UTOPIÆ Uncertainty Treatment and Optimisation in Aerospace Engineering

THE FRAMEWORK PROGRAMME FOR RESEARCH AND INNOVATION HORIZON 2020

MARIE CURIE ACTIONS

Imprecise Filtering for Spacecraft Navigation

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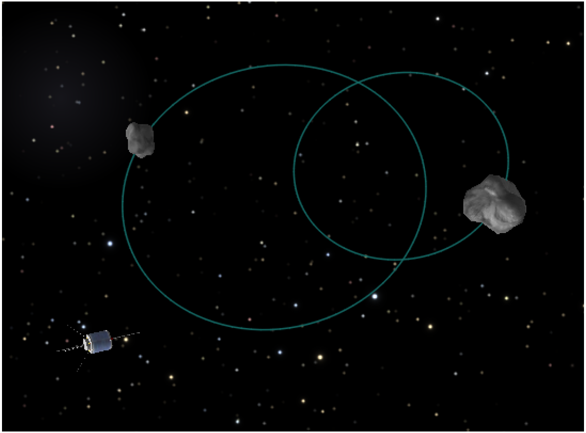
Filtering for Spacecraft Navigation



The General Problem

Dynamical system with state $x_t \in \mathcal{X}$, $t \geq 0$

- For simple orbital mechanics: $\mathcal{X} \simeq [0, 1]^6$

General statement given by (stochastic) differential equation

$$\frac{\mathrm{d}\, x_t}{\mathrm{d}\, t} = \underbrace{f(x_t, t \mid \theta)}_{\text{system dynamics}} + \underbrace{\frac{\mathrm{d}\, W_t}{\mathrm{d}\, t}}_{\text{Brownian motion noise}} \times \underbrace{\sigma(x_t, t \mid \theta)}_{\text{diffusion coefficient}}$$

with stochastic initial condition $x_0 \sim P(X_0 \mid \theta)$.

Collect all parameters in θ

For given θ induces a stochastic process $\{X_t|\theta\}_{t\geq 0}$

- This process is a *Markov chain*

The General Problem

Realisations of $\{X_t|\theta\}_{t\geq 0}$ cannot be observed directly

- I.e. it is a *hidden* Markov chain

But, *noisy measurement model*:

$$Y_t \sim \psi(\cdot \,|\, x_t, \theta)$$

(Just writing θ for the parameters again)

Filtering Problem

Given collection $y_{t_{1:n}} \equiv y_{t_1}, \ldots, y_{t_n}$, compute conditional probability

$$P_\theta(X_t \,|\, y_{t_{1:n}})$$

More generally: conditional expectation of function $h : \mathcal{X} \to \mathbb{R}$:

$$\mathbb{E}_\theta\left[h(X_t) \,\middle|\, y_{t_{1:n}}\right]$$

Imprecise Filtering

What if we don't know θ exactly?

If we can assume it belongs to a *set* ϑ:

- Induces an *imprecise* stochastic process
    - I.e. a *set* of stochastic processes

Imprecise Filtering Problem

Compute *lower*- and *upper* expectation

$$\underline{\mathbb{E}}\left[h(X_t) \,\middle|\, y_{t_{1:n}}\right] = \inf_{\theta \in \vartheta} \mathbb{E}_\theta\left[h(X_t) \,\middle|\, y_{t_{1:n}}\right]$$

$$\overline{\mathbb{E}}\left[h(X_t) \,\middle|\, y_{t_{1:n}}\right] = \sup_{\theta \in \vartheta} \mathbb{E}_\theta\left[h(X_t) \,\middle|\, y_{t_{1:n}}\right]$$

$\Rightarrow$ provide robust bounds on quantity of interest

Some Immediate Difficulties

- Even for fixed θ, no analytical solution for $\{X_t|\theta\}_{t\geq 0}$
  - Let alone for $\mathbb{E}_\theta\left[h(X_t)\,\middle|\,y_{t_{1:n}}\right]$
- Convergence of numerical approaches?
  - Uniformly w.r.t ϑ?
- The optimisation to compute $\underline{\mathbb{E}}$ or $\overline{\mathbb{E}}$ is difficult
  - Multi-modal objective surface, ...
- Semantic issues in problem statement
  - Physical interpretation of certain relaxations,...

Some Existing Approaches

Precise case:

- Kalman Filter (/variants)
  - Gaussian distr., linear dyn. $\Rightarrow P(X_t | y_{t_{1:n}})$ is Gaussian
  - *Extended* Kalman Filter: Just re-linearise over time
- Particle Filters
  - Essentially a Monte Carlo method

Imprecise case:

- Special cases: propagation of p-boxes, set-valued observations, ...
- "Imprecise Kalman Filter" [1]
  - Provides general solution...
  - ...but so far only solved in specific cases
    - E.g. Linear Gaussian-Vacuous Mixture

[1] A. Benavoli, M. Zaffalon, E. Miranda: *Robust filtering through coherent lower previsions*, IEEE Transactions on Automatic Control, 2011

The Formal Setup

Basically we just need to apply Bayes' rule:

$$\mathbb{E}\big[h(X_t)\,|\,y_{t_{1:n}}\big] = \frac{\mathbb{E}\big[h(X_t)\prod_{i=1}^{n}\psi(y_{t_i}\,|\,X_{t_i})\big]}{\mathbb{E}\big[\prod_{i=1}^{n}\psi(y_{t_i}\,|\,X_{t_i})\big]}$$

In the imprecise case (let's say $\underline{\mathbb{E}}\big[\prod_{i=1}^{n}\mathbb{I}_{\{y_{t_i}\}}(Y_{t_i})\big] > 0$):

$$\underline{\mathbb{E}}\big[h(X_t)\,|\,y_{t_{1:n}}\big] = \mu \Leftrightarrow \underline{\mathbb{E}}\left[\big(h(X_t)-\mu\big)\prod_{i=1}^{n}\mathbb{I}_{\{y_{t_i}\}}(Y_{t_i})\right] = 0$$

So, we need to compute a joint expectation.

Iterated (Lower) Expectation

Write $g(X_{t_{1:n}}, X_t, Y_{t_{1:n}}) = \big(h(X_t) - \mu\big) \prod_{i=1}^{n} \mathbb{I}_{\{y_{t_i}\}}(Y_{t_i})$

Use epistemic irrelevance ⇒ "imprecise Markov property"

Recursively decompose the joint:

$$\begin{aligned}
\underline{\mathbb{E}}[g(\cdot)] &= \underline{\mathbb{E}}[\underline{\mathbb{E}}[g(\cdot) \mid X_0]] \\
&= \underline{\mathbb{E}}[\underline{\mathbb{E}}[\underbrace{\underline{\mathbb{E}}[g(\cdot) \mid X_0, X_{t_1}]}_{=\underline{\mathbb{E}}[g(\cdot) \mid X_{t_1}]} \mid X_0]] \\
&= \underline{\mathbb{E}}[\underline{\mathbb{E}}[\underline{\mathbb{E}}[g(\cdot) \mid X_{t_1}] \mid X_0]] \\
&= \underline{\mathbb{E}}[\underline{\mathbb{E}}[\underline{\mathbb{E}}[\underline{\mathbb{E}}[g(\cdot) \mid Y_{t_1}, X_{t_1}] \mid X_{t_1}] \mid X_0]] \\
&\vdots \\
&= \underline{\mathbb{E}}[\underline{\mathbb{E}}[\underline{\mathbb{E}}[\underline{\mathbb{E}}[\cdots \underline{\mathbb{E}}[\underline{\mathbb{E}}[g(\cdot) \mid X_{t_n}, Y_{t_n}] \mid X_{t_n}] \cdots \mid Y_{t_1}, X_{t_1}] \mid X_{t_1}] \mid X_0]]
\end{aligned}$$

Iterated (Lower) Expectation

Now resolve the remaining conditionals "inward-outward":

$$\underline{\mathbb{E}}\big[\underline{\mathbb{E}}\big[\underline{\mathbb{E}}\big[\underline{\mathbb{E}}\big[\cdots\underline{\mathbb{E}}\big[\underbrace{\underline{\mathbb{E}}\big[g(\cdot)\,|\,X_{t_n}, Y_{t_n}\big]}_{\text{start here}}\,|\,X_{t_n}\big]\cdots\,|\,Y_{t_1}, X_{t_1}\big]\,|\,X_{t_1}\big]\,|\,X_0\big]\big]$$

Fix all variables except X_t:

$$\begin{aligned}
&\underline{\mathbb{E}}\left[\left(h(X_t)-\mu\right)\prod_{i=1}^{n}\mathbb{I}_{y_{t_i}}(\gamma_{t_i})\,\middle|\,X_{t_n}=x_{t_n}, Y_{t_n}=\gamma_{t_n}\right]\\
&=\underline{\mathbb{E}}\left[\left(h(X_t)-\mu\right)|\,X_{t_n}=x_{t_n}, Y_{t_n}=\gamma_{t_n}\right]\prod_{i=1}^{n}\mathbb{I}_{y_{t_i}}(\gamma_{t_i})\\
&=\underline{\mathbb{E}}\left[\left(h(X_t)-\mu\right)|\,X_{t_n}=x_{t_n}\right]\prod_{i=1}^{n}\mathbb{I}_{y_{t_i}}(\gamma_{t_i})\\
&=: g'(\cdot)
\end{aligned}$$

Iterated (Lower) Expectation

We have a new function g' that no longer depends on X_t.
After substitution

$$\underline{\mathbb{E}}\Big[\underline{\mathbb{E}}\Big[\underline{\mathbb{E}}\Big[\underline{\mathbb{E}}\Big[\cdots \underbrace{\underline{\mathbb{E}}\big[g'(\cdot)\,\big|\,X_{t_n}\big]}_{\text{this part next}} \cdots \,\Big|\, Y_{t_1}, X_{t_1}\Big] \,\Big|\, X_{t_1}\Big] \,\Big|\, X_0\Big]\Big]$$

Fix all variables except Y_{t_n},

$$\begin{aligned}
&\underline{\mathbb{E}}\big[g'(\cdot)\,\big|\,X_{t_n} = x_{t_n}\big] \\
&= \underline{\mathbb{E}}\Big[\underline{\mathbb{E}}\big[(h(X_t) - \mu)|X_{t_n} = x_{t_n}\big]\, \mathbb{I}_{y_{t_n}}(Y_{t_n}) \textstyle\prod_{i=1}^{n-1} \mathbb{I}_{y_{t_i}}(\gamma_{t_i}) \Big| X_{t_n} = x_{t_n}\Big] \\
&= \textstyle\prod_{i=1}^{n-1} \mathbb{I}_{y_{t_i}}(\gamma_{t_i}) \underline{\mathbb{E}}\big[(h(X_t) - \mu)|X_{t_n} = x_{t_n}\big] \\
&\quad \times \begin{cases} \underline{\mathbb{E}}\big[\mathbb{I}_{y_{t_n}}(Y_{t_n})|X_{t_n} = x_{t_n}\big] & \text{if } \underline{\mathbb{E}}\big[(h(X_t) - \mu)|X_{t_n} = x_{t_n}\big] \geq 0 \\ \overline{\mathbb{E}}\big[\mathbb{I}_{y_{t_n}}(Y_{t_n})|X_{t_n} = x_{t_n}\big] & \text{otherwise} \end{cases} \\
&=: g''(\cdot)
\end{aligned}$$

Iterated (Lower) Expectation

We have

$$\underline{\mathbb{E}}\big[\underline{\mathbb{E}}\big[\underline{\mathbb{E}}\big[\underline{\mathbb{E}}\big[\cdots g''(\cdot)\cdots \mid Y_{t_1}, X_{t_1}\big] \mid X_{t_1}\big] \mid X_0\big]\big]$$

$g''(\cdot)$ depends on $X_{t_1}, \ldots, X_{t_{n-1}}$, X_{t_n} and $Y_{t_1}, \ldots, X_{t_{n-1}}$

$\Rightarrow$ This is what we started with, but reduced

Now just recurse

$$\underline{\mathbb{E}}\big[\underbrace{\underline{\mathbb{E}}\big[\underline{\mathbb{E}}\big[\underline{\mathbb{E}}\big[\cdots g''(\cdot)\cdots \mid Y_{t_1}, X_{t_1}\big] \mid X_{t_1}\big] \mid X_0\big]}_{=:g^*(\cdot)}\big]$$

$$= \underline{\mathbb{E}}\big[g^*(X_0)\big]$$

Remaining Problem, in Summary

Lower- and upper likelihoods of states given measurements:

- $\underline{\mathbb{E}}\big[\mathbb{I}_{y_t}(Y_t)|X_t = x_t\big]$ and $\overline{\mathbb{E}}\big[\mathbb{I}_{y_t}(Y_t)|X_t = x_t\big]$

For arbitrary $h : \mathcal{X} \to \mathbb{R}$, need to evaluate

- $\underline{\mathbb{E}}\big[h(X_0)\big]$ (okay)
- $\underline{\mathbb{E}}\big[h(X_{t_n})\,|\,X_{t_{n-1}}\big]$ (not okay)

Some Simplifying Assumptions

Assume dynamics are *precise* and *time-homogeneous*:

$$\frac{\mathrm{d}\, x_t}{\mathrm{d}\, t} = f(x_t) + \frac{\mathrm{d}\, W_t}{\mathrm{d}\, t} \times \sigma(x_t)$$

Let $\mathcal{X} \simeq [0,1]^d$ and $h_t : x \mapsto \mathbb{E}[h(X_t) \,|\, X_0 = x]$. Then

$$\frac{\mathrm{d}\, h_t(x)}{\mathrm{d}\, t} = \sum_{i=1}^{d} f_i(x) \frac{\partial\, h_t(x)}{\partial\, x_i} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} (\sigma(x)\sigma(x)^\top)_{ij} \frac{\partial^2\, h_t(x)}{\partial x_i \partial x_j}$$

with $h_0 = h$.

To get $h_t(x) = \mathbb{E}[h(X_t) \,|\, X_0 = x]$ we need to solve this PDE.

In general no analytical solution.

Some Further Simplifying Assumptions

Assume the dynamics are also *deterministic*

- I.e. the process is degenerate and described by $f(x_t)$
- We keep stochastic (and imprecise) initial distribution and measurements

We get

$$\frac{\mathrm{d}\, h_t(x)}{\mathrm{d}\, t} = \sum_{i=1}^{d} f_i(x) \frac{\partial\, h_t(x)}{\partial\, x_i}$$

with $h_0 = h$.

Still need to solve a PDE, and still no analytical solution

We're going to approximate this

Approximate Solution

Easy enough to solve pointwise: for any $x_0 \in \mathcal{X}$,

$$\mathbb{E}[h(X_t) \,|\, X_0 = x_0] = h(x_t)\,,$$

with

$$x_t = x_0 + \int_0^t f(x_\tau)\,\mathrm{d}\tau$$

$\Rightarrow$ Just do this for a lot of different x_0

Then extend pointwise estimates to entire domain $\mathcal{X}$

Dynamics Propagation

Collection of starting points $\mathbf{Z} = z_1, \ldots, z_m$

Compute $\mathbf{Z}^* = z_1^{(t)}, \ldots, z_m^{(t)}$, with

$$z_i^{(t)} = z_i + \int_0^t f\left(z_i^{(\tau)}\right) \mathrm{d}\tau, \quad i = 1, \ldots, m$$

This is just an ODE $\Rightarrow$ use your favourite numerical integrator

- I.e. Explicit/implicit Euler, Runge-Kutta schemes,...

Note: this could be "embarrassingly parallelized"

- A modern GPU can solve hundreds to thousands in parallel

Propagated Dynamics

We now know

$$h_t(z_i) = h\left(x_i^{(t)}\right) = \mathbb{E}[h(X_t) \mid X_0 = z_i] \quad \text{for } i = 1, \ldots, m$$

Now extend this to $\hat{h}_t$ on entire $\mathcal{X}$

Radial Basis Function Interpolation

Training points $\mathbf{Z} = z_1, \ldots, z_m \in \mathcal{X}$.

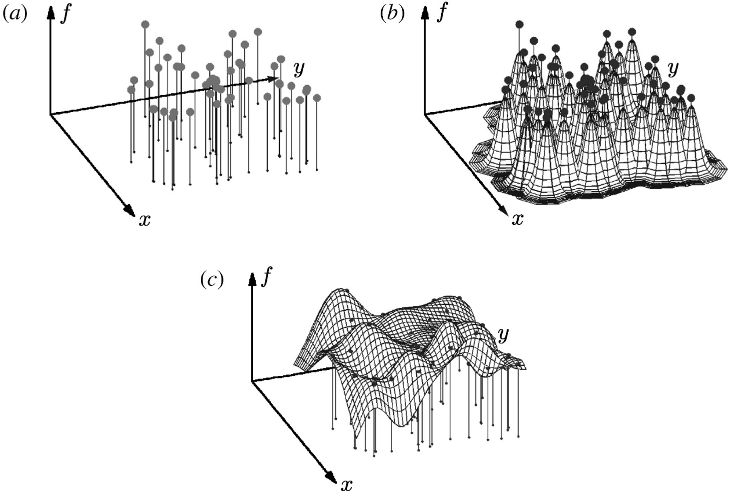
Radial basis functions $\phi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, $\phi(z, c) = \phi(\|z - c\|)$

Basis expansion $\Phi_z \in \mathbb{R}^m$ of any $z \in \mathcal{X}$:

$$\Phi_z = [\phi(z, z_1) \cdots \phi(z, z_m)]^\top$$

Can find $\mathbf{w}_h \in \mathbb{R}^m$ to approximate any ("nice") function $h : \mathcal{X} \to \mathbb{R}$:

$$h(z) \approx \hat{h}(z) = \sum_{i=1}^{m} w_i \phi(z, z_i) = \Phi_z^\top \mathbf{w}_h$$



N. Flyer, G.B. Wright: *A radial basis function method for the shallow water equations on a sphere*, Proceedings Royal Soc A, 2009

Radial Basis Function Interpolation

Construct design(/kernel/feature/...) matrix

$$\mathbf{\Phi}_\mathbf{Z} = \begin{bmatrix} \Phi_{z_1} \\ \vdots \\ \Phi_{z_m} \end{bmatrix} = \begin{bmatrix} \phi(z_1, z_1) & \cdots & \phi(z_1, z_m) \\ \vdots & \ddots & \vdots \\ \phi(z_m, z_1) & \cdots & \phi(z_m, z_m) \end{bmatrix}$$

Then $\mathbf{\Phi}_\mathbf{Z}$ is usually invertible if all $z_1, \ldots, z_m$ are unique.

Let $h(\mathbf{Z}) = [h(z_1) \cdots h(z_m)]^\top$

Find $\mathbf{w}_h$ that interpolates h on $\mathbf{Z}$:

$$\mathbf{\Phi}_\mathbf{Z}\mathbf{w}_h = h(\mathbf{Z})$$

In other words, we get $\mathbf{w}_h = \mathbf{\Phi}_\mathbf{Z}^{-1} h(\mathbf{Z})$

Radial Basis Function Interpolation

Pros:

- Straightforward (conceptually and to implement)
  - Meshless (i.e. arbitrary $\mathbf{Z}$), so no need to partition $\mathcal{X}$
  - Just solve a linear system
- Well-developed theory
- Good performance for approximating PDE solutions
  - Near spectral (exponential) convergence
- Many parameters to fine tune performance
  - Basis functions, hyperparameters, regularisation, ...

Cons:

- Many parameters to fine tune
- Numerical instability (system is typically ill-conditioned)

Approximation in Summary

We want to compute $\hat{h}_t(X_0) \approx h_t(X_0) := \mathbb{E}[h(X_t) \mid X_0]$

We know that $h_t(x_0) = h(x_t)$, where

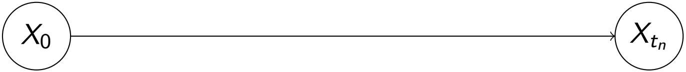
$$x_t = x_0 + \int_0^t f(x_\tau)\,\mathrm{d}\tau$$

We proceed as follows

1. Choose some $\mathbf{Z}$
2. Compute $\mathbf{Z}^*$ for time t
3. Compute design matrix $\mathbf{\Phi}_{\mathbf{Z}}$
4. Compute $\mathbf{w}_{h_t} = \mathbf{\Phi}_{\mathbf{Z}}^{-1} h(\mathbf{Z}^*)$
5. Let $\hat{h}_t(x) := \Phi_x^\top \mathbf{w}_{h_t}$ for any $x \in \mathcal{X}$

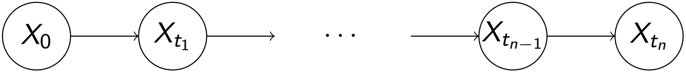
Iterated Expectation, Again

$$\mathbb{E}[h(X_{t_n})] = \mathbb{E}\big[\mathbb{E}\big[h(X_{t_n})|X_0\big]\big]$$



Iterated Expectation, Again

$$\mathbb{E}[h(X_{t_n})] = \mathbb{E}\Big[\mathbb{E}\Big[\mathbb{E}\Big[\cdots \mathbb{E}[h(X_{t_n})|X_{t_{n-1}}] \cdots |X_{t_1}\Big]|X_0\Big]\Big]$$



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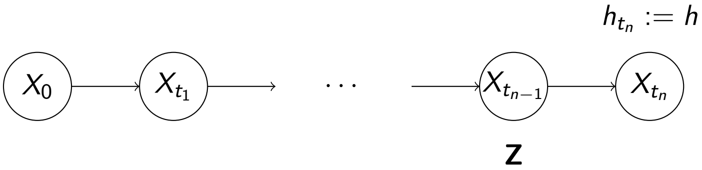
Iterated Expectation, Again

$$\mathbb{E}[h(X_{t_n})] = \mathbb{E}\Big[\mathbb{E}\Big[\mathbb{E}\Big[\cdots \mathbb{E}[h(X_{t_n})|X_{t_{n-1}}]\cdots |X_{t_1}\Big]|X_0\Big]\Big]$$



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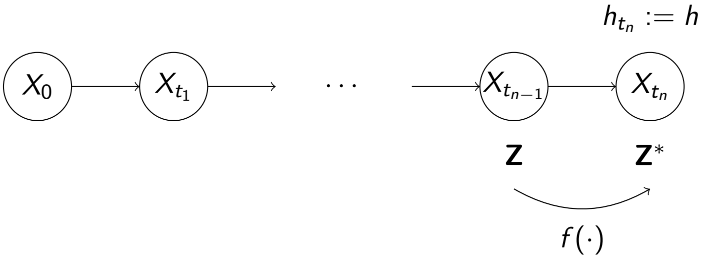

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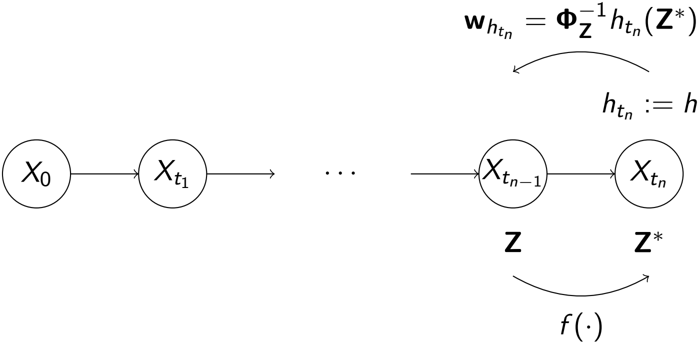

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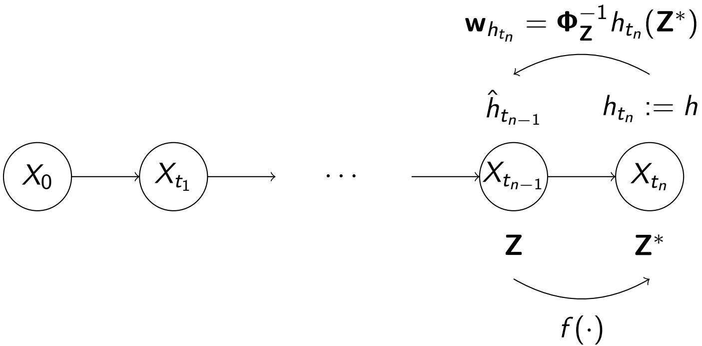
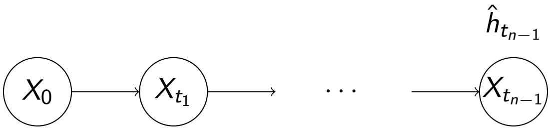


Iterated Expectation, Again

$$\mathbb{E}[h(X_{t_n})] \approx \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\cdots \hat{h}_{t_{n-1}}(X_{t_{n-1}}) \cdots | X_{t_1}\right] | X_0\right]\right]$$



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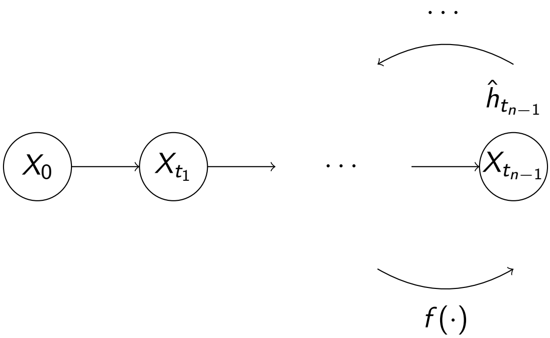

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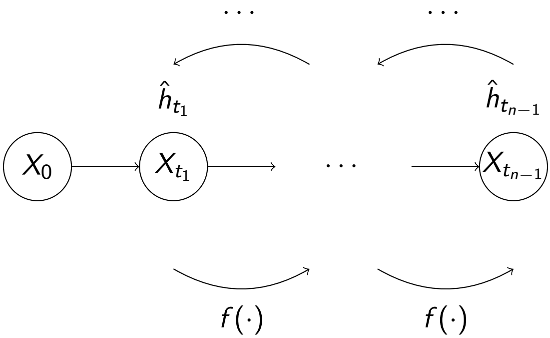
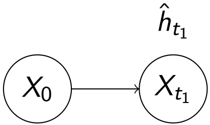


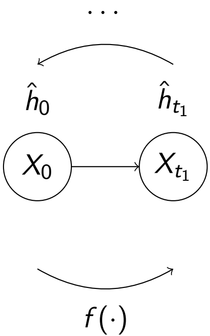
Iterated Expectation, Again

$$\mathbb{E}[h(X_{t_n})] \approx \mathbb{E}\left[\mathbb{E}\left[\hat{h}_{t_1}(X_{t_1})|X_0\right]\right]$$



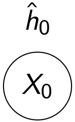
Iterated Expectation, Again

$$\mathbb{E}[h(X_{t_n})] \approx \mathbb{E}\left[\mathbb{E}\left[\hat{h}_{t_1}(X_{t_1})|X_0\right]\right]$$



Iterated Expectation, Again

$$\mathbb{E}[h(X_{t_n})] \approx \mathbb{E}\left[\hat{h}_0(X_0)\right]$$



Iterated Expectation, Again

$$\mathbb{E}[h(X_{t_n})] \approx \mathbb{E}\left[\hat{h}_0(X_0)\right] = \int_{\mathcal{X}} \hat{h}_0(x_0)\,\mathrm{d}P(X_0 = x_0)$$

The Crucial Part

How to make this tractable:

1. Make $0 =: t_0, t_1, \ldots, t_n$ a *uniform* partition of $[0, t]$
   - So $t_i - t_{i-1} = \Delta$ for all $i = 1, \ldots, n$
2. Choose the same $\mathbf{Z}$ at every step

The Crucial Part

- $\mathbf{Z}^*$ is the same at each step (by homogeneity)
- You can precompute $\mathbf{Z}$, $\mathbf{Z}^*$, $\boldsymbol{\Phi}_{\mathbf{Z}}$, $\boldsymbol{\Phi}_{\mathbf{Z}^*}$, $\boldsymbol{\Phi}_{\mathbf{Z}}^{-1}$, $(\boldsymbol{\Phi}_{\mathbf{Z}^*}\boldsymbol{\Phi}_{\mathbf{Z}}^{-1}) =: \mathbf{K}$
    - Only *once*, *offline*, and $\mathcal{O}(\Delta m) + \mathcal{O}(m^3)$

The Crucial Part

- $\mathbf{Z}^*$ is the same at each step (by homogeneity)
- You can precompute $\mathbf{Z}$, $\mathbf{Z}^*$, $\mathbf{\Phi_Z}$, $\mathbf{\Phi_{Z^*}}$, $\mathbf{\Phi_Z^{-1}}$, $(\mathbf{\Phi_{Z^*}\Phi_Z^{-1}}) =: \mathbf{K}$
  - Only *once*, *offline*, and $\mathcal{O}(\Delta m) + \mathcal{O}(m^3)$
- Remains to solve the recursion:

$$
\begin{aligned}
\hat{h}_{t_n}(x) &= h(x) \\
\hat{h}_{t_{n-1}}(x) &= \Phi_x^\top \mathbf{w}_{\hat{h}_{t_n}} = \Phi_x^\top \mathbf{\Phi_Z^{-1}} h(\mathbf{Z}^*) = \Phi_x^\top \mathbf{\Phi_Z^{-1}} \mathbf{K}^0 h(\mathbf{Z}^*) \\
\hat{h}_{t_{n-2}}(x) &= \Phi_x^\top \mathbf{w}_{\hat{h}_{t_{n-1}}} = \Phi_x^\top \mathbf{\Phi_Z^{-1}} \mathbf{\Phi_{Z^*}} \mathbf{\Phi_Z^{-1}} h(\mathbf{Z}^*) = \Phi_x^\top \mathbf{\Phi_Z^{-1}} \mathbf{K}^1 h(\mathbf{Z}^*) \\
&\vdots \\
\hat{h}_{t_{n-\ell}}(x) &= \Phi_x^\top \mathbf{\Phi_Z^{-1}} \mathbf{K}^{\ell-1} h(\mathbf{Z}^*) \quad \text{for } \ell = 1, \dots, n
\end{aligned}
$$

After precomputing, can find $\hat{h}_0$ for *any* h in $\mathcal{O}(nm^2)$

But does it work?

Dynamics of a Spacecraft

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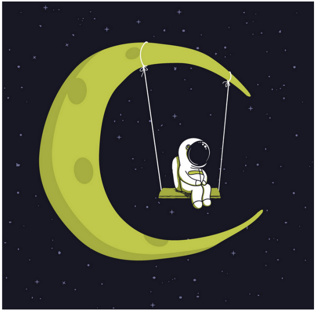
Dynamics of a ~~Spacecraft~~ Pendulum



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Pendulum Setup

Pendulum dynamics model:

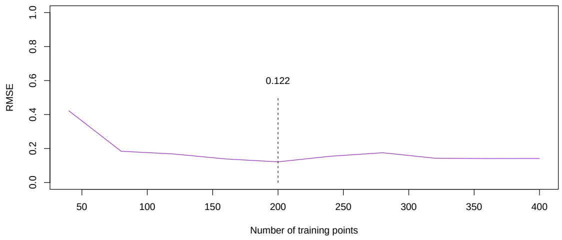
$$\frac{\mathrm{d}\,x_t}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\alpha_t \\ \dot{\alpha}_t\end{bmatrix} = \begin{bmatrix}\dot{\alpha}_t \\ {}^{g}\!/\!{}_{\ell} \sin \alpha_t - \rho\dot{\alpha}_t\end{bmatrix}$$

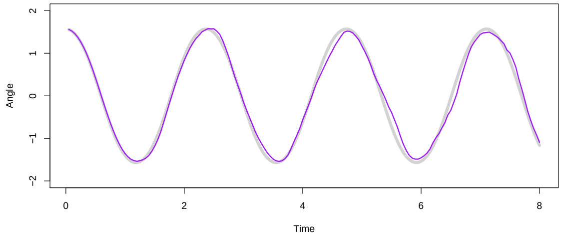
With $\ell = 1, g \approx -9.8$ and $\rho \in \{0, 0.5\}$.

Known true initial position $x_0 = [{}^{\pi}\!/\!{}_{2}\ 0]^\top$

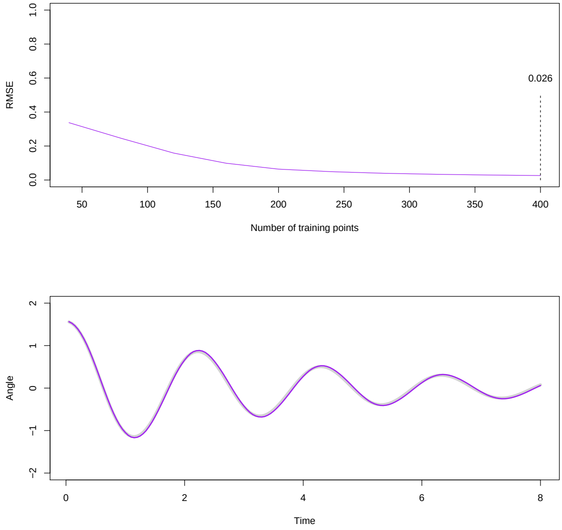
Sequential reinitialisation with best guess after 1 second

No Friction, Known Start, No Measurements





With Friction, Known Start, No Measurements



Pendulum Setup, With Measurements

Model thinks friction term $\rho = 0$

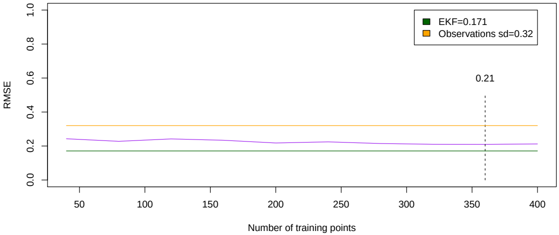
Measurement model:

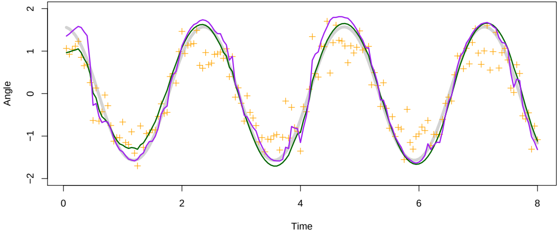
$$Y_t \sim \mathcal{N}\left(\cdot \mid \ell \sin \alpha_t, 0.32^2\right)$$

Unknown initial position with uniform $P(X_0) = U(-2, 2) \times U(-5, 5)$

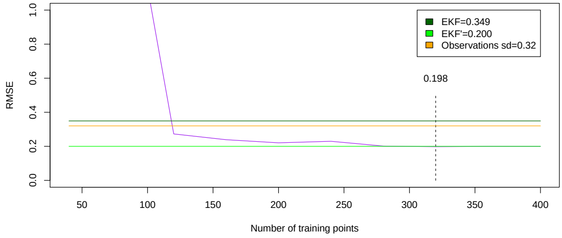
Sequential reinitialisation with same uniform after 1 second

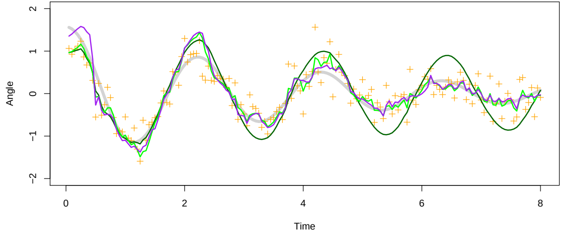
No Friction, Uniform Start, 20 Measurements/s





Friction, Uniform Start, 20 Measurements/s





Pendulum Setup, Imprecise

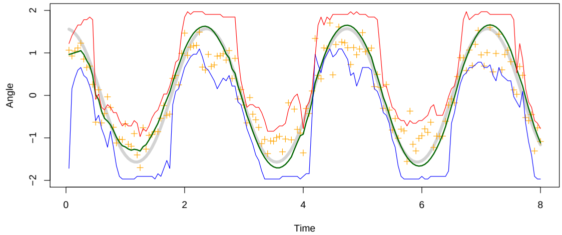
Imprecise measurement model with lower- and upper likelihood:

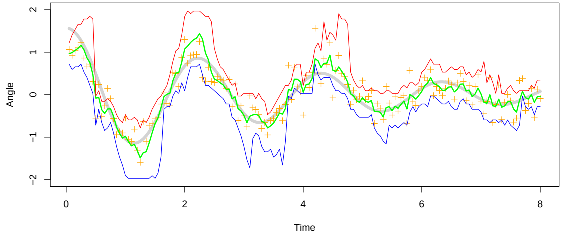
$$\underline{\psi}(y_t \mid x_t) = 0.5\mathcal{N}\left(y_t \mid \ell \sin \alpha_t, 0.32^2\right)$$
$$\overline{\psi}(y_t \mid x_t) = 1.5\mathcal{N}\left(y_t \mid \ell \sin \alpha_t, 0.32^2\right)$$

Unknown initial position with *vacuous* on $[-2, 2] \times [-5, 5]$

Sequential reinitialisation with same vacuous after 1 second

Vacuous Start, 20 Measurements/s, Imprecise Measurement Models





Open Questions

- Does this scale??
- Numerical stability issues?
    - Preconditioning
    - Smart selection/pruning of training points?
    - E.g. regularisation, LASSO
- Generalisation to stochastic (and/or imprecise) dynamics?
- Find informative "sequential prior"?
- Interpretation of $\mathbf{K} = \mathbf{\Phi}_{\mathbf{Z}^*}\mathbf{\Phi}_{\mathbf{Z}}^{-1}$?