An Experimental Study of Updating Ambiguous Beliefs

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Abstract

"Ambiguous beliefs" are beliefs which are inconsistent with a unique, additive prior. The problem of their update in face of new information has been dealt with in the theoretical literature, and received several contradictory answers. In particular, the "maximum likelihood update" and the "full Bayesian update" have been axiomatized. This experimental study attempts to test the descriptive validity of these two theories by using the Ellsberg experiment framework.

Keywords. Decision Making, Uncertainty, Capacities, Updating, Conditioning rules.

1 Introduction

The problem of decision making under uncertainty (or ambiguity) as opposed to risk is receiving growing attention in statistics, economics and decision theory, as well as in artificial intelligence and game theory.

While the body of literature on the subject is motivated by experimental studies which seem to refute the universality of the classical Bayesian paradigm (subjective expected utility theory), it seems that, at this point of time, theory precedes experiments. In particular, the question of updating "ambiguous" beliefs was raised in economic theory and artificial intelligence. It received several contradictory answers, which were axiomatically justified by theoretical works of the authors. However, both the axioms and the derived rules can only be judged by introspection[1]. Despite their behavioral nature, there has not been any attempt to empirically or experimentally test them.

The aim of the present research is to start filling up this gap. The simple experiment designed, following a suggestion of David Kreps, is able to test the descriptive validity of the main two updating rules. Beside its simplicity, this experiment presents the interest of being a natural and direct complement to the famous Ellsberg (1961) experiment.

The importance of Ellsberg's findings came from the fact that, in addition to providing an empirical refutation of the expected utility paradigm (which had already been obtained with Allais' (1953) "paradox"), they undermined the concept of subjective probability per se, by being incompatible with the very notion of an (additive) probability measure as representing beliefs. That is to say, neither expected utility maximization nor any other reasonable procedure which relies only on the outcome distributions induced by an additive probability could account for observed choices.

Although his original motivation was somewhat different, Schmeidler (1982, 1984, 1986, 1989) suggested a generalization of expected utility which could accommodate Ellsberg's evidence. He provided an axiomatic derivation of both utilities and not-necessarily-additive probabilities, such that a decision maker's preferences are equivalent to "expected" utility maximization, where expectation with respect to a non-additive measure is computed by the Choquet integral (Choquet, 1953-4 ; Denneberg, 1994).

Following this line, many authors have provided axiomatizations of "Choquet Expected Utility" maximization, or of related models, in a variety of frameworks and contexts. Among these are Gilboa (1987),

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[1]This remark does not apply to the third approach, used by Walley (1991), which derives his updating rule from principles of self consistency.

Fishburn (1988), Wakker (1989, 1991, 1993), Jaffray (1989a&b), Nakamura (1990), Sarin and Wakker (1992) and Chew and Karni (1992), Jaffray and Wakker (1994).

In a different model, Gilboa and Schmeidler (1989) characterized preferences which may be represented by a utility function and a set of additive measures, in the sense that preference obey maximization of the minimal expected utility over all measures in the given set. (See also Bewley (1986) who deals with a set of probabilities with partially ordered preferences). Related works are Levi (1980, 1986) and Kelsey (1990). These preferences can also be represented by the non-additive model (with maximization of the Choquet integral) in case the set of measures is the core of a convex non-additive measure.

2 Formalization

Formally, if decisions are identified with acts f mapping a set of states of nature S, endowed with an algebra of events $\mathcal{A}$, into a consequence space $\mathcal{X}$:

(i) in the non-additive probability model, decisions are made so as to maximize Choquet expected utility:

$$\int u(f)\ dv = \int_o^\infty v(u(f) \geq t)dt + \int_{-\infty}^o [v(u(f) \geq t) - 1]dt \tag{1}$$

where $u : \mathcal{X}. \rightarrow \mathbb{R}$ is a "von Neumann-Morgenstern type" utility and v is a capacity on $\mathcal{A}$, i.e.,

$v(\phi) = 0$; $v(S) = 1$; $A \subset B \Rightarrow v(A) \leq v(B)$ for all $A, B \in \mathcal{A}$;

note that, when $f(\mathcal{X})$ is finite , and $f(\mathcal{X}) = \{x_1, ..., x_i, ..., x_n\}$ is indexed so that $u(x_i) \geq u(x_{i+1})$ for all i,

$$\int u(f)\ dv = \sum_{i=1}^{n-1} [u(x_i) - u(x_{i+1})] v(\bigcup_{j=1}^{i} A_j) + u(x_n) \tag{2}$$

where $A_i = f^{-1}(x_i),\ i = 1, ..., n$;

(ii) in the multiple prior model, decisions are made so as to maximize

$$inf_{P \in C} \int u(f)\ dP \tag{3}$$

where C is interpretable as the decision maker's set of priors.

Links between the two models can be established when the capacity v in the first model is *convex* , in the sense that

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B)\ for\ all\ A, B \in \mathcal{A} \tag{4}$$

in which case it can be shown that the second model with set of priors

$C = Core(v), i.e.$

$$\begin{aligned} C &= \{P : P\ is\ a\ probability \\ and\ P(A) &\geq v(A)\ for\ all\ A \in \mathcal{A}\} \end{aligned} \tag{5}$$

describes the same decision rule as the first model.

We shall also consider a natural generalization of the second model, the family of Hurwicz α-criteria (with $\alpha \in [0, 1]$ interpretable as a pessimism or ambiguity aversion index), where an act f is evaluated by

$$\alpha\ inf_{P \in C} \int u(f)\ dP + (1 - \alpha)\ sup_{P \in C} \int u(f)\ dP \tag{6}$$

Hurwicz α-criteria are also partially consistent with the first model since, whenever the capacity $w = inf_{P \in C} P$ is convex and $C = Core(w)$, expression (6) reduces to the Choquet integral $\int u(f)\ dv$ with the capacity v defined by

$$v(A) = \alpha w(A) + (1 - \alpha)[1 - w(A^c)]\ for\ all\ A \in \mathcal{A} \tag{7}$$

(see Jaffray and Philippe (1997)).

Potential applications, in the theoretical as in the practical domains, are likely to involve sequential and conditional decision making. It is therefore crucial to provide an appropriate updating rule. When the capacity v in the first model is additive or, equivalently, when the prior in the second model is unique, the only way to update is to use Bayes' rule. By contrast, the problem of updating ambiguous beliefs in the face of new information has received several incompatible answers. In this study we focus on two :

a) the *Maximum Likelihood Update* (MLU), axiomatized by Gilboa and Schmeidler (1993), also known as the Dempster rule, since it was first proposed by Dempster (1967) in the framework of Dempster-Shafer belief function theory (Shafer, 1976):

Given ambiguous beliefs characterized by a capacity v and an event $A \in \mathcal{A}$, the MLU of v given A, $v_1(./A)$, is given by

$$v_1(B/A) = \frac{v(B \cup A^c) - v(A^c)}{1 - v(A^c)}\ for\ all\ B \in \mathcal{A} \tag{8}$$

Equivalently, it can be described in the multiple-prior language as follows: consider only those $P \in \mathcal{C}$ which maximize $P(A)$ and update probabilities according to Bayes' rule to obtain a new set of probabilities on A; they form exactly the set $Core(v_1(./A))$, where $v_1(./A)$ is given by (8) and is a convex capacity when v is itself convex (Gilboa and Schmeidler (1993)).

b) the *Full Bayesian Update* (FBU), in which the new beliefs are given by

$$v_2(B/A) = \frac{v(B \cap A)}{v(B \cap A) + 1 - v(B \cup A^c)} \text{ for all } B \in \mathcal{A} \tag{9}$$

This rule has been proposed and studied by Walley (1981, 1991), de Campos, Larnata and Moral (1990) and Fagin and Halpern (1990). From the multiple prior point of view, it amounts to updating all priors in C according to Bayes' rule and using the set of posteriors on $A, C^A = \{P^A, P \in C\}$, as the new set C.

Note that $v_2(./A)$ is convex as soon as $v(.)$ is convex (see Walley (1981)) and that $v_2(B/A) = inf_{P \in C} P(B/A)$; however, C^A is generally only a strict subset of $Core(v_2(./A))$ (Jaffray, 1992).

Note also that, in the case of the Hurwicz α-criteria, it is the lower probability w of (7) which is updated either as $w_1(B/A)$, given by (8), or as $w_2(B/A)$, given by (9), resulting in the updating of v by

$$v_i(B/A) = \alpha w_i(B/A) + (1-\alpha)[1 - w_i(B^c/A)] (i = 1, 2) \tag{10}$$

As we shall see, the same Choquet expected utility criterion, depending on its combination with either the MLU rule or the FBU rule, can predict different choices even in the most elementary decision situations, thus providing a simple test of their relative descriptive validity.

However, since "given event A" or "conditionally to event A" can receive more than one interpretation, the relevance of the MLU or the FBU rule can certainly depend on the prevailing interpretation in the specific decision problem considered. This question is discussed, in relation with the design of our experiment, in section 6.

3 The main test

Consider Ellsberg's (1961) example. There is an urn with 90 balls, out of which 30 are red and 60 blue or yellow. A ball is to be drawn at random, and the decision maker faces the alternatives described in Table 1:

	R	B	Y	$\inf EU$	$\sup EU$
d_R	m	0	0	$\frac{1}{3}u(m)$	$\frac{1}{3}u(m)$
d_B	0	m	0	$(\frac{1}{3} - \lambda)u(m)$	$(\frac{1}{3} + \lambda)u(m)$
$d_{R \cup Y}$	m	0	m	$(\frac{2}{3} - \lambda)u(m)$	$(\frac{2}{3} + \lambda)u(m)$
$d_{B \cup Y}$	0	m	m	$\frac{2}{3}u(m)$	$\frac{2}{3}u(m)$

($d_R, d_B, d_{R \cup Y}, d_{B \cup Y}$ are the available acts ; R, B and Y stand for the events red, blue and yellow ball, respectively, where the matrix determines the payoff with $m = 1\ 000\ FF$; λ parametrizes the intensity of subjective ambiguity ; the expression of $inf\ EU$ and $sup\ EU$ assume $u(0) = 0$); .

Table 1

Ellsberg has observed (in a similar set-up) the predominant preference pattern $d_R \succ d_B, d_{R \cup Y} \prec d_{B \cup Y}$, which is inconsistent with expected utility theory with an unambiguous prior or any other theory satisfying Savage's (1954) "sure-thing principle". It is, however, explainable by the models discussed above: assume, for simplicity, that the DM (implicitly or explicitly) considers all priors $(P(R), P(B), P(Y))$ in a subjective set of priors $\{(1/3, p, 2/3 - p)/1/3 - \lambda \leq p \leq 1/3 + \lambda\}$ where $0 < \lambda \leq 1/3$; note that it is a subset (a strict subset for $\lambda \neq 1/3$) of the (objective) whole set of priors consistent with the data. Then the minimal expected utility of each of the four acts are is given in the $inf\ EU$ column of Table 1. The $sup\ EU$ column allows one to check that the Hurwicz α-criteria would yield the same preference pattern ($d_R \succ d_B, d_{R \cup Y} \prec d_{B \cup Y}$) for $\alpha > 1/2$ and the opposite pattern ($d_R \prec d_B, d_{R \cup Y} \succ d_{B \cup Y}$) (also inconsistent with Savage's axiom) for $\alpha < 1/2$.

Next consider the same DM and assume that some information is revealed to him/her: after the ball was drawn, the DM is told that it is not yellow: Namely, the event $R \cup B$ is known to have occured[2].

What will be the decision maker's preferences now between d_R and d_B?

Let us consider the predictions of the two update models.

- MLU. Only one prior maximizes $P(R \cup B)$ and that is$(\frac{1}{3}, \frac{1}{3} + \lambda, \frac{1}{3} - \lambda)$. Updating this prior given $R \cup B$ leads to the new evaluations shown in Table 2. Thus we *would expect to observe* $d_B \succ d_R$ (as predicted by all previously mentioned criteria, including Hurwicz for all $\alpha \in [0, 1]$).
- FBU. According to this rule, we obtain the fol-

[2] $d_{R \cup Y}$ and $d_{B \cup Y}$ have become logically equivalent to d_R and d_B, and we need only consider the last alternatives.

lowing set of posteriors

$(P(R/R\cup B), P(B/R\cup B) : \{(\frac{1/3}{1/3+p};), \frac{p}{1/3+p} / \frac{1}{3} - \lambda \leq p \leq \frac{1}{3} + \lambda\}$.

Table 2 gives the evaluation based on the minimal and maximal expected utility for each act

	MLU $+any\ criterium$	FBU $\inf\ EU$	FBU $+\sup\ EU$
d_R	$\frac{1}{2+3\lambda}u(m)$	$\frac{1}{2+3\lambda}u(m)$	$\frac{1}{2-3\lambda}u(m)$
d_B	$\frac{1+3\lambda}{2+3\lambda}u(m)$	$\frac{1-3\lambda}{2-3\lambda}u(m)$	$\frac{1+3\lambda}{2+3\lambda}u(m)$

Table2

according to which we *expect to observe* $d_R \succ d_B$ (as predicted by all the decision criteria considered, including Hurwicz's for all $\alpha \in [0, 1]$).

Thus a clear tendency of the subjects to prefer d_R to d_B would lend support to FBU theory of updates; a reverse tendency would support MLU theory (the interpretation of the results is further discussed in section 6).

4 Experiment Design

Subjects. Subjects were undergraduate economics students with no background in decision theory, divided into two groups, with $N_I = 44$ subjects in sample I and $N_{II} = 62$ subjects in sample II.

Procedure. Sample I subjects were interviewed individually and responses, like stimuli, were expressed verbally, in a single session. Sample II subjects were interviewed and gave their answers simultaneously in a written form[3].

Subjects were described the initial data concerning the Ellsberg urn (30 red balls, 60 blue or yellow balls) and required to choose:

(i) between alternatives d_R and d_B (Question 1);

(ii) between alternatives $d_{R\cup B}$ and $d_{B\cup Y}$, (Question 2),

which constitutes the original Ellsberg experiment.

Then, they were asked to choose again between d_R and d_B after receiving the additional information that the ball had already been drawn and was "non yellow" (Question 3).

[3]Thus eliminating the possibility of a verbal influence of the experimenter on the results.

Apart from strict preferences, subjects had the possibility to declare that they were indifferent (answer "*indif.*") between the two alternatives or that they felt unable to make a choice (answer "*undec.*").

Pay-offs were hypothetical; the same prize, $m = FF$ 1 000, was used throughout the experiments[4].

The questionnaire also contained additional questions of two types:

(i) preliminary questions intended to verify that the subjects did not misunderstand the data and respected fundamental rationality requirements, e.g., declared prefer d_R to d_B when the proportions of red and blue balls in the urn where exactly known and equal to 1/3 and 1/9 respectively;

(ii) complementary questions, similar to the third question, where the additional information was brought under a different form, such as " a non yellow ball has been drawn and then *replaced* in the urn".

Neither the preliminary nor the complementary questions are part of the experiment proper, and they did not contribute to the results; they were only used either to check the subjects comprehension or to get insights into the interpretation of their behavior; for the same reason, subjects were asked to document their reasoning for the choices.

5 Results

The answers to the first two questions (Ellsberg's original experiment) are given in Table 3 :

$Qu1$ / $Qu2$	d_R	d_B	$indif.$	$undec.$
$d_{R\cup Y}$	$\underline{3}$	*2*	1	0
$d_{B\cup Y}$	**30**	$\underline{1}$	1	0
$indif.$	0	1	$\overline{4}$	0
$undec.$	1	0	0	0

Sample I ($N_I = 44$)

[4]On the possible differences between pairwise choices with real and hypothetical payments, see Camerer (1995) and Beatie and Loomes (1997) .

$Qu2$ \ $Qu1$	d_R	d_B	$indif.$	$undec.$
$d_{R\cup Y}$	$\underline{6}$	*2*	2	0
$d_{B\cup Y}$	**40**	$\underline{7}$	1	0
$indif.$	2	0	$\overline{1}$	0
$undec.$	1	0	0	0

Sample II ($N_{II} = 62$)

Bold : Pessimists ; Italics : Optimists ; Underlined : Bayesians ; Overlined : Bayesians or $\alpha = 1/2$

Table 3

The ambiguity averse (pessimistic) pattern $(d_R, d_{B\cup Y})$, found dominant by Ellsberg, is exhibited by 30 subjects in sample I (68%) and 40 subjects in sample II (65%); the opposite ambiguity prone (optimistic) pattern, $(d_B, d_{R\cup Y})$, is only found in 2 subjects in sample I (5%) and also in 2 subjects in sample II (3%).

Patterns consistent with Savage's axiom (and SEU theory), $(d_R, d_{R\cup Y})$, $(d_B, d_{B\cup Y})$, and $(indif., indif.)$ describe the choices of only 8 subjects in sample I (16%) and 14 subjects in sample II (23%)[5].

The discrimination between the two possible updating rules can only be done for those subjects whose behavior is consistent with a multiple prior interpretation : the $(30 + 40 =)$ 70 pessimistic subjects, the $(2 + 2 =)$ 4 optimistic subjects, and the $(4 + 1 =)$ 5 $(indif., indif.)$ pattern subjects (whose choices can be explained by the Hurwicz criterion with $\alpha = \frac{1}{2}$ as well as by SEU theory).

Their choices of these $(36+43 =)$ 79 subjects in Question 3 are given in Table 4 below:

$Qu1\&2$ \ $Qu3$	d_B	d_R	$indif.$	$undec.$
$d_R,\ d_{B\cup Y}$ $Pessimists$	9	19	2	0
$d_B,\ d_{R\cup Y}$ $Optimists$	2	0	0	0
$indif., indif.$	1	0	3	0

Sample I ($N_I = 36$)

[5]For these 8 subjects of sample I and for 10 out of 14 subjects of sample II, answers to Question 1, 2 and 3 were globally consistent with the Bayesian model.

$Qu1\&2$ \ $Qu3$	d_B	d_R	$indif.$	$undec.$
$d_R,\ d_{B\cup Y}$ $Pessimists$	8	23	6	3
$d_B,\ d_{R\cup Y}$ $Optimits$	1	0	1	0
$indif., indif.$	1	0	0	0

Sample II ($N_{II} = 43$)

Table 4

These data first settle the question concerning the $(indif., indif.)$ pattern people: 3 out of 4 subjects of sample I can only be Bayesians, whereas the fourth one, as well as the one in sample II, may have multiple priors[6].

Thus, out of the remaining 33 subjects who must have multiple priors in sample I, 12 chose in accordance with MLU prediction $(36, 36\%)$, whereas 19 of them made choices consistent with FBU $(57, 58\%)$, and 2 were inconsistent with either theory $(6, 06\%)$.

Results in sample II are similar: out of 43 subjects, MLU accounts for the choices of 10 of them $(23, 25\%)$, FBU for the choices of 23 others ($53, 5\%$), whereas the choices of the remaining 10 $(23, 25\%)$ are unexplained.

Table 5 summarizes and pulls together the results:

	# Subj.	MLU	FBU	Others
SpleI	*33*	*12*	*19*	*2*
SpleII	*43*	*10*	*23*	*10*
Total	*76*	*22*	*42*	*12*
	100%	*28.9%*	*55.3%*	*15.8%*

Table 5

6 Discussion

The similarity between the results of sample I and of sample II are striking, but for one thing: the high proportion of subjects 21% $(6 + 3 + 1 = 10\ )$, in sample II, exhibiting in question 3, indifference or inability to choose between the alternatives. A tentative explanation is that they were not sure that they understood

[6]These two subjects may also be Bayesians with symmetric priors on the content of the urn: given "non-yellow", the posterior of B is greater than of R.

the choice situation well, which was less likely to happen to sample I people, who could get clarifications from the Experimenter. In other words, the presence of the experimenter seems to have made decision making easier but not to have influenced the choices of the subjects.

In both samples the preponderence of the FBU rule over the MLU rule is quite strong, the ratio being close to $2:1$. This result can perhaps be explained by the fact that each of the two rules is particularly adapted to a particular type of conditioning and that the FBU rule fits that encountered in the experiment better than does the MLU rule.

As a matter of fact, Dubois and Prade (1994) have proposed to make a distinction between "focusing" where beliefs given an event A are interpretable as "present beliefs focused on the sub-events of A" and "updating" (proper) where they are interpretable as "updated beliefs after the event A has been observed".

Further Dubois and Prade argue that FBU is the appropriate rule for focusing while MLU is more in line with updating; whereas our simple experiment is neither pure focusing nor pure updating (in the Dubois-Prade sense), their classification suggests that different update rules may be appropriate under different circumstances.

Further experiment, involving pure focusing and pure updating, is planned and should clarify this question.

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