How Sets of Coherent Probabilities May Serve as Models for Degrees of Incoherence

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Abstract

We introduce two indices for the degree of incoherence in a set of lower and upper previsions: maximizing the *rate of loss* the incoherent bookmaker experiences in a Dutch Book, or maximizing the *rate of profit* the gambler achieves who makes Dutch Book against the incoherent bookmaker. We report how efficient bookmaking is achieved against these two indices in the case of incoherent previsions for events on a finite partition, and for incoherent previsions that include also a simple random variable. We relate the epsilon-contamination model to efficient bookmaking in the case of the rate of profit.

Keywords. Dutch Book, coherence, ε-contamination model

1 Introduction

It is a familiar remark that deFinetti's *Dutch Book* argument provides a simple dichotomy between coherent and incoherent previsions. For our presentation here, consider the following version of his argument, which we present as a two-person, zero-sum game between a *Bookie*, who is the subject of the argument, and a *Gambler*, who is the opponent.

Let $\boldsymbol{X}$ be a (bounded) random variable defined on some space $\boldsymbol{S}$ of possibilities. The Bookie is required to offer his/her prevision p(X) on the condition that the Gambler may then choose a real quantity $\alpha_{X,p(X)}$ resulting in a payoff to the Bookie of

$$\alpha_{\mathbf{X,p(X)}}[\boldsymbol{X} - \mathbf{p(X)}]$$

with the opposite payoff to the Gambler – a zero-sum game. The Bookie's previsions for a set of random variables are <u>*incoherent*</u> if there is a (finite) selection of non-zero α's by the Gambler that results, by summing, in a (uniformly) negative payoff to the Bookie and a (uniformly) positive payoff to the Gambler. The *Bookie's* previsions are <u>*coherent*</u>, otherwise. This leads to deFinetti's **Dutch Book Theorem** – The Bookie's previsions are coherent if and only if they are the expectations of a (finitely additive) probability distribution.

deFinetti extends his analysis to include assessments of conditional previsions, given an event F, through called-off wagers using the indicator for F, χ_F, of the form

$$\alpha_{\mathbf{X,p(X),F}}\chi_{\mathbf{F}}[\boldsymbol{X} - \mathbf{p(X)}]$$

Moreover, when the random variables $\boldsymbol{X}$ are restricted to indicator functions for events, E, the Bookie's previsions are coherent if and only if they are the conditional probabilities of a single (finitely additive) probability. In this case, the magnitude $|\alpha_{E,p(E),F}|$ is the stake for each wager, and the sign of $\alpha_{E,p(E),F}$, positive or negative, determines whether the Bookie bets respectively, on or against E, called-off if F fails to occur.

It is a familiar concern, appreciated by many at this conference, that deFinetti's criterion of coherence requires that the Bookie posts a *single* prevision, or called-off prevision given F, for each $\boldsymbol{X}$. For betting on events, this amounts to stating his/her "fair (called-off) odds": odds that the Gambler may use regardless the sign of the coefficient α. In response to this concern, the game has been relaxed to permit what C.A.B.Smith [4] called lower and upper "pignic" odds. That is, in the case of indicator variables, the Bookie may post one prevision p – a "lower" probability used with positive α – for wagering on E, and another prevision q – an "upper" probability used with negative α – for wagering against E. In effect, the Bookie asserts that at odds of p : 1- p or less he/she will bet on E, whereas at odds of q : 1-q or greater he/she will bet against E.

deFinetti's Dutch Book theorem generalizes in this setting to assert, roughly, that the Bookie's lower and upper previsions are coherent if and only if they are, respectively, the lower and upper expectations of a convex set of (finitely additive) probability distribution. (See [3] for a precise statement of this result.) This generalization, however, retains the initial dichotomy: the Bookie's previsions are coherent or else the Gambler can make a Dutch Book an achieve a sure return.

2 Degrees of Incoherence

In [2], we introduce two indices of incoherence: a *rate of loss* for the Bookie and a *rate of profit* for the Gambler. These index the amount of the Gambler's sure-gain against either of two "escrow" accounts, accounts that reflect the portion of the total stake each player contributes. The rate of loss indexes the Gambler's guaranteed sure gain (i.e., the minimum of the Bookie's assured loss) against the proportion of the total stake contributed by the Bookie. The rate of profit indexes the Gambler's guaranteed sure gain against his/her own contribution to the total stake. In what follows, we focus on the second of these two indices: the rate of profit achieved by the Gambler. Of course, there are more than these two ways of formalizing degrees of incoherence. Nau [1] gives a flexible framework that incorporates our "rate of loss" as a special case, for example.

2.1 Incoherence for events in a partition

Let $\{A_i: i = 1, ...n\}$ be a partition of the sure-event by n non-empty events, with n > 1. Let $0 = p_i = q_i =1$ be the Bookie's lower and upper previsions for the A_i (i = 1, ..., n). Let $s^+ = \Sigma\, q_i$ and let $s^- = \Sigma\, p_i$, so that the Bookie is incoherent if either $s^+ < 1$ or $1 < s^-$.

Theorem 2 (from [2]):

(1) If $s^+ < 1$ then the rate of guaranteed profit equals $(1-s^+)/s^+$ and is achieved when the Gambler sets all the $\alpha_i = -1/ s^+$.

(2) If $s^- > 1$, then the Gambler maximizes the minimum rate of profit by choosing the stakes according to the following rule: Let k* be the first k such that

$$\sum_{i=n-k+1}^{n} p_i \le 1 + (k-1)\, p_{n-k}$$

with k* = n if this equality always fails. Then the Gambler sets α_i all equal and positive for i ≥ n k*+1, and sets $\alpha_i = 0$ for all i < n-k*.

The Figure below illustrates this result for the case with n = 3 atoms, a ternary partition. The set of coherent deFinetti-previsions is represented by the trianglular hyperplane: the simplex with extreme values {(1,0,0), (0,1,0), (0,0,1)}. The set of incoherent "lower" probabilities, where $s^- > 1$, lies above it. The selected hyperplane in the figure is comprised of lower probabilities with $s^- = 1.5$. For those lower previsions in the white-region, outside the projection of the coherent simplex, the Gambler maximizes his/her rate of profit (which equals 3/7) by ignoring the Bookie's prevision on A_3, and achieving book by having the Bookie bet on each of A_1 and A_2, at equal stakes. That would be the case if the Bookie's lower previsions were (.6, .7, .2). If, however, the Bookie's previsions were inside the projection of the coherent simplex, e.g., (.5, .5, .5), then the Gambler's rate of profit is only 1/3, achieved with equal stakes on each of the three atoms.

2.2 Incoherence with previsions for a simple random variable

Next, consider the addition of a single random variable defined by a (finite) partition, $\{A_i: i = 1, ...n\}$, as in the subsection above. Let X be a (simple) random variable defined on these n events. For the next result, we assume that the the Bookie gives previsions $p_i = p(a_{(i)})$, ordered to be increasing in p, which are singly coherent, $0 \le p_i \le 1$. Also, the Bookie gives a prevision p_X for X. . For simplicity, we state the following result for the case s ≤ 1.

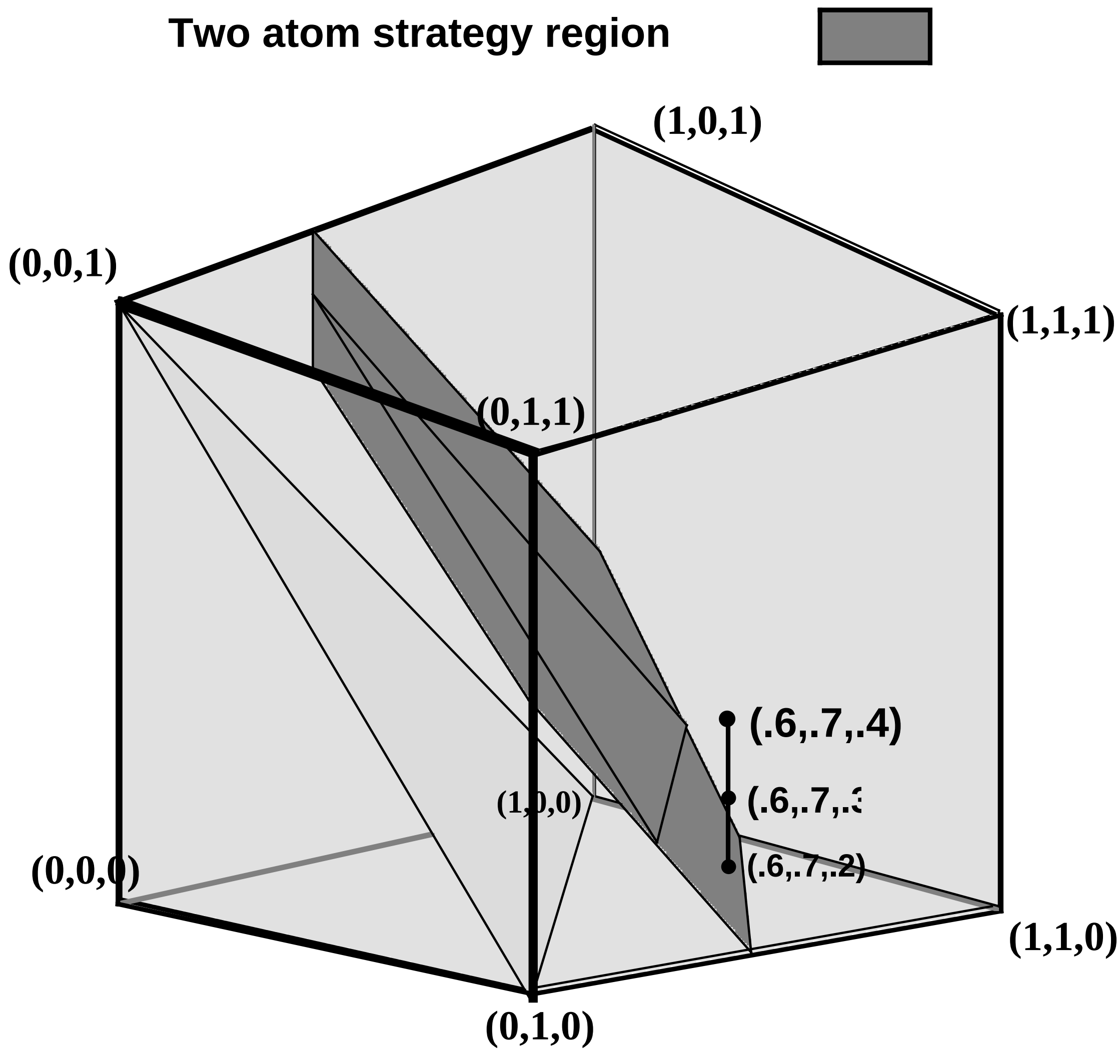
Define these seven quantities,

$$s = \Sigma_i\, p_i \qquad \mu = \Sigma_i\, x_i p_i \qquad \text{and} \quad \delta = p_X - \mu.$$

$$s_{-,k} = \sum\nolimits_{i=1}^{k} p_i \qquad s_{+,k} = \sum\nolimits_{i=n-k+1}^{n} p_i$$

$$v_-(k) = p_X - \sum\nolimits_{i=1}^{k} x_i p_i - (1 - s_{-,k})x_1$$

$$v_+(k) = (1 - s_{+,k})x_n + \sum\nolimits_{i=n-k+1}^{n} x_i p_i - p_X$$



Theorem 6 (of [2]) The Gambler achieves the maximum guaranteed rate of profit, as follows:

1) If $\delta < (1\text{-}s)x_1$, let k* be the smallest value of k such that $v_-(k) < 0$. Then set $\alpha_X = -1$, set $\alpha_i = x_i - x_{k^*}$ for $i \leq k^*$ and set $\alpha_i = 0$ for $i > k^*$.

2) If $\delta > (1\text{-}s)x_n$, let k* be the smallest value of k such that $v_+(k) < 0$. Then set $\alpha_X = 1$, set $\alpha_i = -x_i + x_{n-k^*}$ for $i \geq n\text{-}k^*$ and $\alpha_i = 0$ for $i < n - k^*$.

3) If $(1\text{-}s)x_1 \leq \delta \leq (1\text{-}s)x_n$, then set $\alpha_X = 0$ and apply the previous theorem, i.e., ignore the Bookie's prevision for X but, instead, use solely the incoherence among the p_i.

A Corollary to this Theorem is interesting and intelligible on its own. Having already given the (possibly incoherent) previsions p_i, and now obliged to provide the additional prevision p_X, the Bookie can ask how to avoid increasing the rate of profit that the Gambler may achieve.

Corollary The Gambler's rate of profit after learning the Bookie's prevision p_X does not increase if and only if p_X satisfies:

$$\mu + (1\text{-}s)\, x_1 \leq p_X \leq \mu + (1\text{-}s)\, x_n.$$

That is, the corollary identifies the Bookie's minimax strategies for augmenting the previsions p_i for the events A_i, with a single new prevision p_X for X.

This corollary applies to "called-off" betting as a special case:

Consider the ternary partition and random variable X whose values are given in the second row of the following table.

a_1	a_2	a_3
p_X	1	0

Thus,

0	$\alpha[1\text{-}p(X)]$	$-\alpha p(X)$

are the three corresponding payoffs to the Bookie associated with the wager $\alpha[X - p(X)]$.

Then, e.g., with $s \leq 1$, having already announced the previsions p_i (i = 1, 2, 3), the Bookie's minimax strategies for restraining the Gambler's rate-of-profit satisfies:

$$p_1 p_X + p_2 \leq p_X \leq p_1 p_X + p_2 + 1 - s.$$

It is interesting to note that choosing the pseudo-Bayes' "conditional' value $p_X = p_2 /(p_2 + p_3)$ always satisfies these inequalities. In other words, the incoherent Bookie can take advantage of the fact that the pseudo-Bayes' solutionn is minimax. You don't have to be coherent to like Bayes' solutions! Of course, if s = 1, so that the Bookie is coherent, the sole minimax solution is just the Bayes' solution.

3 Epsilon-contamination and the rate of guaranteed profit

The Gambler's decisions in the first of the two Theorems, in section 2.1 above, can be explained with an ε-contamination model, through the Bayesian "dual" to the minimax strategies for this case. For the Gambler to accept wagers when the Bookie offers "upper" probabilities, the Gambler must find these wagers acceptable as "lower" probabilities in a rational decision. Similarly, for the Gambler to accept wagers when the Bookie offers "lower" probabilities, the Gambler must find these wagers acceptable as "upper" probabilities in a rational decision.

Given a fixed probability distribution, p*, an ε-contamination model of probabilities is a set of probabilities $M_{p*} = \{(1\text{-}\varepsilon)p^* + \varepsilon q$:

$0 < \varepsilon < 1\}$, with q an arbitrary probability. Equivalently for finite algebras, an ε-contaminated model is given by specifying a coherent set of "lower" probabilities for the atoms of the algebra.

When the Bookie is incoherent with "upper" probabilities, $s^+ < 1$, these may be the coherent lower probabilities for the Gambler using an ε-contamination model. In fact, the Gambler maximizes his/her expected rate of profit according to this (convex) set by wagering as indicated in (1) of the Theorem.

When the Bookie is incoherent with "lower" probabilities, $s^- > 1$, it is **not** always the case that these can be the coherent "upper" probabilities for an ε-contamination model. Precisely when the Bookie's "lower" probabilities fall within the projection of the coherent simplex, when they fall within the triangular region illustrated in the Figure, then the Gambler may use these as the coherent "upper" probabilities from an e-contamination model. Otherwise, the Gambler fits the "largest" ε-contamination model that is allowed by the Bookie's offers. Expressed in other words, the strategies reported by the Theorem are those which give the Gambler a positive expected value for each component wager used to make the Dutch Book, and these relate to an ε-contamination model, as just explained.

The same analysis applies to the second of the two Theorems, in section 2.2 above. This case involves the Gambler's rate of guaranteed profit when the Bookie's previsions include a set of bets on a finite partition and a prevision for one (simple) random variable defined on that partition. The inequalities of Theorem 6 correspond, in precisely the same way, to the upper and lower expectations from an ε-contamination model, based on the Bookie's incoherent "upper" previsions, i.e., when $s < 1$.

The conference presentation includes, also, results for the parallel case when $s > 1$. Then Gambler's maximin strategies for securing an efficient Dutch Book, reflect the added complication of truncation of the ε-contamination model, just as in the corresponding case ($s > 1$) for the Theorem of section 2.2.

4 Conclusion

This presentation introduces the use of a convex set of coherent probabilities, the ε-contamination model, as the Bayes' dual solutions to a Gambler's maximin strategies for what we call the guaranteed rate of profit in making efficient Dutch Book against an incoherent Bookie. The two cases discussed here include (1) incoherent upper and lower previsions for events in a finite partition, and (2) a context where the Bookie includes a prevision for a simple random variable defined on this same partition.

Ongoing work (to be reported at the conference) specificies the corresponding convex set of probabilities that are dual to the Gambler's maximin strategies for maximizing the Bookie's guaranteed rate of loss in each of these two cases. These sets involve fixing both upper and lower probability bounds on the atoms of the finite algebra, rather than merely fixing the lower probabilities, as is done in an ε-contamination model.

References

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