Evaluating betting odds and free coupons using desirability

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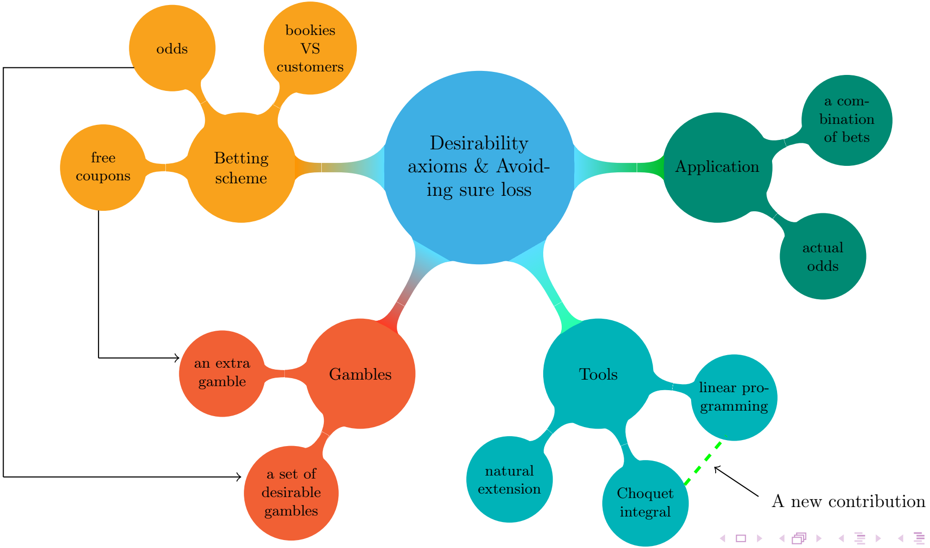
Outline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Desirability axioms

- A possibility space Ω
- A gamble $f : \Omega \to \mathbb{R}$

Q: How should we reason with desirable gambles?

Suppose we are offered:

Outcomes	James	Chen	None
f_1	-5	-1	-2
f_2	30	20	0
f_3	-1	2	-1
f_4	-50	100	-50
$f_2 + f_4$	-20	120	-50

Desirability axioms [3]

(D1) Do not accept sure loss.

(D2) Accept sure gain.

(D3) Positive scaling invariance.

(D4) Accept combination of desirable gambles.

Avoiding sure loss

Definition 1

A set of desirable gambles $\mathcal{D}$ is said to *avoid sure loss* if for all $n \in \mathbb{N}$, $\lambda_1, \cdots, \lambda_n \geq 0$ and $f_1, \cdots, f_n \in \mathcal{D}$ [4]:

$$\sup_{\omega \in \Omega} \left(\sum_{i=1}^{n} \lambda_i f_i(\omega) \right) \geq 0. \tag{1}$$

Betting with bookies

- Odds: a bookmaker offers odds, say a/b, on outcomes of an event.

Friday 10th August 2018				
		Home	Draw	Away
20:00 [redacted]	Man Utd Leicester	7/17	4/1	19/2

- The odds a/b on ω can represent an upper probability mass function: $\overline{p}(\omega) = \frac{b}{a+b}$.
- These odds are unfair.
- Bookies profit is $\sum_{i=1}^{n} \dfrac{b_i}{a_i + b_i} - 1 > 0$ [2].
- $\frac{17}{7+17} + \frac{1}{4+1} + \frac{2}{19+2} \approx 1.003 \geq 1$.

Odds and avoiding sure loss

Odds a/b on x can be viewed as a desirable gamble to the bookmaker:

$$g(\omega) = \begin{cases} -a & \text{if } \omega = x \\ b & \text{otherwise.} \end{cases} \tag{2}$$

Lemma 2

Let a/b be desirable odds on ω. Then, for all $\lambda > 0$, the odds $\lambda a/\lambda b$ on ω are also desirable.

Theorem 3 ([5])

Let $\Omega = \{\omega_1, \ldots, \omega_n\}$. Suppose a_i/b_i are betting odds on ω_i. For each $i = 1, \ldots, n$, let $g_i(\omega)$ be the corresponding gamble for the odds a_i/b_i. Then the set of desirable gambles $\mathcal{D} = \{g_1, \ldots, g_n\}$ avoids sure loss if and only if $\sum_{i=1}^{n} \dfrac{b_i}{a_i + b_i} \geq 1$.

Betting with bookies

Sign Up Offers	£	£20	£50	£30		£100	£30	£20	£20	£50	£30	£25	£10	£20	£30	£50	£100	£15	£75	£30	£60	£10		£20	£20
Special Offers																									
Each-way terms	2 1/3	2 1/3	2 1/3		2 1/3			2 1/3			2 1/3	2 1/3		2 1/3	2 1/3	2 1/3			2 1/3				2 1/3		
Man City	4/7	2/3	8/13		8/13			8/13		8/13	8/13	8/13		8/13	8/13	4/6	8/13	8/13	5/8		8/13	8/13	4/7	4/7	4/7
Liverpool	5	9/2	5		5			9/2		5	5	5		5	5	5	11/2	5	5		5	11/2	5	5	11/2
Man Utd	7	7	6		6			6		6	13/2	6		7	13/2	6	29/5	7	13/2		6	29/5	6	7	33/5
Chelsea	12	12	9		12			11		9	12	12		10	12	11	11	12	12		12	11	14	12	18
Tottenham	16	12	11		16			11		11	12	12		12	16	11	14	14	14		14	14	14	16	18
Arsenal	28	25	25		25			20		25	25	25		18	25	Bet 25	28	22	25		20	28	25	28	25
Everton	200	250	200		200			100		200	200	200		200	200	200	391/2	250	200		250	391/2	200	200	389
Wolves	250	250	100		200			200		100	250	300		250	300	100	977/4	300	300		300	977/4	250	300	284
Leicester	300	250	200		400			200		200	250	300		250	200	200	977/4	300	300		400	977/4	250	300	949
West Ham	500	500	500		500			400		500	500	500		500	500	500	48975	500	500		500	48975	500	500	408
Southampton	400	500	500		400			500		500	400	500		500	250	500	500	500	500		500	500	500	400	949
Crystal Palace	750	750	500		750			500		500	750	500		750	750	500	733	750	750		750	733	750	750	949
Newcastle	500	750	500		750			500		500	500	500		750	500	500	48975	500	750		750	48975	500	500	949

$$a := a/1$$

$$\sum_{i=1}^{13} \frac{b_i}{a_i + b_i} = 1.4393$$

Free coupons

- Free coupon = a free stake, but not truly free.
- To claim a free coupon there are standard requirements:
  1. It only applies to the customer's first bet with the bookmaker.
  2. The value of the coupon = the value of the bet that he placed.
  3. There is a maximum value of the free coupon.
  4. The free coupon can be spent only on a single outcome with the same bookmaker.

Free coupons (example 1)

Outcomes		Draw	
odds	7/17	4/1	19/2

- Tim bets £6 on the odd 4/1 on Draw, so a corresponding desirable gamble to bookies is:

Outcomes		D	
f_1	-24	6	6

- Tim gets a free coupon valued £6 and suppose that he bets his free coupon on . We scale odds $19/2 \to 57/6$. A corresponding desirable gamble to bookies is:

Outcomes		D	
f_2	0	-57	0

- Adding them together, we have:

Outcomes		D	
f	-24	-51	6

Avoiding sure loss with extra gambles

Definition 4 (Natural extension [1])

For any set $\mathcal{D} \subseteq \mathcal{L}(\Omega)$ and $f \in \mathcal{L}(\Omega)$, we define:

$$\overline{E}_{\mathcal{D}}(f) := \inf \left\{ \alpha \in \mathbb{R} \colon \alpha - f \geq \sum_{i=1}^{n} \lambda_i g_i, n \in \mathbb{N}, g_i \in \mathcal{D}, \lambda_i \geq 0 \right\}. \quad (3)$$

Theorem 5

Let $f \in \mathcal{L}(\Omega)$ and let $\mathcal{D} = \{g_1, \ldots, g_n\}$ be a set of desirable gambles that avoids sure loss. Then, $\mathcal{D} \cup \{f\}$ avoids sure loss if and only if $\overline{E}_{\mathcal{D}}(f) \geq 0$.
If $\mathcal{D} \cup \{f\}$ does not avoid sure loss, then there exists a combination of $f + \sum_{i=1}^{n} \lambda_i g_i$ for $\lambda_i \geq 0$ such that the loss is at least $|\overline{E}_{\mathcal{D}}(f)|$.

Choquet integration

Theorem 6 (modified [3])

Let f be decomposed in terms of its level sets of Ω:

$$f = \sum_{i=0}^{n} \lambda_i I_{A_i} \tag{4}$$

where $\lambda_0 \in \mathbb{R}$, $\lambda_1, \ldots, \lambda_n > 0$ and $\Omega = A_0 \supsetneq A_1, \ldots, \supsetneq A_n \supsetneq \emptyset$. If $\mathcal{D}$ is a set $\{g_1, \ldots, g_n\}$ of desirable gambles for odds, then

$$\overline{E}_{\mathcal{D}}(f) = \sum_{i=0}^{n} \lambda_i \overline{E}_{\mathcal{D}}(A_i) \tag{5}$$

where

$$\overline{E}_{\mathcal{D}}(A) = \min\{\sum_{\omega \in A} \overline{p}(\omega), 1\}. \tag{6}$$

Choquet integration (example 2)

From

Outcomes	MU	D	LC
odds	7/17	4/1	19/2
f	-24	-51	6

, we decompose a gamble in terms of its level sets as

$$f = -51 I_{A_0} + 27 I_{A_1} + 30 I_{A_2} \tag{7}$$

where $A_0 = \{\text{MU}, \text{D}, \text{LC}\}$ and $A_1 = \{\text{MU}, \text{LC}\}$. $A_2 = \{\text{LC}\}$. By eq. (6), we have

$$\overline{E}(A_2) = \min\{\overline{p}(\text{LC}), 1\} = \min\left\{\frac{2}{19+2}, 1\right\} = \frac{2}{21}$$

$$\overline{E}(A_1) = \min\{\overline{p}(\text{LC}) + \overline{p}(\text{MU}), 1\} = \min\left\{\frac{2}{19+2} + \frac{17}{7+17}, 1\right\} = \frac{45}{56}$$

$$\overline{E}(A_0) = \min\{\overline{p}(\text{LC}) + \overline{p}(\text{D}) + \overline{p}(\text{MU}), 1\} = 1.$$

Substitute $\overline{E}(A_i)$, $i = 0, 1, 2$ into eq. (7). By theorem 6, we have

$$\overline{E}(f) = \overline{E}(-51 I_{A_0} + 27 I_{A_1} + 30 I_{A_2}) = -51\overline{E}(A_0) + 27\overline{E}(A_1) + 30\overline{E}(A_2) \approx -26.45$$

Finding a combination of bets

Q: How to find λ_i in eq. (3) of theorem 5?

(P) $\quad \min \quad \alpha$

subject to $\begin{cases} \forall \omega \in \Omega \colon \alpha - \sum_{i=1}^{n} g_i(\omega)\lambda_i \geq f(\omega) \\ \forall i = 1, \ldots, n \colon \lambda_i \geq 0. \end{cases}$

$\overline{E}_{\mathcal{D}}(f)$ is equal to the optimal value of (P).

The dual of (P) is:

(D) $\quad \max \quad \sum_{\omega \in \Omega} f(\omega)p(\omega)$

subject to $\begin{cases} \forall \omega \colon 0 \leq p(\omega) \leq \overline{p}(\omega) \\ \sum_{\omega \in \Omega} p(\omega) = 1. \end{cases}$

Theorem 7 (new theoretical contribution)

1. *State an optimal solution of* (D) *from the Choquet integral.*
2. *Exploit the optimal solution of* (D) *with the complementary slackness to write a system of equalities.*
3. *Solve this system to find an optimal solution of* (P).

Finding an optimal solution of (P) and (D)

1. Order the elements $\omega_1, \omega_2, \ldots, \omega_n$ such that $\forall i \leq j \colon A(\omega_i) \subseteq A(\omega_j)$, where $A(\omega) = \bigcap_{i=0, \omega \in A_i}^{m} A_i$.
2. Let k be the smallest index such that $\sum_{j=1}^{k} \overline{p}(\omega_j) \geq 1$. Define p as follows:

$$p(\omega_i) := \begin{cases} \overline{p}(\omega_i) & \text{if } i < k \\ 1 - \sum_{j=1}^{i-1} \overline{p}(\omega_j) & \text{if } i = k \\ 0 & \text{if } i > k, \end{cases} \tag{8}$$

   then $(p(\omega_1), \ldots, p(\omega_n))$ is an optimal solution of (D) and α is the optimal value.
3. By the complementary slackness, a system of equalities is:
   1. if $p(\omega_j) > 0$, then $\alpha - \sum_{i=1}^{n} g_i(\omega_j)\lambda_i = f(\omega_j)$, and
   2. if $p(\omega_j) < \overline{p}(\omega_j)$, then $\lambda_j = 0$.
4. We solve these equations as a system of equalities in $\lambda_1, \ldots, \lambda_n$ to obtain an optimal solution of (P).

A combination of bets (example 3)

From example 2, the corresponding linear programs are as follows:

$$\text{(D1)} \quad \max \quad -24p(\boxed{\text{Manchester United}}) - 51p(\text{D}) + 6p(\boxed{\text{Leicester City}})$$

$$\text{subject to} \quad \begin{cases} 0 \le p(\boxed{\text{Manchester United}}) \le \frac{17}{24} \\ 0 \le p(\text{D}) \le \frac{2}{7} \\ 0 \le p(\boxed{\text{Leicester City}}) \le \frac{2}{21} \\ p(\boxed{\text{Manchester United}}) + p(\text{D}) + p(\boxed{\text{Leicester City}}) = 1. \end{cases}$$

As $A(\boxed{\text{Leicester City}}) \subseteq A(\boxed{\text{Manchester United}}) \subseteq A(D)$, we order these outcomes and apply eq. (8) to obtain an optimal solution of (D1), which is

$$p(\boxed{\text{Leicester City}}) = \frac{2}{21}$$
$$p(\boxed{\text{Manchester United}}) = \frac{17}{24}$$
$$p(\text{D}) = 1 - \frac{45}{56} = \frac{11}{56},$$

where the optimal value is -26.45.

$$\text{(P1)} \quad \min \quad \alpha$$

$$\text{subject to} \quad \begin{cases} \alpha + 7\lambda_{\boxed{\text{Manchester United}}} - \lambda_D - 2\lambda_{\boxed{\text{Leicester City}}} \ge -24 \\ \alpha - 17\lambda_{\boxed{\text{Manchester United}}} + 4\lambda_D - 2\lambda_{\boxed{\text{Leicester City}}} \ge -51 \\ \alpha - 17\lambda_{\boxed{\text{Manchester United}}} - \lambda_D + 19\lambda_{\boxed{\text{Leicester City}}} \ge 6 \end{cases}$$

$$\text{and} \quad \lambda_{\boxed{\text{Manchester United}}},\ \lambda_D,\ \lambda_{\boxed{\text{Leicester City}}} \ge 0,\ \alpha \text{ free.}$$

Note that $\alpha = -26.45$ and $\lambda_D = 0$.
An optimal solution of (P1) is

$$\lambda_{\boxed{\text{Manchester United}}} = \frac{27}{10}, \quad \lambda_D = 0, \quad \lambda_{\boxed{\text{Leicester City}}} = \frac{19}{7}.$$

So, Tim should additionally bet £$\frac{27}{10}$ on $\boxed{\text{Manchester United}}$ and £$\frac{19}{7}$ on $\boxed{\text{Leicester City}}$ in order to gain £26.45 from the bookies.

Results

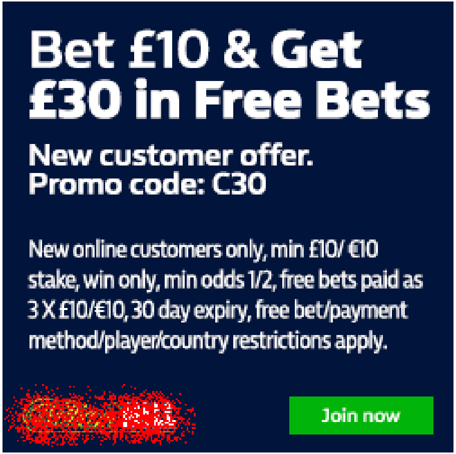
To sum up, we conclude that:

- A set of desirable gambles avoids sure loss if and only if the natural extension is non-negative.
- For this specific problem, if the set does not avoid sure loss, then a combination of bets can be derived through the Choquet integral.
- In the actual market, we found that a set of desirable gambles derived from those odds usually avoids sure loss.
- With a free coupon, the set of desirable gambles no longer avoids sure loss. Consequently, there is a combination of bets for which the customer can make a sure gain.

Open questions

There is still an open question about:

- Many choices of free coupons.
- Extend this approach to solve linear programs involving 2-monotone lower probabilities.



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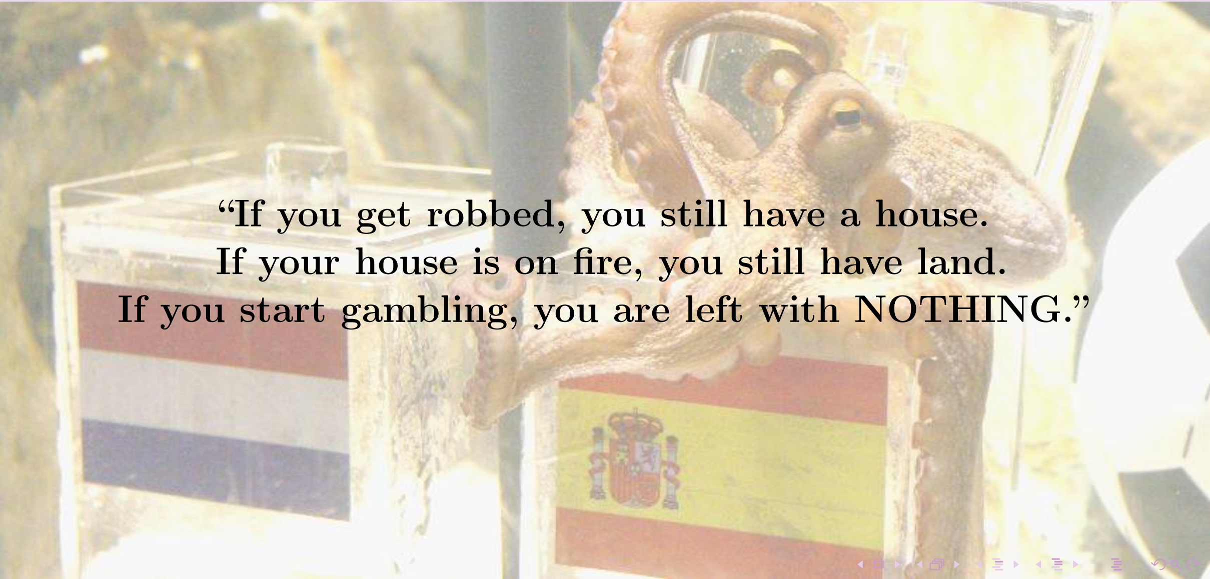

"If you get robbed, you still have a house.
If your house is on fire, you still have land.
If you start gambling, you are left with NOTHING."

References

- Thomas Augustin, Frank P. A. Coolen, Gert De Cooman, and Matthias C. M. Troffaes, editors.
  Introduction to Imprecise Probabilities.
  Wiley Series in Probability and Statistics. Wiley, 2014.
- Joseph Buchdahl.
  Fixed Odds Sports Betting: Statistical Forecasting and Risk Management.
  Oldcastle Books, 2003.
- Enrique Miranda and Gert de Cooman.
  Introduction to Imprecise Probabilities, chapter Lower prevision, pages 28–55.
  Wiley, 2014.
- Matthias C. M. Troffaes and Gert de Cooman.
  Lower Previsions.
  Wiley Series in Probability and Statistics. Wiley, 2014.
- Peter Walley.
  Statistical Reasoning with Imprecise Probabilities.
  Chapman and Hall, London, 1991.