A Toy Example

a) Basic situation

a_1	:	treatment
a_2	:	no treatment
ϑ_1	:	disease
ϑ_2	:	no disease

$+$	:	test positive (x_1)
$-$	:	test negative (x_2)

	ϑ_1	ϑ_2
a_1	0	5
a_2	20	0

	$+$	$-$
ϑ_1	0.9	0.1
ϑ_2	0.05	0.95

b) Prior

$$\begin{aligned}\pi(\{\vartheta_1\}) &= [0.6, 0.8]\\ \pi(\{\vartheta_2\}) &= [0.2, 0.4]\end{aligned}$$

$$\mathcal{M} = \left\{p(\cdot)\,\middle|\,p(\cdot)=\nu\begin{pmatrix}0.6\\0.4\end{pmatrix}+(1-\nu)\begin{pmatrix}0.8\\0.2\end{pmatrix},\; \nu \in [0;1]\right\}$$

$$\begin{aligned}\mathbb{E}_{\mathcal{M}}l(a_1) &= [1;2]\\ \mathbb{E}_{\mathcal{M}}l(a_2) &= [2;16]\end{aligned}$$

$$\left.\begin{aligned}{}^{\cup}\mathbb{E}_{\mathcal{M}}l(a_1) &= 2\\ {}^{\cup}\mathbb{E}_{\mathcal{M}}l(a_2) &= 16\end{aligned}\right\} \Rightarrow a_1$$

c) Data Problem with updated posterior

c1) after observation 'test positive' (+)
(posterior rounded)

$$\begin{aligned}\pi(\{\vartheta_1\}|\{+\}) &= [0.964, 0.986] \\ \pi(\{\vartheta_2\}|\{+\}) &= [0.014, 0.036]\end{aligned}$$

$$\mathcal{M}_+ = \left\{ p(\cdot) \,\middle|\, p(\cdot) = \nu \begin{pmatrix} 0.964 \\ 0.036 \end{pmatrix} + (1-\nu) \begin{pmatrix} 0.986 \\ 0.014 \end{pmatrix}, \right.$$
$$\left. \nu \in [0;1] \right\}$$

$$\left. \begin{aligned} \mathbb{E}_{\mathcal{M}_+} l(a_1) = [0.068, 0.179] \\ \mathbb{E}_{\mathcal{M}_+} l(a_2) = [19.286, 19.726] \end{aligned} \right\} \Rightarrow a_1$$

$$\left. \begin{aligned} {}^{\mathsf{U}}\mathbb{E}_{\mathcal{M}_+} l(a_1) &= 0.018 \\ {}^{\mathsf{U}}\mathbb{E}_{\mathcal{M}_+} l(a_2) &= 19.73 \end{aligned} \right\} \Rightarrow a_1$$

c2) after observation 'test negative' (-)
(posterior rounded)

$$\pi(\{\vartheta_1\}|\{-\}) = [0.134, 0.296]$$
$$\pi(\{\vartheta_2\}|\{-\}) = [0.704, 0.863]$$

$$\mathcal{M}_- \;=\; \left\{ p(\cdot) \,\middle|\, p(\cdot) = \nu \begin{pmatrix} 0.134 \\ 0.863 \end{pmatrix} + (1-\nu) \begin{pmatrix} 0.296 \\ 0.704 \end{pmatrix}, \right.$$
$$\left. \nu \in [0;1] \right\}$$

$$\left. \begin{array}{l} \mathbb{E}_{\mathcal{M}_-} l(a_1) = [3.519, 4.318] \\ \mathbb{E}_{\mathcal{M}_-} l(a_2) = [2.727, 5.926] \end{array} \right\} \Rightarrow a_1$$

$$\left. \begin{array}{lcl} {}^{\mathsf{U}}\mathbb{E}_{\mathcal{M}_-} l(a_1) & = & 4.318 \\ {}^{\mathsf{U}}\mathbb{E}_{\mathcal{M}_-} l(a_2) & = & 5.926 \end{array} \right\} \Rightarrow a_1$$

d) Data Problem:

Decision Functions

d	$\begin{pmatrix} R(d,\vartheta_1) \\ R(d,\vartheta_2) \end{pmatrix}$	$E_{\mathcal{M}}(R(d,\vartheta))$
$\begin{pmatrix} + & - \\ a_1 & a_1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 5 \end{pmatrix}$	$[1, 2]$
$\begin{pmatrix} + & - \\ a_1 & a_2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 0.25 \end{pmatrix}$	$[1.3, 1.65]$
$\begin{pmatrix} + & - \\ a_2 & a_1 \end{pmatrix}$	$\begin{pmatrix} 18 \\ 4.75 \end{pmatrix}$	$[12.7, 15.35]$
$\begin{pmatrix} + & - \\ a_2 & a_2 \end{pmatrix}$	$\begin{pmatrix} 20 \\ 0 \end{pmatrix}$	$[12, 16]$

$$ {}^{U}\mathbb{E}_{\mathcal{M}}\left(R\begin{pmatrix} + & - \\ a_1 & a_1 \end{pmatrix}\right) = 2 $$

$$ {}^{U}\mathbb{E}_{\mathcal{M}}\left(R\begin{pmatrix} + & - \\ a_1 & a_2 \end{pmatrix}\right) = 1.65 $$

e.g., for $d_2 = \begin{pmatrix} + & - \\ a_1 & a_2 \end{pmatrix}$

$$R(d_2, \vartheta_j) = \begin{array}{l} l(a_1, \vartheta_j) \cdot p(+|\vartheta_j) + \\ l(a_2, \vartheta_j) \cdot p(-|\vartheta_j) \end{array}$$

$$\begin{aligned}
&{}^{\mathrm{U}}\mathbb{E}_{\mathcal{M}} \left(R \left(d_2, \vartheta \right) \right) \\
&= \max_{\pi} \Big(R(d_2, \vartheta_1) \cdot \pi(\{\vartheta_1\}) \\
&\qquad + R(d_2, \vartheta_2) \cdot \pi(\{\vartheta_2\}) \Big) \\
&= \max \left(\begin{pmatrix} 0 + 20 \cdot 0.1 \\ 5 \cdot 0.05 + 0 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}, \right. \\
&\qquad \left. \begin{pmatrix} 0 + 20 \cdot 0.1 \\ 5 \cdot 0.05 + 0 \end{pmatrix} \cdot \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} \right) \\
&= 1.65
\end{aligned}$$