1st International Symposium on Imprecise Probabilities and Their Applications, Ghent, Belgium, 29 June – 2 July 1999

Imprecise Probabilities Relating to Prior Reliability Assessments

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Abstract

The paper summarizes the author's experience in dealing with the Dempster-Shafer theory relating to reliability assessments and demonstrates how to make component and system reliability assessments based on the theory of coherent imprecise previsions. The procedure of prior imprecise probability elicitation of components is based on analogical reasoning, and two cases of precise and imprecise probabilities of prototypes are considered. Cases of combining different reliability judgements on the same component are analyzed. The formulae obtained for system reliability assessments allow getting the lower and upper probabilities without the presumption of a conditional independence. An example of system reliability calculating was considered.

Keywords. Coherent imprecise probabilities, reliability assessments, belief functions, analogy

1 Introduction

Obtaining grounded, explanatory and credible prior reliability assessments is an important task. As a matter of fact, the decision to put a technical object into operation is based on exclusively prior assessments of reliability, risk and other attributes. This issue is most prominent for large-scale and unique installations and actions, failures of which can lead to dramatic losses. No specific failures precede putting a technical installation into operation. Furthermore, for non-repeatable actions, prior assessments will be final, being unchangeable in the future due to the absence of failures. In many cases, rare occurrences cannot constitute representative samples with the aim of making substantially more precise posterior assessments compared to prior ones. That is, the expectations of representative samples will not come true and we cannot count on post-action qualifications and, therefore, have to seek faithful methods and theories to get prior reliabilities of components and systems.

Some introspective analysis can demonstrate that in complicated situations of decision making we cannot determine what actions are more preferable. This happens when we suffer from a lack of subject matter knowledge. This indeterminacy is natural and should be taken into consideration when modeling. A precise probabilistic analysis always gives a determinate decision, and it cannot be considered a faithful reflection of reality. The use of precise probabilities might be dangerously misleading when making crucial decisions. This issue is related directly to reliability and risk analyses of large-scale potentially hazardous installations.

Imprecise probabilities are intended to make prior assessments and allow for reliability experience and model indeterminacy that is caused by the state of information at hand. They are therefore sometimes referred to as the models of the state-of-knowledge uncertainty. There are several theories of imprecise probabilities and they work differently in practice, in reliability practice in particular. The Dempster-Shafer theory of evidence and the theory of possibility have been considered to be the most promising theories for reliability and safety assessments, however they have been under frequent serious criticism by experts in the area of safety and reliability analyses.

The objective of the paper is to give some summaries of the author's experience in dealing with the Dempster-Shafer theory relating to reliability assessments and to demonstrate how to solve some reliability problems with the help of the theory of coherent imprecise previsions.

2 Experiences in the Dempster-Shafer Theory

The criticism of the Dempster-Shafer theory is based on several points. The first is the failure of Dempster's rule of combination to produce rational results in the case of inconsistent combined pieces of information. Another crucial argued disadvantage of this rule is its inability to combine opinions of different people with overlapping

experiences, making it hardly applicable in safety analysis practice [8]. Furthermore, it turned out that the theory could produce inferences that are formally incoherent [5]. The conclusion in [8] is that the usefulness of the Dempster-Shafer theory and the theory of possibility in probabilistic safety assessments is very doubtful at this time and that this theory should be subjected to the same degree of scrutiny as that applied to the theory of probability.

Attempting to implement the Dempster-Shafer and possibility theories into risk and reliability analyses, the authors of the paper also encountered some difficulties that could not be solved in the frameworks of these theories. The main experienced drawbacks are described below. (Details on the concepts of the Dempster-Shafer theory can be found, for example, in [4]).

2.1 Combination of Homogeneous Bodies of Evidence

The combination of knowledge in a rational way is essential to prior reliability and safety assessments. Therefore a reliable rule of combination is required to work with homogeneous, heterogeneous, consistent and inconsistent pieces of information. Let us consider a case of the combination of homogeneous judgements where it seemed the rule produces good results.

There are n judgements on an event A in the form of simple support functions, $Bel_k(A)=p_k$, where $k=1,\ldots,n$. Source basic assignments m_k obtained from the support functions are: $m_k(A)=p_k$, $m_k(\Omega)=1-p_k$, $k=1,\ldots,n$. From Dempster's rule of combination the following result for combined value of basic probability assignment $m_{1\ldots n}(\Omega)$ can be obtained

$$m_{1\ldots n}(\Omega)=\prod_{i=1}^{n} m_i(\Omega)=\prod_{i=1}^{n}(1-p_i)\,.$$

It is observed that $m_{1\ldots n}(\Omega)$ tends to zero as n tends to infinity. Since $m_{1\ldots n}(\Omega)\to 0$ as $n\to\infty$, hence $m_{1\ldots n}(A)\to 1$, since $m_{1\ldots n}(A)=1-m_{1\ldots n}(\Omega)$. This result tells us that independently of the values $Bel_k(A)$ the combined belief function will be close to 1 when the number of combined belief functions is large. This conclusion renders the results of the combination irrational when combining many experts' opinions.

2.2 Combination of Inconsistent Pieces of Information

Let there be two conflicting bodies of evidence for the same event B, $Bel_1(B)=P_{A1}$, $Pl_1(B)=1$, $Bel_2(B)=0$, $Pl_2(B)=P_{A2}$ and $P_{A1}>P_{A2}$. When P_{A1} is close to 0 and P_{A2} is close to 1, these belief functions represent the fact that expert 1 strongly feels that event A will occur; on the other hand, expert 2 strongly feels that it will not occur. Basic probability assignments for this case are $m_1(B)=P_{A1}$, $m_1(B\cup B^c)=m_1(\Omega)=1-P_{A1}$, $m_2(B^c)=1-P_{A2}$, $m_2(B\cup B^c)=m_2(\Omega)=P_{A2}$, where B^c is the complement of B. In combining these two bodies of evidence we can see that there is one conjunction equal to $\varnothing$, $B\cap B^c=\varnothing$. Therefore the probability mass $m_1(B)m_2(B^c)$ should be distributed between the rest of mass probabilities by renormalization of the basic probability assignment through the constant $\rho=(1-m_1(B)m_2(B^c))^{-1}=(1-P_{A1}(1-P_{A2}))^{-1}$. Thus, the new assignment is $m_{1,2}(B)=\rho P_{A1}P_{A2}$, $m_{1,2}(B^c)=\rho(1-P_{A1})(1-P_{A2})$, $m_{1,2}(B\cup B^c)=m_{1,2}(\Omega)=\rho(1-P_{A1})P_{A2}$. Eventually we get $Bel(B)=\rho P_{A1}P_{A2}$, and $Pl(B)=\rho(P_{A1}P_{A2}+(1-P_{A1})P_{A2})=\rho P_{A2}$. The result tells us if, for example, $Bel_1(B)=P_{A1}=0.9$ and $Bel_2(B^c)=P_{A2}=0.1$, then the combination is $Bel(B)=0.47$ and $Pl(B)=0.526$. This should be considered intuitively inconsistent, since the two extremely conflicting opinions are combined to form quite strong consensus.

2.3 Judgements Admitted in Elicitation

The diversity of direct judgements admitted in a theory of imprecise probabilities is an important point relating to reliability assessment practice. When making judgements in the framework of the Dempster-Shafer theory we must bear in mind the following interpretations of belief functions and basic assignment: $Bel(A)$ measures the total belief that the special element is in A, whereas $m(A)$ measures the amount of belief that one commits exactly to A alone (Shafer [4] and Yager [9]). In fact, the analysts must pose their questions to the experts in such a manner that the experts have to judge whether a special unknown element is confined in A or not. In some practical situations it would be difficult to adjust an issue in a manner matching the idea of the theory. For example, I could not find a way of constructing a belief function from the evidence: 'The reliability of component B is at least as probable as the reliability of prototype A', where A's reliability is imprecise, that is, $Bel(A)$ and $Pl(A)$ were known for A. This kind of comparative evidence appears when comparing the reliabilities of two analogous components. It is obvious that this is evidence that could be useful for prior assessments, but it is difficult to construct the belief and plausibility functions. This is just one example, and there are many more situations where we can suffer from the poor variety of probability judgements to express beliefs in whatever forms are most natural and meaningful for the experts.

The judgement of conditional independence is needed in most problems to cope with systems reliability assessments. In fact, this judgement is a strong structural constraint that should be justified and not taken for granted as is often the case. In doing prior assessments it is difficult sometimes to decide in advance whether events are dependent or not. It is a worthy feature of a theory of probability to leave room for making systems reliability assessments without constraining the analysts with the necessity of making arbitrary or ungrounded

conclusions. The Dempster-Shafer theory does not possess this feature, but the theory of possibility and coherent imprecise probabilities does. It is clear that the cost of a weaker judgement is a lower precision, but in some problems it might be enough and definitely more faithful representation.

2.4 Dependence of Imprecision on the Amount of Information.

One of big advantages that are expected from employing a theory of imprecise probability is the dependence of upper and lower probabilities on the amount of information at hand. For reliability and risk assessments the most important post-action source of information is the number of failures that have happened. The question arises: How can we model the dependence of *Bel* and *Pl* functions on the number of occurrences that have happened? Since the theory is built on the terms of basic assignments, we have to embed this dependence on the number of occurrences in the basic probability numbers *m*. But these probability masses are precise, which makes it difficult to find a straightforward way to allow for this obvious dependence.

As a final remark, Dempster's rule of combination can produce formally incoherent inferences [5], which can negatively effect some practical decisions through incurring sure loss.

The above observations regarding to the Dempster-Shafer theory of evidence should be known for those who are going to use the theory in reliability and safety assessments.

3 Imprecise Prior Reliability Assessments

The latest theory of coherent imprecise probabilities [5], [6], [7] appears to take into account the previous experience in the field, avoid the disadvantages of the predecessors and represent more comprehensive and flexible tools for practitioners.

3.1 Key Concepts of the Theory

(Details on the theory can be found in [5]). The mathematical theory of imprecise probabilities is based on a *behavioral interpretation* and the three fundamental principles: *avoiding sure loss, coherence and natural extension*. The basic concept relating to the behavioral interpretation is the concept of a *gamble*.

A *gamble* is a bounded real-valued function defined on domain Θ. A gamble should be interpreted as a reward whose value depends on the uncertain state $\theta_i \in \Theta$, $i=1,\ldots,n$. If you accept the gamble A, then at some later time the true state θ_k will be determined and you will receive the reward $A(\theta_k)$, in units of utility.

The probabilistic models on which the theory is based are *lower previsions* and their corresponding *upper previsions*. In this paper we will consider a particular case of gambles for which the reward can be either 0 or 1. In this case lower and upper previsions are called *lower* and *upper probabilities* correspondingly. A *lower probability* $\underline{P}$ is a real-valued function defined on some class of gambles $\mathcal{K}$, where $\mathcal{K}$ is called the domain of $\underline{P}$. $\underline{P}(A)$ is interpreted as a supremum price you are willing to pay for the gamble A, which pays 1 unit if event A occurs (and nothing otherwise). Thus each event is identified with a gamble, and for both an event and a gamble we will keep the same notation. The *upper probability* can be written as $\overline{P}(A) = 1 - \underline{P}(A^c)$, where A^c is the set-theoretic complement of A.

$G(A) = A - \underline{P}(A)$ is called the *marginal gamble* on A, since $G(A) + \varepsilon$ is desirable to you for all positive ε. That is, $G(A)$ is "almost desirable", we will denote also $G(A) \in \mathcal{D}$, where $\mathcal{D}$ is a set of almost desirable gambles.

Throughout this paper the following definition of the natural extension is implicit (it is a consequence of the lower envelope theorem [6]). So, the class $\mathcal{M} = \mathcal{M}(\mathcal{D})$ consists of all linear previsions P such that $P(A) \geq 0$ for all A in $\mathcal{D}$. (When the lower and upper probabilities coincide and are coherent, they are called *linear probabilities* and denoted by $P(A)$). Provided $\mathcal{D}$ avoids sure loss, $\mathcal{M}$ is a non-empty set and the natural extension consists of all gambles A, such that $P(A) \geq 0$ for all P in $\mathcal{M}$. The corresponding lower and upper previsions are defined by $\underline{P}(A) = \min\{P(A): P \in \mathcal{M}\}$ and $\overline{P}(A) = \max\{P(A): P \in \mathcal{M}\}$.

The rule of combination of several sources of information discriminates between consistent and inconsistent judgements/models. The combined lower and upper previsions $\underline{P}$ and $\overline{P}$ for the two consistent judgements $\underline{P}_1(A)$, $\overline{P}_1(A)$ and $\underline{P}_2(A)$, $\overline{P}_2(A)$ are defined as follows

$$\left.\begin{aligned} \underline{P}(A) &= \max\{\underline{P}_1(A), \underline{P}_2(A)\} \\ \overline{P}(A) &= \min\{\overline{P}_1(A), \overline{P}_2(A)\} \end{aligned}\right\} \qquad (1)$$

This rule is called the *conjunction rule*.

For inconsistent judgements an alternative rule can be used that is called the *unanimity rule*. According to this rule, the combined lower and upper previsions $\underline{P}$ and $\overline{P}$ are:

$$\left.\begin{aligned} \underline{P}(A) &= \min\{\underline{P}_1(A), \underline{P}_2(A)\} \\ \overline{P}(A) &= \max\{\overline{P}_1(A), \overline{P}_2(A)\} \end{aligned}\right\} \qquad (2)$$

3.2 Analogy-Based Procedure of Reliability Elicitation

In [2], [3] an analogical approach to prior reliability elicitation was offered and briefly described. The main idea of the approach was to analyze analogous components (prototypes), the reliabilities of which are known. Then, through a procedure of comparing dissimilarities and similarities of the component of interest with the analogs, to elicit reliability assessments that, how it was shown, can be only of imprecise nature. We will consider below that the analogical analysis has been done and we deal with the outcomes of it. The outcomes of this analysis can be of four types

$$\left.\begin{array}{l} 1)\ Pr(B) \geq Pr(A), \\ 2)\ Pr(B) \leq Pr(A), \\ 3)\ Pr(B) \sim Pr(A), \\ 4)\ \text{indeterminacy}, \end{array}\right\} \quad (3)$$

where symbol "~" means indifference between the two gambles A and B (lack of difference or distinction between them, that is, we cannot see any reason to assert that the reliability of B will be either higher or lower than A). $Pr(A)$ is a shortened designation of the phrase "Probability of an event A", and A denotes a success state of the prototype A and A^c is the failure of A. B and B^c denote success and failure states for the analogous component B. In a broad sense, if an event or a kind of behavior meets a predetermined criterion, whatever the criterion is, then we say it is a success. If the criterion is violated, then a failure occurs [10].

Consider two different cases of prototypes' reliabilities: when they are precise and imprecise.

3.3 Precise Prototypes' Reliabilities

Let P_A denote a known prototype's precise reliability. For precise probabilities of prototypes the first two outcomes of (3) can be rewritten 1) $Pr(B) \geq P_A$, 2) $Pr(B) \leq P_A$. Behavioral interpretation and results in terms of lower and upper probabilities of these judgements are presented respectively as follows: **1)** P_A is a maximal price which can be paid for gamble B, that is $(B - P_A) \in \mathcal{D}$ ($\mathcal{D}$ is a set of almost desirable gambles) $\Rightarrow P(B - P_A) = P(B) - P_A \geq 0 \Rightarrow \underline{P}(B)=min\{P(B):P(B)\geq P_A\}=P_A$, and $\overline{P}(B)=max\{P(B):P(B) \geq P_A\}=1$; **2)** P_A is a minimal selling price for B, that is $(P_A - B) \in \mathcal{D} \Rightarrow P(P_A - B) = P_A - P(B) \geq 0 \Rightarrow \underline{P}(B)=min\{P(B):P(B)\leq P_A\}=0$, and $\overline{P}(B)=max\{P(B):P(B) \leq P_A\}= P_A$; **3)** any of two gambles is desirable $P_A - B$ and $B - P_A$, that is $(P_A - B) \in \mathcal{D} \Rightarrow$ (see No. 2) $\underline{P}(B)=0$ and $\overline{P}(B)=P_A$, and $(B-P_A) \in \mathcal{D} \Rightarrow$ (see No. 1) $\underline{P}(B)=P_A$, and $\overline{P}(B)=1$. The combination of the two generated models according to (1) gives $\underline{P}(B)=\overline{P}(B)=P_A$; **4)** beliefs about two events (gambles) are indeterminate when the events A and B are not equivalent for us, but between which we cannot figure out preference. Experts' and our choices are simply not determined by the current state of mind or knowledge.

We can only see the result of prior reliability elicitation by use of two forms $[0,P_A]$ or $[P_A,1]$.

Consider now two prototypes A_1 and A_2 with reliabilities P_{A1} and P_{A2} ($P_{A1} \neq P_{A2}$) and a component B, analogous to both A_1 and A_2. To take advantage of having the two sources of reliability information on the component of interest B, the two evidences have to be combined into a single imprecise probability model. The possible cases of relations between P_{A1}, P_{A2} and $Pr(B)$ can be as follows:

1. $Pr(B) \geq P_{A1}$, $Pr(B) \geq P_{A2}$, $P_{A1} \leq P_{A2} \Rightarrow \underline{P}_1(B) = P_{A1}$, $\overline{P}_1(B)=1$ and $\underline{P}_2(B) = P_{A2}$, $\overline{P}_2(B) = 1$
2. $Pr(B) \leq P_{A1}$, $Pr(B) \leq P_{A2}$, $P_{A1} \leq P_{A2} \Rightarrow \underline{P}_2(B) = 0$, $\overline{P}_1(B) = P_{A1}$ and $\underline{P}_1(B) = 0$, $\overline{P}_2(B) = P_{A2}$
3. $Pr(B) \geq P_{A1}$, $Pr(B) \leq P_{A2}$, $P_{A1} \leq P_{A2} \Rightarrow \underline{P}_1(B) = P_{A1}$, $\overline{P}_1(B) = 1$, and $\underline{P}_2(B) = 0$, $\overline{P}_2(B) = P_{A2}$
4. $Pr(B) \geq P_{A1}$, $Pr(B) \leq P_{A2}$, $P_{A1} \geq P_{A2} \Rightarrow \underline{P}_1(B) = P_{A1}$, $\overline{P}_1(B) = 1$, and $\underline{P}_2(B) = 0$, $\overline{P}_2(B) = P_{A2}$
5. Indifference with two prototypes means $Pr(B) = P_{A1}$ and $Pr(B) = P_{A2}$, which is nonsense and cannot be further processed

Case 1. In this case the two judgements are consistent and we must use the conjunction rule of combination (1). The result of combining the two intervals, one of which is included within the other $[\underline{P}_1(B),1] \supset [\underline{P}_2(B),1]$, is the interval $[\underline{P}(B),1] = [\underline{P}_2(B),1] = [P_{A2},1]$.

Case 2. In this case the two judgements are also consistent and we must use the conjunction rule of combination (1). The result of combining the two intervals $[0,\overline{P}_1(B)] \subset [0,\overline{P}_2(B)]$ is the interval $[0,\overline{P}(B)] = [0,\overline{P}_2(B)] = [0,P_{A2}]$.

Case 3. The source underlying intervals are not included within one another but their intersection is not equal to $\varnothing$, i.e., $[0, \overline{P}_2(B)] \cap [\underline{P}_1(B),1] \neq \varnothing$. This means the two judgements are consistent and we have to use the same conjunction rule of combination (1). The result of combining the two intervals is the interval $[\underline{P}(B),\overline{P}(B)]=[\underline{P}_1(B),\overline{P}_2(B)]=[P_{A1},P_{A2}]$.

Case 4. For this case $[0, \overline{P}_2(B)] \cap [\underline{P}_1(B),1] = \varnothing$, which means that the first model says (or the first expert) that the event B^c is highly probable, but the second model (expert) considers the opposite event B as highly probable. The two models are inconsistent, and we have to use the unanimity rule (2). The result of combining is maximally imprecise $[\underline{P}(B),\overline{P}(B)]=[0,1]$ and states our complete ignorance concerning the reliability of the analogous component B. Complete ignorance is modeled by *vacuous probabilities* $\underline{P}(B)=0$ and $\overline{P}(B)=1$. (How we could see earlier this case of combination of the two

inconsistent models was crucial for Dempster's rule of combination).

Consider three prototypes A_i, $i=1,2,3$ with their known precise reliabilities $P_{A1}<P_{A2}<P_{A3}$. Let the analogy-based elicitation give the following results: $Pr(B)\geq P_{A1}$, $Pr(B)\leq P_{A2}$, and $Pr(B)\geq P_{A3}$. Results of combining these three intervals are different depending on the sequence of combination. If the sequence is $\{[0,\overline{P}_2(B)]\oplus[\underline{P}_1(B),1]\}\oplus[\underline{P}_3(B),1]$, where $\oplus$ denotes "combination", then the result of the combination is the interval $[P(B),1]=[\underline{P}_1(B),1]=[P_{A1},1]$. If the sequence is $\{[\underline{P}_1(B),1]\oplus[\underline{P}_3(B),1]\}\oplus[0,\overline{P}_2(B)]$, then the result is the interval $[0,1]$, i.e., vacuous probabilities (complete ignorance).

When all of the three intervals are consistent, for example, $Pr(B)\geq P_{A1}$, $Pr(B)\geq P_{A2}$, $Pr(B)\leq P_{A3}$, and $P_{A1}<P_{A2}<P_{A3}$, the combined interval is unique and more precise compared to all the three combining underlying judgements, that is $[\underline{P}_1(B),1]\oplus[\underline{P}_2(B),1]\oplus[0,\overline{P}_3(B)] = [\underline{P}_2(B), \overline{P}_3(B)]$. It should be stressed that when combining more than two judgements, we must do it pairwise.

Having analyzed the above results, the following generalization can be accomplished.

There are n judgements on the reliability of the component B - $[0,\overline{P}_i]$, $i=1,\ldots l$, and $[\underline{P}_j,1]$, $j=l+1,\ldots,n$. The combined intervals for different possible cases are presented below:

1. $[0,\overline{P}_1]\subseteq\ldots\subseteq[0,\overline{P}_n]\Rightarrow \underline{P}(B)=0,\ \overline{P}(B)=\overline{P}_1$

2. $[\underline{P}_l,1]\subseteq\ldots\subseteq[\underline{P}_n,1]\Rightarrow \underline{P}(B)=\underline{P}_1,\ \overline{P}(B)=1$

3. $[0,\overline{P}_1]\subseteq\ldots\subseteq[0,\overline{P}_l]$ and $[\underline{P}_{l+1},1]\subseteq\ldots\subseteq[\underline{P}_n,1]$:
   a) if $\underline{P}_n\geq\overline{P}_1$ and $\overline{P}_l\leq\underline{P}_{l+1}\Rightarrow \underline{P}(B)=0,\ \overline{P}(B)=1$,
   b) if $\underline{P}_n\geq\overline{P}_1$ and $\overline{P}_l\geq\underline{P}_{l+1}\Rightarrow \underline{P}(B)=0,\ \overline{P}(B)=\min_i\{\overline{P}_i:\underline{P}_{l+1}\leq\overline{P}_i\}$,
   c) if $\underline{P}_n\leq\overline{P}_1$ and $\overline{P}_l\leq\underline{P}_{l+1}\Rightarrow \underline{P}(B)=\max_j\{\underline{P}_j:\overline{P}_1\geq\underline{P}_j\},\ \overline{P}(B)=1$,
   d) if $\underline{P}_n\leq\overline{P}_1$ and $\overline{P}_l\geq\underline{P}_{l+1}\Rightarrow \underline{P}(B)=\max_j\{\underline{P}_j:\overline{P}_1\geq\underline{P}_j\},\ \overline{P}(B)=\min_i\{\overline{P}_i:\underline{P}_{l+1}\leq\overline{P}_i\}$.

These results can be extended for general intervals of $[\underline{P}_i,\overline{P}_i]$, where $0\leq\underline{P}_i\leq\overline{P}_i\leq 1$ for any $i=1,\ldots,n$. The generalization can be done if we take into account that according to the conjunction rule (1), any interval $[\underline{P}_i,\overline{P}_i]$ may be represented as the combination of two intervals $[0,\overline{P}_i]\oplus[\underline{P}_i,1]$. That is, source combined probability intervals always may be represented as set of intervals of the forms $[0,\overline{P}]$ and $[\underline{P},1]$, and, hence, the above algorithm can be considered as general algorithm of interval combination.

3.4 Imprecise Prototypes' Reliability

The above-described procedure of eliciting and combining reliabilities from analogous components dealt with the precise probabilities of prototypes. Let us consider a more general case when prototypes are qualified by imprecise models; that is, lower and upper reliabilities $\underline{P}(A_i)$ and $\overline{P}(A_i)$ of the success states of components A_i, $i=1,\ldots,n$ are known.

When the reliability of prototype A is qualified by the two numbers $\underline{P}(A)$ and $\overline{P}(A)$ the analogical procedure of prior reliability elicitation must be constructed to be able to distinguish cases compared to precise prototypes' reliability. If, for example, the result of the judgements is $Pr(B)\geq Pr(A)$, then we should be sure whether it is possible to make a more precise judgement of $Pr(B)\geq\overline{P}(A)$. If not, we have to accept the judgement of $Pr(B)\geq Pr(A)$. Thus, the following outcomes must be considered: **1)** $Pr(B)\geq\overline{P}(A)$, **2)** $Pr(B)\geq Pr(A)$, **3)** $Pr(B)\leq\underline{P}(A)$, **4)** $Pr(B)\leq Pr(A)$, **5)** $Pr(B)\sim Pr(A)$ and **6)** indeterminacy.

Once we have arrived at one of these conclusions, we can translate them into statements about class $\mathcal{D}$, of the almost-desirable gambles and corresponding lower and upper probabilities $\underline{P}$ and $\overline{P}$. So, correspondingly we have:

1. $(B-\overline{P}(A))\in\mathcal{D}\Rightarrow P(B-\overline{P}(A))=P(B)-\overline{P}(A)\geq 0 \Rightarrow \underline{P}(B)=\min\{P(B):P(B)\geq\overline{P}(A)\}=\overline{P}(A)$, and $\overline{P}(B)=\max\{P(B):P(B)\geq\overline{P}(A)\}=1$, that is $Pr(B)=[\overline{P}(A),1]$;
2. $(B-A)\in\mathcal{D}\Rightarrow P(B-A)=P(B)-P(A)\geq 0\Rightarrow \underline{P}(B)=\min\{P(B):P(B)\geq P(A)\}=\underline{P}(A)$, and $\overline{P}(B)=\max\{P(B):P(B)\geq P(A)\}=1$, that is $Pr(B)=[\underline{P}(A),1]$;
3. $(\underline{P}(A)-B)\in\mathcal{D}\Rightarrow P(\underline{P}(A)-B)=\underline{P}(A)-P(B)\geq 0 \Rightarrow \underline{P}(B)=\min\{P(B):\underline{P}(A)\geq P(B)\}=0$, and $\overline{P}(B)=\max\{P(B):\underline{P}(A)\geq P(B)\}=\underline{P}(A)$, that is $Pr(B)=[0,\underline{P}(A)]$;
4. $(A-B)\in\mathcal{D}\Rightarrow P(A-B)=P(A)-P(B)\geq 0\Rightarrow \underline{P}(B)=\min\{P(B):P(A)\geq P(B)\}=0$, and $\overline{P}(B)=\max\{P(B):P(A)\geq P(B)\}=\overline{P}(A)$, that is $Pr(B)=[0,\overline{P}(A)]$;
5. $(A-B)\in\mathcal{D}\Rightarrow \underline{P}(B)=0$ and $\overline{P}(B)=\overline{P}(A)$ (see No. 4)
   $(B-A)\in\mathcal{D}\Rightarrow \underline{P}(B)=\underline{P}(A)$ and $\overline{P}(B)=1$ (see No.2). The combination of these two models according to (1) yields $\underline{P}(B)=\underline{P}(A)$ and $\overline{P}(B)=\overline{P}(A)$.
6. Indeterminacy does not suppose any solution.

4 Systems Reliability

Quantitative systems reliability analysis is based on Boolean algebra, where the events either occur or do not occur. Most systems can be viewed as sets of series and parallel subsystems from reliability standpoint. Defining how to calculate the reliability of series and parallel systems with imprecise probabilities will cover a wide range of practical tasks and lay down a principle foundation for systems reliability assessing.

Let us say that there are two events $A, B \subset \Theta$. Imprecise probabilities $\underline{P}(A)$, $\overline{P}(A)$, $\underline{P}(B)$ and $\overline{P}(B)$ are known and satisfy the coherence constraints $0 \le \underline{P}(A) \le \overline{P}(A) \le 1$ and $0 \le \underline{P}(B) \le \overline{P}(B) \le 1$. What are the formulae for calculating $\underline{P}(A \cup B)$, $\underline{P}(A \cap B)$, $\overline{P}(A \cup B)$, and $\overline{P}(A \cap B)$ depending on $\underline{P}(A)$, $\overline{P}(A)$, $\underline{P}(B)$ and $\overline{P}(B)$? (The events $A \cup B$ and $A \cap B$ characterize a series and parallel system success states correspondingly).

Unlike conventional theory, the lower and upper probabilities of the unions and intersections (depending on the lower and upper probabilities of X and Y) can be obtained without the judgement of conditional dependence or independence. This possibility can be important when we are completely ignorant of conditional dependence. When considering the array of judgements, the judgement of logical independence is the weakest structural constraint for a system. It is reasonable to expect that the results based on the stronger structural judgement are more precise compared to the weaker one.

4.1 Series and Parallel Reliability Structures

The knowledge of system S consists of two components A and B, each of which has two states A, A^c and B, B^c, generates a structure on Θ such that all of the events $A \cap B$, $A \cap B^c$, $A^c \cap B$ and $A^c \cap B^c$ are non-empty. In this case the two events A and B are called *logically independent* [1] and [5].

We say that the two components are connected in series if the failure of either one of the components causes an immediate failure of the system, or, otherwise, the system S_s is in a success state if the both components A **and** B are in a success state, that is $S_s = X \cap Y$ (see, for example, [10]). For this system the task is to calculate $\underline{P}(A \cap B)$ and $\overline{P}(A \cap B)$. The expressions for doing it are the following [5]:

$$\underline{P}_l(A \cap B) = max[0, \underline{P}(A) + \underline{P}(B) - 1]$$
$$\overline{P}_l(A \cap B) = min[\overline{P}(A), \overline{P}(B)]$$

Designation 'P_l' with the sub l indicates that the resulting probabilities are obtained based on the judgement of logical independence.

A system S_p of two components A and B is connected in parallel if the system fails only if both components fail, or, otherwise, the system S_p is in a success state if either A **or** B is in a success state, that is, $S_p = A \cup B$. For this system the task is to calculate $\underline{P}(A \cup B)$ and $\overline{P}(A \cup B)$.

$$\underline{P}_l(A \cup B) = max[\underline{P}(A), \underline{P}(B)],$$
$$\overline{P}_l(A \cup B) = min[1, \overline{P}(A) + \overline{P}(B)].$$

General expressions for calculating the imprecise probabilities of the two kinds of the systems have been obtained. So, for a system consisting of n components connected in **series** we have

$$\underline{P}_l\left(\bigcap_{i=1}^{n} A_i\right) = \begin{cases} 0 & \text{if} \quad \sum_{i=1}^{n} \underline{P}(A_i) \le n-1 \\ \sum_{i=1}^{n} \underline{P}(A_i) - (n-1) & \text{if} \quad \sum_{i=1}^{n} \underline{P}(A_i) > n-1 \end{cases},$$

$$\overline{P}_l\left(\bigcap_{i=1}^{n} A_i\right) = \min_{1 \le i \le n} [\overline{P}(A_i)] \qquad (4)$$

For a system consisting of n components connected in **parallel**

$$\underline{P}_l\left(\bigcup_{i=1}^{n} A_i\right) = \max_{1 \le i \le n} [\underline{P}(A_i)],$$

$$\overline{P}_l\left(\bigcup_{i=1}^{n} A_i\right) = \begin{cases} \sum_{i=1}^{n} \overline{P}(A_i) & \text{if} \quad \sum_{i=1}^{n} \overline{P}(A_i) \le 1 \\ 1 & \text{if} \quad \sum_{i=1}^{n} \overline{P}(A_i) > 1 \end{cases} \qquad (5)$$

The imprecision $\Delta = \overline{P} - \underline{P}$ of the both systems is nonlinear function components imprecise probabilities. Qualitatively the behaviour of the imprecise probabilities of series and parallel systems can be characterized as follows. If component reliability for series systems is close to 1, then the lower probability of the system converges to zero very quickly as n increases; the upper probability differs from 1 if there is at least one component in the system the upper reliability of which is less then 1. The upper reliability of a parallel system converges very quickly to 1 as n increases; now the lower probability is greater than zero if at least one component has a lower reliability greater than 0.

Now consider the case of conditional independence.

Using the generalized Bayes rule (for details on the rule see [2]) for two independent events A and B through natural extension we can get the following equations:

$$\left.\begin{array}{c} \underline{P}_c(A\cap B)=\underline{P}(A)\underline{P}(B), \\ \overline{P}_c(A\cap B)=\overline{P}(A)\overline{P}(B), \\ \underline{P}_c(A\cup B)=\underline{P}(A)+\underline{P}(B)-\underline{P}(A)\underline{P}(B), \\ \overline{P}_c(A\cup B)=\overline{P}(A)+\overline{P}(B)-\overline{P}(A)\overline{P}(B), \end{array}\right\} \quad (6)$$

where the sub c indicates that the resulting probabilities are obtained based on the judgement of conditional independence.

It is easy to check that $\underline{P}_l(A\cap B) \le \underline{P}_c(A\cap B)$, $\overline{P}_c(A\cap B) \le \overline{P}_l(A\cap B)$ and $\underline{P}_l(A\cup B) \le \underline{P}_c(A\cup B)$, $\overline{P}_c(A\cup B) \le \overline{P}_l(A\cup B)$. This means that imprecision in the case of the judgement of conditional independence is less that what would have been expected.

Consider an example of a systems of three components, the structure function of which is $S=A_1\cap(A_2\cup A_3)$, where S is a success state of the system S, see Fig. 1.

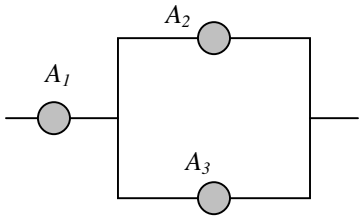

Figure 1: Reliability structure of a system of three components

Under the conditions $\underline{P}(A_1)=\underline{P}(A_2)=\underline{P}(A_3)=\underline{P}$ and $\overline{P}(A_1)=\overline{P}(A_2)=\overline{P}(A_3)=\overline{P}$ the imprecise probabilities of S are calculated through the expressions $\underline{P}_l(S)=max\{0, [\underline{P}(A_1)+max(\underline{P}(A_2),\underline{P}(A_3))-1]\}=max(0,2\underline{P}-1)$, $\overline{P}_l(S) = min\{\overline{P}(A_1),\overline{P}(A_2),\overline{P}(A_3)\}=\overline{P}$, $\underline{P}_c(S)=\underline{P}(A_1)[\underline{P}(A_2)+\underline{P}(A_3)-\underline{P}(A_2)\underline{P}(A_3)]=\underline{P}(2\underline{P}-\underline{P}^2)$ and $\overline{P}_c(S)=\overline{P}(A_1)[\overline{P}(A_2)+\overline{P}(A_3) - \overline{P}(A_2)\overline{P}(A_3)]=\overline{P}(2\overline{P}-\overline{P}^2)$. These functions are graphed in Fig. 2. It is seen that models based on the judgement of conditional independence are substantially more precise compared to those based on logical independence. So, for arbitrary $\underline{P}_l$ and $\overline{P}_l$ $\Delta_{cl}=\overline{P}_{cl}(S) - \underline{P}_{cl}(S) < \Delta_{ll}=\overline{P}_{ll}(S) - \underline{P}_{ll}(S)$. Even though the reliability of components in a system are precise, the judgement of logical independence generates imprecise probabilities.

On the basis of formulae (4) and (5) an algorithm for quickly calculating the lower and upper reliabilities of a system composed of components connected both in series and parallel has been worked out and its performance has been checked for some systems tasks.

5 Conclusions

1. The work described in this paper summarizes briefly some of the author's experiences in employing imprecise probabilities for reliability and risk assessments. Multiple attempts to implement the Dempster-Shafer theory of evidence in reliability practice have failed due to the serious disadvantages found. Some of them, for example, the combination of inconsistent judgements can be corrected in principle, but most of them, not. The disadvantages of the theory make it problematical for doing reliability assessments.

2. The other theory of coherent imprecise previsions appeared to be a practical and applicable reliability analysis tool. It provides clear and tractable results when combining reliabilities from different prototypes even though the bodies of evidence are inconsistent. A way of transition from the results of analogical comparative judgements to imprecise probabilities with behavioral interpretation was demonstrated.

3. System reliability assessments were restricted by considering the sets of components connected in series and parallel. On the basis of the theory of coherent imprecise probabilities we now have some practical results that cannot be obtained within the framework of conventional probability and the Dempster-Shafer theory. So, the formulae were obtained that allow for getting the lower and upper probabilities without the presumption of conditional independence, which can be useful for doing rough and quick assessments when we are ignorant or in doubt about the independence of components in a system, or we know that the components are dependent but do not know to what extent. For some practical tasks it can be satisfactory and a cheap way of solving the issue. If there are grounds to judge conditional independence of components in a system, the lower and upper are more precise and can be calculated according to the formulae provided.

Acknowledgement

This work was completed at the Center for Intelligent Systems, State University of New York at Binghamton, where the author, hosted by Professor George Klir, studied as a Fulbright Scholar. The author expresses his deep gratitude to the Lipsmeyers for their versatile support and help, to George Klir, whose aura of creativity and kind advice were a source of constant encouragement, and to Yevgeny Filimonov for his valuable comments.

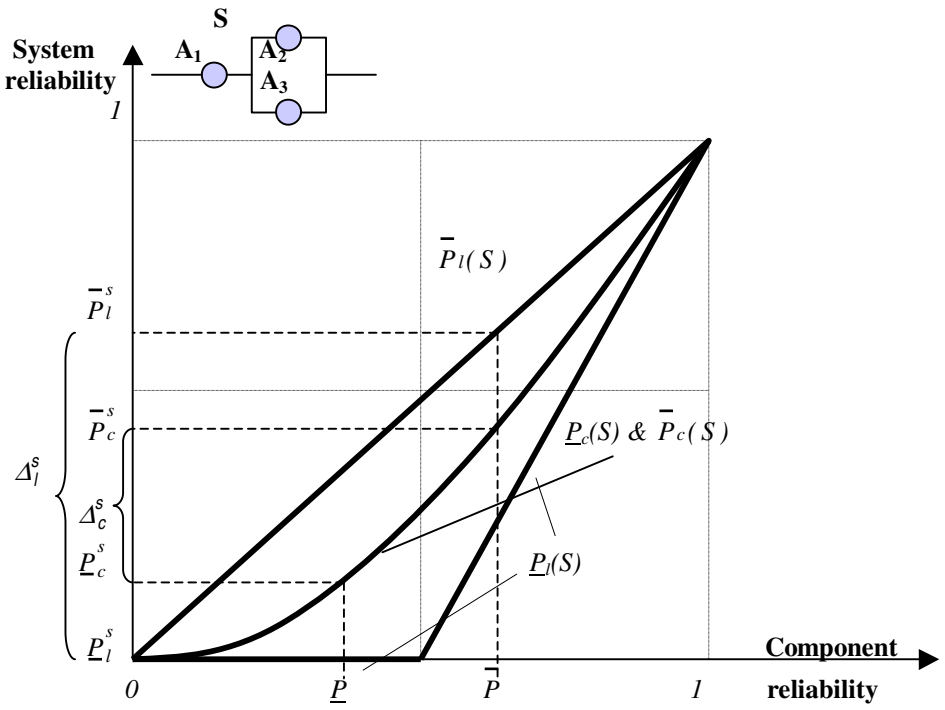

Figure 2: Upper and lower probability functions for both conditionally independent and logically independent components for a system of three components

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