Efficient algorithms for finding maximal gambles

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August 2018

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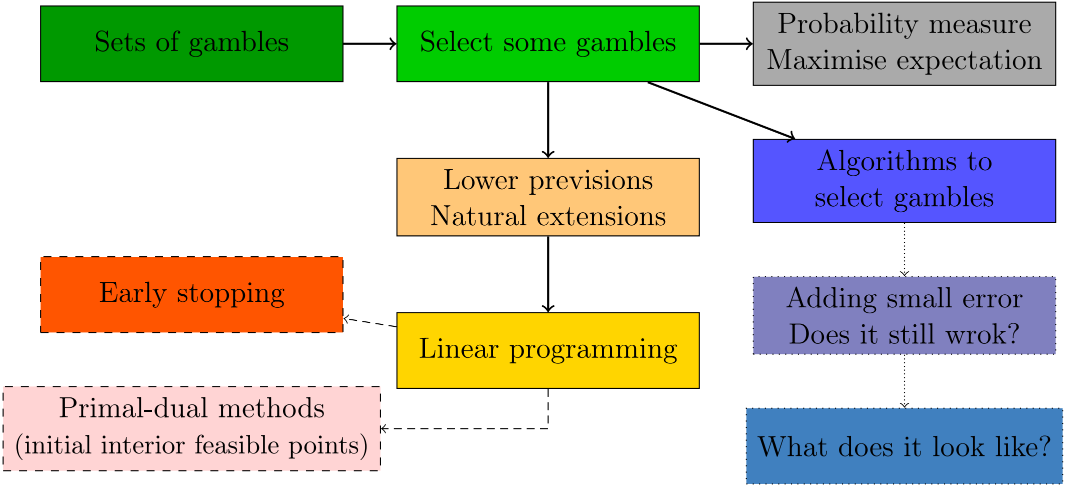
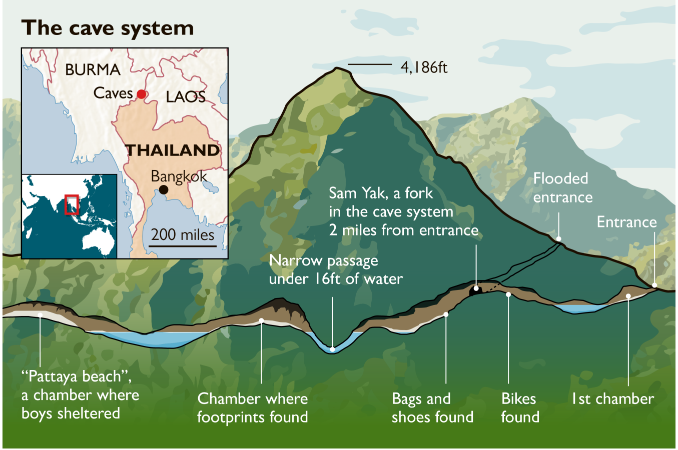
Outline


Sets of gambles
Select some gambles
Probability measure
Maximise expectation
Lower previsions
Natural extensions
Algorithms to
select gambles
Early stopping
Linear programming
Adding small error
Does it still wrok?
Primal-dual methods
(initial interior feasible points)
What does it look like?

Gambles and decisions

- A set of possible acts D
- A set of possible outcomes Ω
- A gamble $f := U(d, \omega)$
- A set of gambles $\mathcal{K}$
- An optimal set $\mathrm{opt}(\mathcal{K})$



Maximality

Definition 1 (Natural extension [1])

For any set $\mathcal{K} \subseteq \mathcal{L}(\Omega)$ and $f \in \mathcal{L}(\Omega)$, we define:

$$\underline{E}(f) := \sup\left\{\alpha \in \mathbb{R} \colon f - \alpha \geq \sum_{i=1}^{n} \lambda_i g_i, n \in \mathbb{N}, g_i \in \mathcal{K}, \lambda_i \geq 0\right\}. \tag{1}$$

Definition 2 (Maximality [1])

Let $>_{\underline{E}}$ be the partial order given by: for any two gamble f and g, $f >_{\underline{E}} g$ whenever

$$\underline{E}(f - g) > 0. \tag{2}$$

Let $\mathrm{opt}_{>}$ be the choice function corresponds to this partial order, called *maximality*, and defined by

$$\mathrm{opt}(\mathcal{K}) := \{f \in \mathcal{K} \colon (\forall g \in \mathcal{K})(g \not>_{\underline{E}} f)\}. \tag{3}$$

Algorithms for finding maximal gambles

From the definition of maximality, we can write a pariwise comparison algorithm for finding maximal gambles as follows:

- $\mathrm{opt}(\mathcal{K}) = \mathrm{Filter}(f : f \in \mathcal{K}, \mathrm{isopt}(f, \mathcal{K}) = \mathrm{True})$
- where $\mathrm{isopt}(f, \mathcal{K}) = \mathrm{all}\ (\overline{E}(f - g) \geq 0,\ \text{ for } g \in \mathcal{K})$.

Algorithms for finding maximal gambles

There are many efficient algorithm for finding maximal gambles, e.g. Algorithm 16.4 in [1]:

Require: set of gambles $\mathcal{K}$
Ensure: a set of maximal gambles $\text{opt}(\mathcal{K})$

$\mathcal{R} \leftarrow \mathcal{K}$ {remaining gambles}
$\text{opt}(\mathcal{K}) \leftarrow \emptyset$ {maximal gambles}
while $\mathcal{R} \neq \emptyset$ **do**
    $f \leftarrow$ next element from $\mathcal{R}$
    $\mathcal{R} \leftarrow \mathcal{R} \setminus \{f\}$
    if $\exists g \in \text{opt}(\mathcal{K}) \colon \underline{E}(g - f) > 0$ **then**
        f is dominated by $g \in \text{opt}(\mathcal{K})$
    else
        if $\exists g \in \mathcal{R} \colon \underline{E}(g - f) > 0$ **then**
            f is dominated by $g \in \mathcal{R}$
        end if
    end if
end while
return $\text{opt}(\mathcal{K})$

Linear programming problems

We can calculate $\underline{E}(f)$ through the following linear program:

$$
\begin{aligned}
\text{(D)} \quad \max \quad & \alpha \\
\text{s.t.} \quad & \forall \omega \in \Omega \colon \sum_{i=1}^{n} g_i(\omega)\lambda_i + \alpha \leq f(\omega) \\
& \forall i \colon \lambda_i \geq 0 \quad (\alpha \text{ free}).
\end{aligned}
$$

$\underline{E}(f)$ is equal to the optimal value of (D).

$$
\begin{aligned}
\text{(P)} \quad \min \quad & \sum_{\omega \in \Omega} f(\omega)p(\omega) \\
\text{s.t.} \quad & \forall g_i \in \mathcal{K} \colon \sum_{\omega \in \Omega} g_i(\omega)p(\omega) \geq 0 \\
& \sum_{\omega \in \Omega} p(\omega) = 1 \\
& \forall \omega \colon p(\omega) \geq 0.
\end{aligned}
$$

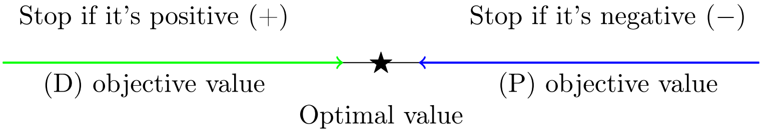


Figure: Early stopping

Solving linear programming problems

- We solve linear programs by the primal-dual interior point method.
- Initial interior feasible points:
  - for (D), we can immediately state interior feasible point.
  - for (P), once we find an interior feasible point of (P), then we can use this point for different objective function, i.e. different $\underline{E}(f)$ with respect to $\mathcal{K}$.

Numerical error

Normally, the primal-dual stops when the duality gap is less than a tolerant. Suppose that the method terminates and

- (D)-value is still negative and (P)-value is still positive, but
- the optimal value is very closed to zero.

What should we conclude in this case?

Points for discussion

- An algorithm for generating a set of gambles $\mathcal{K}$ for benchmarking
- Order gambles in $\mathcal{K}$ before put them in the algorithm.
- Impact of early stopping and initial feasible interior points.
- Numerical error, e.g., the optimal value is closed to zero.
- Linear programming that consider all gambles at once [2, 3].

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