{c}(t_{k+1}^-)}, \quad q = 0, \ldots, N-1, \tag{11}$$

where

$$[\boldsymbol{\Upsilon}_{k+1}]_{q,\ell} = \langle \mathrm{p}(\mathbf{y}_{k+1}|\mathbf{x})\phi_\ell, \phi_q\rangle \quad (12)$$
$$[\boldsymbol{\upsilon}_{k+1}]_{\ell} = \langle \mathrm{p}(\mathbf{y}_{k+1}|\mathbf{x}), \phi_\ell^*\rangle. \quad (13)$$

This is the measurement update step of the continuous-discrete filter within the approximating subspace S_N.

3 The Set-Valued Nonlinear Projection Filter

3.1 A Convex Set of Prior Distributions

Our aim in this section is to define a parameterization for convex sets of prior probability density functions (pdf's) which may be propagated according to the filter equations. Consider a set of r probability density functions

$$\Theta = \{\theta_0, \ldots, \theta_{r-1}\} \quad (14)$$

where the θ_i are arbitrary pdf's. A convex combination of the elements of Θ is a pdf

$$\mathrm{p}_{\boldsymbol{\alpha}} = \alpha_0\theta_0 + \ldots + \alpha_{r-1}\theta_{r-1}, \quad (15)$$

where $\alpha_i \geq 0$ and $\sum_{i=0}^{r-1} \alpha_i = 1$. We wish to construct a convex set of density functions with elements of the form of (15).

The set of probability density functions to be constructed will be a subset of the convex hull[1] of Θ, and will be referred to as $\mathcal{V}$. We seek a parameterization of $\mathcal{V}$ in terms of a convex set of vectors $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{r-1})^T$. The coordinates, α_i, of each vector, $\boldsymbol{\alpha}$, are coefficients of a convex combination, as in (15), of the functions in Θ. Define

$$V = \{\boldsymbol{\alpha} \in \mathbb{R}^r | \mathbf{n}^T(\boldsymbol{\alpha} - \boldsymbol{\eta}) = 0 \text{ and } \|\mathbf{M}(\boldsymbol{\alpha} - \boldsymbol{\eta})\| \leq 1\}, \quad (16)$$

where $\mathbf{n} = (1, \ldots, 1)^T$, and $\boldsymbol{\eta}$ is chosen to be some r vector, $\boldsymbol{\eta} = (\eta_0, \ldots, \eta_{r-1})^T$, such that $\sum_{i=0}^{r-1} \eta_i = 1$ and $\eta_i \geq 0$. We require $\mathbf{M} \in \mathbb{R}^{r\times r}$ to be nonsingular. The set V is a hyperellipsoidal $(r-1)$-dimensional manifold in $\mathbb{R}^r$. This structure ensures that as time evolves and V is subsequently updated, that it retains the same hyperellipsoidal structure, and leads to convenient recursions. The vector $\boldsymbol{\eta}$ and the matrix $\mathbf{M}$ parameterize the imprecision associated with the prior distribution.

We desire each element of V to be a valid parameterization for a probability density function in $\mathcal{V}$ of the form (15). From the definition of V, any $\boldsymbol{\alpha} \in V$ satisfies the requirement that $\sum_{i=0}^{r-1} \alpha_i = 1$. To ensure satisfaction of the requirement that $\alpha_i \geq 0$, for $\boldsymbol{\alpha} \in V$, we must choose $\mathbf{M} \in \mathbb{R}^{r\times r}$ so that V is contained entirely within the positive orthant. In this case the set V defines a set of convex combinations of the densities in Θ since each vector $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{r-1}) \in V$ represents a sequence of valid coefficients for a convex combination of the form of (15). Thus the set of r dimensional real vectors V specifies the convex set of probability density functions $\mathcal{V}$. That is,

$$\mathcal{V} = \{\mathrm{p}_{\boldsymbol{\alpha}} | \boldsymbol{\alpha} \in V \text{ and } \mathrm{p}_{\boldsymbol{\alpha}} = \alpha_0\theta_0 + \ldots + \alpha_{r-1}\theta_{r-1}\}. \quad (17)$$

We have characterized the vectors $\boldsymbol{\alpha}$ as parameters for probability density functions from the convex hull of Θ. What we really need, however, are the orthogonal projections of the elements of $\mathcal{V}$ onto the basis functions of S_N, the approximating subspace of functions on which the filter is defined.

Recall that the approximate continuous-discrete filter consists of a recursion on a vector $\mathbf{c}(t)$. The coordinates of this vector define the conditional density function for the state by (7). We now need to find a representation within S_N for the set of priors $\mathcal{V}$. We require a set of coefficients $c_1(t_0) \ldots c_{N-1}(t_0)$ such that, upon equating projections, we have

$$\begin{aligned}\langle \mathrm{p}_\alpha(x, t_0), \phi_s\rangle &= \left\langle \sum_{l=0}^{r-1} \alpha_l\theta_l, \phi_s \right\rangle \\ &= \left\langle \sum_{q=0}^{N-1} c_q(t_0)\phi_q, \phi_s \right\rangle \\ &= \langle \hat{\mathrm{p}}(x, t_0), \phi_s\rangle, \end{aligned} \quad (18)$$

for $s = 0, \ldots, N-1$. It follows that

$$\sum_{q=0}^{N-1} c_q(t_0)\langle\phi_q, \phi_s\rangle = \sum_{l=0}^{r-1} \alpha_l\langle\theta_l, \phi_s\rangle. \quad (19)$$

Since $\langle\phi_q, \phi_\ell\rangle = \delta_{q,\ell}$, this becomes

$$\mathbf{c}(t_0) = \mathbf{K}_0\boldsymbol{\alpha}, \quad (20)$$

where $\mathbf{c}(t_0) = (c_0, \ldots, c_{N-1})^T$, and $[\mathbf{K}_0]_{s,l} = \langle\theta_l, \phi_s\rangle$.

The relationship (20) maps each density in $\mathcal{V}$, each of which is of the form (15), to its corresponding approximation in the subspace S_N, of the form of (7). That is, each density function in $\mathcal{V}$ is represented by its parameter vector $\boldsymbol{\alpha} \in V \subset \mathbb{R}^r$, and each of these vectors is then mapped by (20) to a vector $\mathbf{c}(t_0) \in \mathbb{R}^N$.

[1] The convex hull of a set is the smallest convex set containing the original set.

3.2 The Set-Valued Nonlinear Projection Filter Equations

We must now find a method for propagating the information contained in the prior set $\mathcal{V}$ conditioned on the data. At this juncture it is useful to combine the prediction equation (10) with the measurement update equation (11) to obtain

$$\mathbf{c}(t_{k+1}) = \frac{\boldsymbol{\Upsilon}_{k+1}\boldsymbol{\Phi}(t_{k+1};t_k)\mathbf{c}(t_k)}{\boldsymbol{\upsilon}_{k+1}^T\boldsymbol{\Phi}(t_{k+1};t_k)\mathbf{c}(t_k)}. \tag{21}$$

By carrying out a few iterations of the recursion it becomes easy to see that in general (21) is equivalent to

$$\mathbf{c}(t_{k+1}) = \frac{\boldsymbol{\Upsilon}_{k+1}\boldsymbol{\Phi}(t_{k+1};t_k)\mathbf{K}_k\boldsymbol{\alpha}}{\boldsymbol{\upsilon}_{k+1}^T\boldsymbol{\Phi}(t_{k+1};t_k)\mathbf{K}_k\boldsymbol{\alpha}}, \tag{22}$$

where $\mathbf{K}_k$ is given by the recursion

$$\mathbf{K}_k = \boldsymbol{\Upsilon}_k\boldsymbol{\Phi}(t_k;t_{k-1})\mathbf{K}_{k-1},\ k = 1, \ldots. \tag{23}$$

Thus, for any prior density in $\mathcal{V}$ we can construct the corresponding approximate density at any time $t > 0$ conditioned on the data sequence $\mathbf{Y}_t$. Each prior in $\mathcal{V}$ has a unique representation in terms of a parameter vector $\boldsymbol{\alpha} \in V$. The transformation (22) produces, at time t, for each $\boldsymbol{\alpha} \in V$, a parametric representation of the current conditional density for the state of the form (7). The significance of this is that all of the information necessary to construct the set of conditional densities is propagated by the linear recursion (23), and that when we want to scrutinize this information at some time instant we may resolve it to a useful form through the nonlinear transformation (22).

4 Computing Sets of First and Second Moments

4.1 The Set of Conditional Means

The set of densities propagated by the set-valued filter, (23) and (22), is parameterized by the set of vectors $\boldsymbol{\alpha} \in V$ where $V \subset \mathbb{R}^r$. We will assume that $r > n$, where n is the dimension of the state vector $\mathbf{x}$, and r is the number of prior density functions in the set Θ which spans $\mathcal{V}$. We wish to derive the set of conditional means at a given time instant directly from the set V.

By definition, the conditional mean of the state at time t given observations, $\mathbf{Y}_\tau$, up to and including time $\tau \leq t$ is

$$\bar{x}_t^\tau = \int \boldsymbol{\xi}\mathrm{p}(\boldsymbol{\xi}, t|\mathbf{Y}_\tau)d\boldsymbol{\xi}. \tag{24}$$

We may compute the mean of any one of the approximate densities generated by (16) and (22) as follows

$$\begin{aligned}\hat{\bar{\mathbf{x}}}_t^\tau &= \int \boldsymbol{\xi}\hat{\mathrm{p}}(\boldsymbol{\xi}, t|\mathbf{Y}_\tau)d\boldsymbol{\xi} \\ &= \int \boldsymbol{\xi}\sum_{l=0}^{N-1} c_l(t)\phi_l(\boldsymbol{\xi})d\boldsymbol{\xi} \\ &= \sum_{l=0}^{N-1} c_l(t)\int \boldsymbol{\xi}\phi_l(\boldsymbol{\xi})d\boldsymbol{\xi} \\ &= \boldsymbol{\Gamma}^T\mathbf{c}(t) \end{aligned}\tag{25}$$

where the columns of

$$\boldsymbol{\Gamma}^T = \left[\ \int \boldsymbol{\xi}\phi_0(\boldsymbol{\xi})d\boldsymbol{\xi},\quad \ldots,\quad \int \boldsymbol{\xi}\phi_{N-1}(\boldsymbol{\xi})d\boldsymbol{\xi}\ \right] \in \mathbb{R}^{n\times r} \tag{26}$$

are mean vectors of the basis distributions. From this and (22) we have

$$\hat{\bar{\mathbf{x}}}_t^{t_{k+1}} = \frac{\boldsymbol{\Gamma}^T\boldsymbol{\Phi}(t;t_{k+1})\boldsymbol{\Upsilon}_{k+1}\boldsymbol{\Phi}(t_{k+1};t_k)\mathbf{K}_k\boldsymbol{\alpha}}{\boldsymbol{\upsilon}_{k+1}^T\boldsymbol{\Phi}(t_{k+1};t_k)\mathbf{K}_k\boldsymbol{\alpha}}. \tag{27}$$

Define

$$\check{\boldsymbol{\Gamma}}^T = \boldsymbol{\Gamma}^T\boldsymbol{\Phi}(t;t_{k+1})\boldsymbol{\Upsilon}_{k+1}\boldsymbol{\Phi}(t_{k+1};t_k)\mathbf{K}_k, \tag{28}$$

and

$$\check{\boldsymbol{\upsilon}}^T = \boldsymbol{\upsilon}_{k+1}^T\boldsymbol{\Phi}(t_{k+1};t_k)\mathbf{K}_k, \tag{29}$$

then (27) becomes

$$\hat{\bar{\mathbf{x}}} = \frac{\check{\boldsymbol{\Gamma}}^T\boldsymbol{\alpha}}{\check{\boldsymbol{\upsilon}}^T\boldsymbol{\alpha}}, \tag{30}$$

where we have suppressed the dependence on time. We may now simplify our problem by applying the linear invertible transformation

$$\tilde{\boldsymbol{\alpha}} = \mathbf{M}\boldsymbol{\alpha} \tag{31}$$

as follows

$$\hat{\bar{\mathbf{x}}} = \frac{\check{\boldsymbol{\Gamma}}^T\mathbf{M}^{-1}\mathbf{M}\boldsymbol{\alpha}}{\check{\boldsymbol{\upsilon}}^T\mathbf{M}^{-1}\mathbf{M}\boldsymbol{\alpha}} = \frac{\tilde{\boldsymbol{\Gamma}}^T\tilde{\boldsymbol{\alpha}}}{\tilde{\boldsymbol{\upsilon}}^T\tilde{\boldsymbol{\alpha}}}, \tag{32}$$

where

$$\tilde{\boldsymbol{\Gamma}} = \mathbf{M}^{-T}\check{\boldsymbol{\Gamma}} \tag{33}$$

$$\tilde{\boldsymbol{\upsilon}} = \mathbf{M}^{-T}\check{\boldsymbol{\upsilon}}. \tag{34}$$

Now with $\boldsymbol{\alpha} = \mathbf{M}^{-1}\tilde{\boldsymbol{\alpha}}$, we have

$$\begin{aligned}\tilde{V} = \big\{\tilde{\boldsymbol{\alpha}} \in \mathbb{R}^r\ \big|\ &\|\mathbf{M}(\mathbf{M}^{-1}\tilde{\boldsymbol{\alpha}} - \boldsymbol{\eta})\|^2 \leq 1 \\ &\text{and } \mathbf{n}^T(\mathbf{M}^{-1}\tilde{\boldsymbol{\alpha}} - \boldsymbol{\eta}) = 0\big\},\end{aligned} \tag{35}$$

which, upon setting

$$\begin{aligned} \tilde{\boldsymbol{\eta}} &= \mathbf{M}\boldsymbol{\eta} && (36) \\ \tilde{\mathbf{n}} &= \mathbf{M}^{-T}\mathbf{n}, && (37) \end{aligned}$$

becomes

$$\tilde{V} = \left\{ \tilde{\boldsymbol{\alpha}} \in \mathbb{R}^r \,\middle|\, \|\tilde{\boldsymbol{\alpha}} - \tilde{\boldsymbol{\eta}}\|^2 \le 1, \text{ and } \tilde{\mathbf{n}}^T(\tilde{\boldsymbol{\alpha}} - \tilde{\boldsymbol{\eta}}) = 0 \right\}. \tag{38}$$

Our problem has now become that of finding the set of all means $\hat{\tilde{\mathbf{x}}}$ generated by (32) as $\tilde{\boldsymbol{\alpha}}$ ranges over all values in the set $\tilde{V}$.

The set $\tilde{V}$ resides in $\mathbb{R}^r$ and r is potentially much greater than n, the dimension of the state space. If we restrict attention only to the first moments of the distributions represented by $\tilde{V}$, however, we may significantly reduce the dimension of the parameterization space. We observe that the only component of any $\tilde{\boldsymbol{\alpha}} \in \tilde{V}$ which affects the value of $\hat{\tilde{\mathbf{x}}}$ in (32) is that component which lies in the column space of the augmented matrix

$$\boldsymbol{\Delta} = (\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\Gamma}}) \in \mathbb{R}^{r\times(n+1)}. \tag{39}$$

To see this, consider the orthogonal projection transformation

$$\wp = \boldsymbol{\Delta}(\boldsymbol{\Delta}^T\boldsymbol{\Delta})^{-1}\boldsymbol{\Delta}^T, \tag{40}$$

which maps any vector in $\mathbb{R}^r$ to its projection in the subspace spanned by the columns of $\boldsymbol{\Delta}$ within $\mathbb{R}^r$. Consider also the orthogonal projection operator

$$\wp^{\perp} = \mathbf{I} - \wp \tag{41}$$

which projects into the orthogonal complement of the column space of $\boldsymbol{\Delta}$ with respect to $\mathbb{R}^r$. We may now write

$$\begin{aligned} \hat{\tilde{\mathbf{x}}} &= \frac{\tilde{\boldsymbol{\Gamma}}^T\tilde{\boldsymbol{\alpha}}}{\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{\alpha}}} \\ &= \frac{\tilde{\boldsymbol{\Gamma}}^T(\wp + \wp^{\perp})\tilde{\boldsymbol{\alpha}}}{\tilde{\boldsymbol{v}}^T(\wp + \wp^{\perp})\tilde{\boldsymbol{\alpha}}} \\ &= \frac{\tilde{\boldsymbol{\Gamma}}^T\wp\tilde{\boldsymbol{\alpha}} + \tilde{\boldsymbol{\Gamma}}^T\wp^{\perp}\tilde{\boldsymbol{\alpha}}}{\tilde{\boldsymbol{v}}^T\wp\tilde{\boldsymbol{\alpha}} + \tilde{\boldsymbol{v}}^T\wp^{\perp}\tilde{\boldsymbol{\alpha}}} \\ &= \frac{\tilde{\boldsymbol{\Gamma}}^T\wp\tilde{\boldsymbol{\alpha}}}{\tilde{\boldsymbol{v}}^T\wp\tilde{\boldsymbol{\alpha}}} && (42) \end{aligned}$$

since $\tilde{\boldsymbol{\Gamma}}^T\wp^{\perp}$ and $\tilde{\boldsymbol{v}}^T\wp^{\perp}$ are both zero due to the construction of $\wp^{\perp}$. Thus we see that only the component of $\tilde{\boldsymbol{\alpha}}$ contained in the column space of $\boldsymbol{\Delta}$ determines $\hat{\tilde{\mathbf{x}}}$.

We may now transform (32) with a (possibly non-invertible) linear transformation which preserves this essential information and which simplifies the form of (32). We proceed by first applying the Gram-Schmidt [4] process to the columns of $\boldsymbol{\Delta}$ to obtain the matrix $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_{n+1})$, which has orthonormal columns spanning the same space as the columns of $\boldsymbol{\Delta}$. The orthogonal projection matrix defined above may now be written

$$\wp = \mathbf{U}\mathbf{U}^T. \tag{43}$$

With $\wp$ in this form, (32) may be written as

$$\begin{aligned} \hat{\tilde{\mathbf{x}}} &= \frac{\tilde{\boldsymbol{\Gamma}}^T\tilde{\boldsymbol{\alpha}}}{\tilde{\boldsymbol{v}}^T\tilde{\boldsymbol{\alpha}}} \\ &= \frac{\tilde{\boldsymbol{\Gamma}}^T\wp\tilde{\boldsymbol{\alpha}}}{\tilde{\boldsymbol{v}}^T\wp\tilde{\boldsymbol{\alpha}}} \\ &= \frac{\tilde{\boldsymbol{\Gamma}}^T\mathbf{U}\mathbf{U}^T\tilde{\boldsymbol{\alpha}}}{\tilde{\boldsymbol{v}}^T\mathbf{U}\mathbf{U}^T\tilde{\boldsymbol{\alpha}}}. && (44) \end{aligned}$$

Now let

$$\begin{aligned} \breve{\boldsymbol{\alpha}} &= \mathbf{U}^T\tilde{\boldsymbol{\alpha}} && (45) \\ \breve{\boldsymbol{\Gamma}} &= \mathbf{U}^T\tilde{\boldsymbol{\Gamma}} && (46) \\ \breve{\boldsymbol{v}} &= \mathbf{U}^T\tilde{\boldsymbol{v}}, && (47) \end{aligned}$$

so that

$$\hat{\tilde{\mathbf{x}}} = \frac{\breve{\boldsymbol{\Gamma}}^T\breve{\boldsymbol{\alpha}}}{\breve{\boldsymbol{v}}^T\breve{\boldsymbol{\alpha}}} = \frac{\breve{\boldsymbol{\Gamma}}^T\breve{\boldsymbol{\alpha}}}{\|\tilde{\boldsymbol{v}}\|\,\mathbf{e}_1^T\breve{\boldsymbol{\alpha}}} \tag{48}$$

where $\breve{\boldsymbol{\alpha}} \in \mathbb{R}^{n+1}$ and $\mathbf{e}_i$ is the i^{th} standard Euclidean basis vector. Note that $\breve{\boldsymbol{v}} = \|\tilde{\boldsymbol{v}}\|\,\mathbf{e}_1$ due to the construction of $\mathbf{U}$.

We must now characterize the set of all $\breve{\boldsymbol{\alpha}} \in \mathbb{R}^{n+1}$ such that $\breve{\boldsymbol{\alpha}} = \mathbf{U}^T\tilde{\boldsymbol{\alpha}}$ for some $\tilde{\boldsymbol{\alpha}} \in \tilde{V}$. We will refer to this set as $\breve{V}$, and we will provide for its definition an inequality constraint on a functional. That is, we will find a description for this set of the form

$$\breve{V} = \left\{ \breve{\boldsymbol{\alpha}} \in \mathbb{R}^{n+1} \,\middle|\, f(\breve{\boldsymbol{\alpha}}) \le \text{a constant} \right\} \tag{49}$$

where f is to be determined. To derive this functional we must determine conditions which are satisfied only by those points of $\tilde{V}$ that map to boundary points of $\breve{V}$.

Theorem 1 *Let the set $\tilde{V}$ be defined as in (38), that is*

$$\tilde{V} = \left\{ \tilde{\boldsymbol{\alpha}} \in \mathbb{R}^r \,\middle|\, \|\tilde{\boldsymbol{\alpha}} - \tilde{\boldsymbol{\eta}}\|^2 \le 1, \textit{ and } \tilde{\mathbf{n}}^T(\tilde{\boldsymbol{\alpha}} - \tilde{\boldsymbol{\eta}}) = 0 \right\}. \tag{50}$$

Define $\mathbf{U} \in \mathbb{R}^{r\times(n+1)}$ to have orthonormal columns that span the column space of $(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\Gamma}})$. With $\wp = \mathbf{U}\mathbf{U}^T$ and $\wp^{\perp} = \mathbf{I} - \wp$, let

$$\mathcal{M} = \mathbf{I} + \frac{\tilde{\mathbf{n}}\tilde{\mathbf{n}}^T}{\tilde{\mathbf{n}}^T\wp^{\perp}\tilde{\mathbf{n}}}, \tag{51}$$

and define

$$\check{V} = \left\{ \check{\boldsymbol{\alpha}} \in \mathbb{R}^{n+1} \left| (\check{\boldsymbol{\alpha}} - \check{\boldsymbol{\eta}})^T \mathbf{U}^T \mathcal{M} \mathbf{U} (\check{\boldsymbol{\alpha}} - \check{\boldsymbol{\eta}}) \leq 1 \right. \right\}, \tag{52}$$

where $r \geq n+1$, *and* $\check{\boldsymbol{\eta}} = \mathbf{U}^T \tilde{\boldsymbol{\eta}}$. *Then* $\check{\boldsymbol{\alpha}} \in \check{V}$ *if and only if* $\check{\boldsymbol{\alpha}} = \mathbf{U}^T \tilde{\boldsymbol{\alpha}}$ *for some* $\tilde{\boldsymbol{\alpha}} \in \tilde{V}$.

The proofs of all theorems in this paper may be found in [7]. We will now proceed to determine a representation of the set of means from the simplified intermediate equation of (48) and (52). Consider the linear transformation

$$\mathbf{F} = \left(\; \mathbf{e}_1{}^T, \check{\boldsymbol{\Gamma}}^T \; \right)^T \in \mathbb{R}^{(n+1)\times(n+1)}. \tag{53}$$

Without loss of generality, the rows of $\mathbf{F}$ may be taken to be linearly independent since dependencies among the rows of $\mathbf{F}$ would allow us to reduce the order of the filter. Hence $\mathbf{F}$ is invertible, and we may make the transformation on the equations for $\hat{\hat{\mathbf{x}}}$ as follows. Define

$$\acute{\boldsymbol{\alpha}} = \mathbf{F}\check{\boldsymbol{\alpha}}. \tag{54}$$

Then

$$\hat{\hat{\mathbf{x}}} = \frac{\check{\boldsymbol{\Gamma}}^T \mathbf{F}^{-1} \mathbf{F} \check{\boldsymbol{\alpha}}}{\|\tilde{\boldsymbol{v}}\| \, \mathbf{e}_1^T \mathbf{F}^{-1} \mathbf{F} \check{\boldsymbol{\alpha}}} = \frac{(\mathbf{0}, \mathbf{I}) \acute{\boldsymbol{\alpha}}}{\|\tilde{\boldsymbol{v}}\| \, \mathbf{e}_1^T \acute{\boldsymbol{\alpha}}} \tag{55}$$

for each $\acute{\boldsymbol{\alpha}} \in \acute{V}$, where

$$\acute{V} = \left\{ \boldsymbol{\alpha} \in \mathbb{R}^{n+1} \left| (\mathbf{F}^{-1} \acute{\boldsymbol{\alpha}} - \check{\boldsymbol{\eta}})^T \mathbf{U}^T \mathcal{M} \mathbf{U} (\mathbf{F}^{-1} \acute{\boldsymbol{\alpha}} - \check{\boldsymbol{\eta}}) \leq 1 \right. \right\}.$$

Setting

$$\begin{aligned} \boldsymbol{\Xi} &= \mathbf{F}^{-T} \mathbf{U}^T \mathcal{M} \mathbf{U} \mathbf{F}^{-1} && (56) \\ \acute{\boldsymbol{\eta}} &= \mathbf{F} \check{\boldsymbol{\eta}}, && (57) \end{aligned}$$

we obtain

$$\acute{V} = \left\{ \acute{\boldsymbol{\alpha}} \in \mathbb{R}^{n+1} \left| \left\| \boldsymbol{\Xi}^{1/2} (\acute{\boldsymbol{\alpha}} - \acute{\boldsymbol{\eta}}) \right\|^2 \leq 1 \, , \right. \right\}. \tag{58}$$

With the problem cast in the form of (55) and (58) we may now state the following theorem that gives us the set of all conditional means generated by (16) and (27).

Theorem 2 *Define*

$$\hat{\hat{V}} = \left\{ \hat{\hat{\mathbf{x}}} \in \mathbb{R}^n \left| (\|\tilde{\boldsymbol{v}}\|^{-1}, \hat{\hat{\mathbf{x}}}^T) \mathcal{N} \begin{pmatrix} \|\tilde{\boldsymbol{v}}\|^{-1} \\ \hat{\hat{\mathbf{x}}} \end{pmatrix} \geq 0 \right. \right\}, \tag{59}$$

where $\tilde{\boldsymbol{v}}$ *is given by (34) and*

$$\mathcal{N} = \boldsymbol{\Xi} \acute{\boldsymbol{\eta}} \acute{\boldsymbol{\eta}}^T \boldsymbol{\Xi} - \acute{\boldsymbol{\eta}}^T \boldsymbol{\Xi} \acute{\boldsymbol{\eta}} \boldsymbol{\Xi} + \boldsymbol{\Xi}. \tag{60}$$

Then $\hat{\hat{\mathbf{x}}} \in \hat{\hat{V}}$ *if and only if*

$$\hat{\hat{\mathbf{x}}} = \frac{(\mathbf{0}, \mathbf{I}) \acute{\boldsymbol{\alpha}}}{\|\tilde{\boldsymbol{v}}\| \, \mathbf{e}_1^T \acute{\boldsymbol{\alpha}}} \tag{61}$$

for some $\acute{\boldsymbol{\alpha}} \in \acute{V}$, *where* $\acute{V}$ *is given by (58).*

4.2 The Set of Covariance Matrices

Theorem 2 provides the final step in defining the set of means produced by the set-valued filter at a given time instant. Notwithstanding the importance of a convex set of estimates such as (59), it only portrays a portion of the information available in the convex set of conditional densities. A measure of the central tendency of a distribution about its mean provides needed information about the reliability of the estimate. Usually the covariance associated with the distribution fills this need.

We will find in this development that by augmenting the mean vector with a row scanned version of the associated correlation matrix, the problem of determining the set of correlation matrix/mean vector pairs reduces to the same form as that of finding the set of means by themselves. This makes the development of the previous section directly applicable to the problem of determining the set of correlations. To find the related set of covariance matrix/mean vector pairs we employ a nonlinear invertible transformation. We will now proceed to characterize this set of covariances.

We first consider the correlation matrix, which is given by definition as

$$\mathbf{COR} = E\{\mathbf{x}\mathbf{x}^T\} = \int \mathrm{p}(\boldsymbol{\xi}) \boldsymbol{\xi} \boldsymbol{\xi}^T d\boldsymbol{\xi}. \tag{62}$$

Replacing $\mathrm{p}(\mathbf{x})$ with $\hat{\mathrm{p}}(\mathbf{x})$, we obtain the approximation

$$\begin{aligned} \widehat{\mathbf{COR}} &= \int \hat{\mathrm{p}}(\boldsymbol{\xi}) \boldsymbol{\xi} \boldsymbol{\xi}^T d\boldsymbol{\xi} \\ &= \int \left(\sum_{\ell=0}^{N-1} c_\ell(t) \phi_\ell(\boldsymbol{\xi}) \right) \boldsymbol{\xi} \boldsymbol{\xi}^T d\boldsymbol{\xi} \\ &= \sum_{\ell=0}^{N-1} c_\ell(t) \int \phi_\ell(\boldsymbol{\xi}) \boldsymbol{\xi} \boldsymbol{\xi}^T d\boldsymbol{\xi}. && (63) \end{aligned}$$

Recall that $\widehat{\mathbf{COR}}$ is a real symmetric matrix, hence there is no loss of information if we concern ourselves only with those entries lying on and above the diagonal. To accomplish this and also to put $\widehat{\mathbf{COR}}$ into a manageable form we will represent $\widehat{\mathbf{COR}}$ as a column vector by stacking up the entries of $\widehat{\mathbf{COR}}$ and then deleting those entries in the resulting vector which are redundant because of symmetry. We will make use of the function $vec(\cdot)$ defined in [3] to achieve this goal. The function $vec(\cdot)$ is a vector valued function of a matrix argument which is defined as follows

$$\begin{aligned} \mathbf{cor} &= vec(\widehat{\mathbf{COR}}) \\ &= \left[\widehat{\mathbf{COR}}_{\cdot 1}^T, \; \widehat{\mathbf{COR}}_{\cdot 2}^T, \; \cdots, \; \widehat{\mathbf{COR}}_{\cdot n}^T \right]^T, && (64) \end{aligned}$$

where $\widehat{\mathbf{COR}}_{\cdot i}$ denotes the i^{th} column of $\widehat{\mathbf{COR}}$. We now may write

$$\begin{aligned}\mathbf{cor} &= \sum_{\ell=0}^{N-1} c_\ell(t) \int \phi_\ell(\boldsymbol{\xi}) \begin{bmatrix} \xi_1 \boldsymbol{\xi} \\ \xi_2 \boldsymbol{\xi} \\ \vdots \\ \xi_n \boldsymbol{\xi} \end{bmatrix} d\xi \\ &= \sum_{\ell=0}^{N-1} c_\ell(t) \int \phi_\ell(\boldsymbol{\xi})(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= \left[\int (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \boldsymbol{\phi}(\boldsymbol{\xi}) d\boldsymbol{\xi}\right] \mathbf{c}(t) \qquad (65)\end{aligned}$$

where $\boldsymbol{\phi}(\boldsymbol{\xi}) = (\phi_0(\boldsymbol{\xi}), \phi_1(\boldsymbol{\xi}), \ldots, \phi_{N-1}(\boldsymbol{\xi}))$, and $\otimes$ is the matrix Kroneker product. We must now remove the redundancy inherent in **cor** as we mentioned before. This will be accomplished by first defining the linear transformation $\mathbf{T} \in \mathbb{R}^{\frac{n(n+1)}{2} \times n^2}$ as

$$\begin{aligned}\mathbf{T} = [&\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{e}_{n+2}, \ldots, \mathbf{e}_{2n}, \mathbf{e}_{2n+3}, \ldots, \\ &\mathbf{e}_{3n}, \mathbf{e}_{3n+4}, \ldots, \ldots, \mathbf{e}_{n^2}], \qquad (66)\end{aligned}$$

where $\mathbf{e}_i$ is the i^{th} standard basis vector of $\mathbb{R}^{n^2}$. Left multiplication of the vector **cor** by the non-square matrix $\mathbf{T}$ will have the effect of removing the coordinates of **cor** which originated from elements below the diagonal of $\widehat{\mathbf{COR}}$. We may now write

$$\begin{aligned}\boldsymbol{cor} &= \mathbf{T}\,\mathbf{cor} \\ &= \left[\mathbf{T} \int (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \boldsymbol{\phi}(\boldsymbol{\xi}) d\boldsymbol{\xi}\right] \mathbf{c}(t), \qquad (67)\end{aligned}$$

and with

$$\boldsymbol{\Lambda}^T = \left[\mathbf{T} \int (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \boldsymbol{\phi}(\boldsymbol{\xi}) d\boldsymbol{\xi}\right] \qquad (68)$$

we have

$$\boldsymbol{cor} = \boldsymbol{\Lambda}^T \mathbf{c}(t). \qquad (69)$$

We may now augment this equation for the correlation with the equation for the conditional mean to obtain

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{cor} \\ \hat{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Lambda}^T \\ \boldsymbol{\Gamma}^T \end{bmatrix} \mathbf{c}(t),$$

and with

$$\boldsymbol{\mathcal{G}}^T = \begin{bmatrix} \boldsymbol{\Lambda}^T \\ \boldsymbol{\Gamma}^T \end{bmatrix} \qquad (70)$$

we may write

$$\boldsymbol{x} = \boldsymbol{\mathcal{G}}^T \mathbf{c}(t), \qquad (71)$$

wherein we will refer to the vector $\boldsymbol{x}$ as the augmented correlation vector. Equation (71) is in identically the same form as (25); therefore, the entire development for determining the set of conditional means applies directly to determining the set of augmented correlation vectors $\boldsymbol{x} = (\boldsymbol{cor}^T, \hat{\hat{\mathbf{x}}}^T)^T$. That is, the set of augmented correlation vectors is given by

$$\left\{\boldsymbol{x} \in \mathbb{R}^{n(n+3)/2} \,\middle|\, (\|\tilde{\boldsymbol{v}}\|^{-1}, \boldsymbol{x}^T)\mathcal{N}\begin{pmatrix} \|\tilde{\boldsymbol{v}}\|^{-1} \\ \boldsymbol{x} \end{pmatrix} \geq 0\right\}, \qquad (72)$$

where we now have

$$\begin{aligned}\mathbf{F} &= \begin{pmatrix} \mathbf{e}_1^T \\ \breve{\boldsymbol{\mathcal{G}}}^T \end{pmatrix} \\ \breve{\boldsymbol{\mathcal{G}}} &= \mathbf{U}^T \tilde{\boldsymbol{\mathcal{G}}} \\ \tilde{\boldsymbol{\mathcal{G}}} &= \mathbf{M}^{-T} \mathbf{K}_k^T \boldsymbol{\Phi}^T(t_{k+1}; t_k) \boldsymbol{\Upsilon}_{k+1}^T \boldsymbol{\Phi}^T(t; t_{k+1}) \boldsymbol{\mathcal{G}} \\ \tilde{\boldsymbol{v}} &= \mathbf{M}^{-T} \mathbf{K}_k^T \boldsymbol{\Phi}^T(t_{k+1}; t_k) \boldsymbol{v}_{k+1} \\ \boldsymbol{\Delta} &= \left(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\mathcal{G}}}\right)\end{aligned}$$

We now turn to the problem of specifying the set of covariances. To accomplish this we define the augmented covariance vector

$$\begin{aligned}\boldsymbol{\chi} &= \begin{bmatrix} \boldsymbol{cov} \\ \hat{\hat{\mathbf{x}}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{T}vec(\mathbf{COV}) \\ \hat{\hat{\mathbf{x}}} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{cor} - \mathbf{T}vec(\hat{\hat{\mathbf{x}}}\hat{\hat{\mathbf{x}}}^T) \\ \hat{\hat{\mathbf{x}}} \end{bmatrix} \\ &= \boldsymbol{x} - \begin{bmatrix} \mathbf{T}vec(\hat{\hat{\mathbf{x}}}\hat{\hat{\mathbf{x}}}^T) \\ \mathbf{0} \end{bmatrix}. \qquad (73)\end{aligned}$$

The right hand side of (73) gives the augmented covariance vector, $\boldsymbol{\chi}$, in terms of the augmented correlation vector, $\boldsymbol{x}$, along with a term which depends only on the mean. Since the mean vector is a sub-vector of $\boldsymbol{\chi}$ we may construct the inverse transformation from $\boldsymbol{\chi}$ to $\boldsymbol{x}$ as follows

$$\boldsymbol{\chi} = \boldsymbol{x} - \begin{bmatrix} \mathbf{T}vec(\hat{\hat{\mathbf{x}}}\hat{\hat{\mathbf{x}}}^T) \\ \mathbf{0} \end{bmatrix}, \qquad (74)$$

which implies

$$\begin{aligned}\boldsymbol{x} &= \boldsymbol{\chi} + \begin{bmatrix} \mathbf{T}vec(\hat{\hat{\mathbf{x}}}\hat{\hat{\mathbf{x}}}^T) \\ \mathbf{0} \end{bmatrix} \\ &= \boldsymbol{\chi} + \begin{bmatrix} \mathbf{T}[(\mathbf{0}, \mathbf{I}_n) \otimes (\mathbf{0}, \mathbf{I}_n)](\boldsymbol{\chi} \otimes \boldsymbol{\chi}) \\ \mathbf{0} \end{bmatrix} \\ &= \boldsymbol{\chi} + \mathbf{W}(\boldsymbol{\chi} \otimes \boldsymbol{\chi}), \qquad (75)\end{aligned}$$

where we define

$$\mathbf{W} = \begin{bmatrix} \mathbf{T}[(\mathbf{0}, \mathbf{I}_n) \otimes (\mathbf{0}, \mathbf{I}_n)] \\ \mathbf{0} \end{bmatrix}. \qquad (76)$$

With the relationship of (75) and the definition of the set of augmented correlation vectors (72) we may now specify the set of augmented covariances as

$$V = \left\{ \chi \in \mathbb{R}^{n(n+1)/2} \;\middle|\; \begin{pmatrix} \|\tilde{\upsilon}\|^{-1} \\ \chi + \mathbf{W}(\chi \otimes \chi) \end{pmatrix}^T \mathcal{N} \begin{pmatrix} \|\tilde{\upsilon}\|^{-1} \\ \chi + \mathbf{W}(\chi \otimes \chi) \end{pmatrix} \geq 0 \right\} \tag{77}$$

which is the desired representation.

5 Simulation Results

The following simulation compares the performance and operating characteristics of the set-valued nonlinear projection filter (SVNPF) developed in this paper with those of the set-valued extended Kalman filter (SVEKF) [9, 10]. The prior basis pdf's for the SVNPF are Gaussian with variances equal to the variance used in the SVEKF and their mean values chosen so that the resulting set of prior means is approximately the same as those of the SVEKF. The same data sequence is applied to both the SVEKF and to the SVNPF. The system dynamics are described by

$$dx_t = \sin\left(x_t + \frac{\pi}{18}\right) dt + d\beta_t \tag{78}$$

where $x_t \in \mathbb{R}$ is the state of the system at time t and $\{\beta_t, t \geq t_0\}$ is a Brownian motion process with variance $Q(t)dt$. This system has an equilibrium point at each zero crossing of the sine function, but only the equilibria associated with a negative slope are stable. Thus, if a probability density function maintains nonzero mass across an unstable equilibrium, it will tend to bifurcate as its mass migrates to the two adjacent stable equilibria unless information in the data overcome this tendency. Suppose, however, that observations are taken according to the nonlinear model

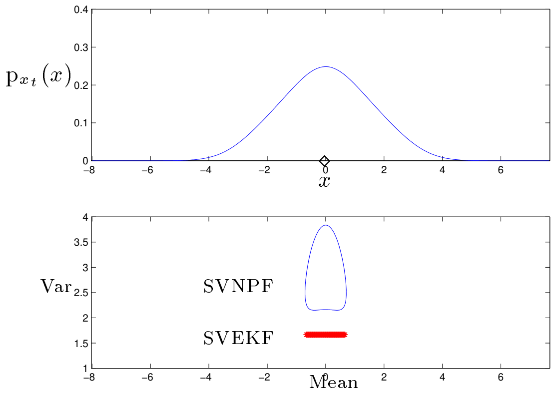
$$y_k = \frac{1}{2}|x_{t_k}| + \mathrm{v}_k, \quad k = 1, 2, \dots, \tag{79}$$

where $\{\mathrm{v}_k, k \geq 1\}$ is a white Gaussian sequence independent of $d\beta_t$ with variance R_k. Since this observation renders it impossible to distinguish between positive and negative values of the state, there will not be sufficient information to overcome the bifurcation.

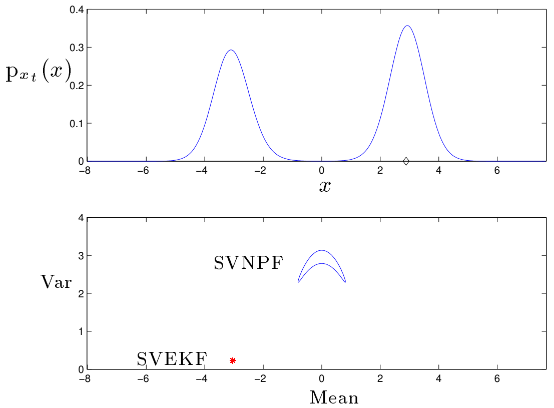
For this example, $n = 1$, $N = 64$, and $r = 5$. Θ consists of Gaussian distributions with means in $\{-2, -1, 0, 1, 2\}$ with unit variance. The orthonormal basis functions for the Hilbert space are taken as

$$\phi_\ell(x) \begin{cases} \frac{1}{\sqrt{b-a}} & \ell = 0 \\ \sqrt{\frac{2}{b-a}} \cos\left(\frac{\pi\ell}{b-a}(x-a)\right) & \ell = 1, 2, \dots, \end{cases} \tag{80}$$

where $\Omega = [a, b]$. For this simulation, a and b are chosen to ensure that several equilibria are included in the support.



(a) $t = 0$



(b) $t = 2.25$

Figure 1: Simulation results: (a) conditional density and mean/variance pairs at $t = 0$; (b) conditional density and mean/variance pairs at $t = 2.25$. (For this simulation, $Q(t) = 1$ and $R_t = 2$.)

Figure 1 shows the state of this system at time $t = 0$ and at time $t = 2.25$. A particular density function is chosen from the set of priors and its evolution in time is plotted, with the true system state is marked with a diamond. Cross plots of the set of mean/variance pairs is plotted for both the SVEKF and the SVNPF. In the case of the SVEKF the variance is the same for each density in the set, hence the associated set of mean variance pairs is always a line, plotted in bold face. The true initial state is zero, which is the best possible circumstance for the SVEKF since that is the initial valued of its central mean, $\boldsymbol{c}_{0|0}$ ($\boldsymbol{c}_{k|k}$ is the point about which the SVEKF linearizes the system and measurement functions). Notice that the initial mean set of the SVEKF has been chosen to have about the same width as does that of the SVNPF.

Figure 1(b) illustrates the situation at time $t = 2.25$. This representative distribution has a distinctive bimodal character, with the two modes converging to the two stable equilibria. As expected by the system's inability to distinguish the polarity of the observations, the set of means produced by the SVNPF is enlarging, reflecting the fact that some of the densities in the set have more mass converging to one side than the other. The SVEKF, however, has incorrectly converged to a point estimate at the wrong equilibrium point since the actual state converged to the other stable equilibrium.

This example illustrates that, with nonlinear systems, it is possible to generate multi-modal conditional distributions. In such situations, the first two moments do not convey sufficient information to indicate the behavior of the estimator. The SVNPF, however, provides the set of all such conditional distributions, from which sets of all high-order moments can be extracted.

6 Summary

We have extended the nonlinear filter of [1] to propagate a convex set of probability distributions. A parameterization of a convex set of probability density functions is defined in (16). This parameterization is shown to represent the uncountably infinite set of priors with a finite number of parameters. The approximate filter equations (8) and (11) from [1] are extended to evolve the set of conditional distributions arising at each time t from the set of priors parameterized by (16). It is seen that the set of conditional density functions produced by the filter is equivalent to propagating each density in the prior set individually with the filter of [1]. A method is developed in Section 4.1 for describing the set of conditional means at any chosen time t. This set of conditional means contains the mean values of each of the pdf's propagated by the filter, and is given by a closed form expression in Theorem 2. To further aid in the interpretation of the filter output, the set of conditional covariances associated with the set of conditional distributions is also derived. This set is defined by the closed form expression of (77) which defines the set of mean vector/covariance matrix pairs.

Since nonlinear filtering requires the propagation of the entire distribution in contrast to the need to propagate only the first two moments with linear filtering, it is to be expected that the computational burden is severe. This burden can be mitigated by judicious choice of Hilbert space basis functions and employing fast algorithms, such as the fast cosine transform, to perform inner products. These innovations notwithstanding, however, the practical implementation of set-valued non-linear projection filtering is not yet a feasible option for systems with high state space dimensionality.

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