MEASURES OF UNCERTAINTY AND INFORMATION

DAVID HARMANEC

Abstract. This contribution overviews the approaches, results and history of attempts at measuring uncertainty and information in the various theories of imprecise probabilities. The main focus, however, is on the theory of belief functions (or the Dempster-Shafer theory) [62] and the possibility theory [7] as most of the development so far has happened there. Due to the limited space I am focusing on the main ideas and point to references for details. There are several other published overviews of the subject (e.g., [27, 31, 33, 35, 38, 39]), one monograph [44], and parts of two other monographs touch upon the subject also [36, 46]. This writing summarizes my personal view of the subject. I do not cover measuring fuzziness here. Please, see one of the above references for an introduction and relevant references.

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1. Approaches to measuring uncertainty

Before we can measure uncertainty or information, we have to be clear what exactly we are trying to measure. This is not as straightforward as it might seem. Even in the framework of classical probability theory there are at least two distinct approaches to measuring "information": classical (Shannonian) information theory and algorithmic information theory [2].

Measuring uncertainty or information means assigning a number or a value from some ordinal scale to a given model of an epistemic state (in our case imprecise probability). For some work on ordinal measures of uncertainty and information see [70]. In the rest of this document, I assume that we are interested in a function of the form

$$\mathcal{U} : \mathcal{B}(X) \to \mathbb{R},$$

where $\mathcal{B}(X)$ denotes the set of all lower previsions (possibly, from a certain class) on a finite set X.

To qualify as an acceptable measure of the total uncertainty contained in a given lower prevision, function $\mathcal{U}$ must satisfy certain requirements. The five requirements below has been identified as basic requirements for uncertainty measure for belief functions. I believe, however, that their appropriate generalizations are applicable in general. The only potentially controversial point with generalization of the requirements might be the definition of independence or non-interactivity in the last requirement. For an introduction to belief functions see [62] or the introduction at this site.

(1) When Bel is a probability measure, $\mathcal{U}$ collapses to the Shannon entropy, i.e.,

$$\mathcal{U}(Bel) = -\sum_{x \in X} Bel(\{x\}) \log_2 Bel(\{x\}).$$

(2) When the basic probability assignment m corresponding to Bel has only one focal element, $\mathcal{U}$ becomes the Hartley measure of the unique focal element. Formally, for any $\emptyset \neq A \in \mathcal{P}(X)$

$$\mathcal{U}(Bel_A) = \log_2 |A|,$$

where Bel_A denotes the belief function on X defined as

$$Bel_A(C) = \begin{cases} 1 & \text{if } A \subseteq C \\ 0 & \text{otherwise} \end{cases}.$$

(3) When the uncertainty is measured in bits, which is the usually accepted unit, the range of $\mathcal{U}$ should be $[0, \log_2 |X|]$. ($\mathcal{U}(Bel) \in [0, \log_2 |X|]$ for any belief function Bel on X.)

(4) The measure $\mathcal{U}$ is subadditive, i.e., whenever the system described by X can be decomposed into two subsystems, the overall uncertainty is smaller than or equal to the sum of uncertainties about the subsystems. Formally, let $Y_1 = \{A_1, A_2, \ldots, A_P\}$ and $Y_2 = \{B_1, B_2, \ldots, B_Q\}$ be two distinct partitions of X, such that $|A_i \cap B_j| = 1$ for all $i \in \{1, 2, \ldots, P\}$ and all $j \in \{1, 2, \ldots, Q\}$. Then for an arbitrary belief function Bel on X

$$\mathcal{U}(Bel) \leq \mathcal{U}(Bel \downarrow Y_1) + \mathcal{U}(Bel \downarrow Y_2),$$

where $Bel \downarrow Y_i$ denotes the projection of Bel onto Y_i.

(5) The measure $\mathcal{U}$ is additive, i.e., whenever the system described by X can be decomposed into two "independent" or "non-interactive" subsystems, the overall uncertainty is equal to the sum of uncertainties about the subsystems. Formally, let $Y_1 = \{A_1, A_2, \ldots, A_P\}$ and $Y_2 = \{B_1, B_2, \ldots, B_Q\}$ be two (different) partitions of X, such that $|A_i \cap B_j| = 1$ for all $i \in \{1, 2, \ldots, P\}$ and all $j \in \{1, 2, \ldots, Q\}$. Moreover, let Bel be a belief function on X with the corresponding basic probability assignment m such that for every focal element A of m,

$$A = \{y^1 \cap y^2 \mid y^1 \in A \downarrow Y_1, y^2 \in A \downarrow Y_2\}$$

and $m(A) = m \downarrow Y_1(A \downarrow Y_1) \cdot m \downarrow Y_2(A \downarrow Y_2)$. Then,

$$\mathcal{U}(Bel) = \mathcal{U}(Bel \downarrow Y_1) + \mathcal{U}(Bel \downarrow Y_2). \tag{1}$$

The first two requirements say that the sought uncertainty measure should generalize already established uncertainty measures for classical probability theory and set theory. The third requirement is just a normalization requirement. The fourth requirement says that if we can break the problem into two "orthogonal" subproblems than the uncertainty about the original problem should be less than or equal to the sum of uncertainties about the subproblems. The rationale is the fact that by breaking the problem into two parts we may loose some knowledge (or information) about the dependency or interaction between the subproblems, i.e., our uncertainty would increase by the break up. The last requirement says that in the situation of the fourth requirement, under the additional assumption that there is no interaction or dependency between the subproblems, the uncertainty about the original problem is equal to the sum of uncertainties about the subproblems. The last two requirements could be conceivably weakened by requiring only existence of a (continuous) function playing the role of addition in the requirements, but there seems to be no real need for doing so.

Most attempts at measuring uncertainty in belief functions divided the problem into two parts: nonspecificity and conflict. The motivation was the observation that uncertainty ingrained in a belief function can be attributed to two distinct phenomena. One source of uncertainty, called nonspecificity, is our inability to distinguish which of several possible alternatives is the true one in a particular situation. The larger the set of possible alternatives is, the larger is the nonspecificity. The second source of uncertainty, called conflict, is present whenever there is inconsistency or disagreement in our evidence or information. One piece of evidence points in one direction and second piece points in another direction. The more mutually disagreeing pieces of evidence we have, the larger the conflict. Also, the more even the strength of the disagreeing pieces of evidence is, the larger the conflict. The total uncertainty was taken to be the sum of nonspecificity and conflict. The measurement of nonspecificity was settled relatively easily by a measure generalizing the Hartley measure [17]. Most proposals for the measure of conflict were generalizations of classical Shannon entropy [63]. For this reason, they are sometimes called entropy-like measures. However, none of the proposals was fully satisfactory. The most common drawback was the lack of subadditivity property. Consequently, the corresponding measures of total uncertainty were unsatisfactory also.

It appears that, although there is an obvious relationship between the concepts, the amounts of nonspecificity and conflict are features or measurements of 'quality' of evidence at hand rather than measurements of uncertainty about the system of interest. If one is trying to decide on a message coding scheme, for example, it seems irrelevant whether the uncertainty is caused by imprecision or by a discord in one's information. To require subadditivity for a measure of uncertainty is quite natural, but it is not clear that it is also a desirable property of a measure of 'quality' of evidence. Given these observations, it is not surprising that the only proposed measure of uncertainty, which satisfies all the five requirements stated above, did not follow the path of dividing the uncertainty to nonspecificity and conflict, but rather approached the problem directly.

2. Measures of uncertainty

This section overviews the main proposals for measures of uncertainty found in the literature with their brief motivation and basic properties. As explained in the previous section most of the research in the area aimed to generalize classical results of Hartley [17] and Shannon [63]. For the sake of completeness, I include their definitions here also. Hartley argued for the use of the function

$$I(A) = \log_2 |A|,$$

where $\emptyset \neq A \subseteq X$, as a measure of uncertainty when our information is expressed in terms of a subset of a universal set. Here, the set A represents the smallest subset of X such that we are certain that the actual state is in A.

For a probability distribution p on X, the *Shannon entropy*, K, is defined as

$$K(p) = -\sum_{x \in X} p(x) \log_2 p(x).$$

The measure of nonspecificity was first proposed for possibility and necessity measures By Higashi and Klir [18]. It was later generalized by Dubois and Prade for belief functions [6]. The *measure of nonspecificity* (or the nonspecificity, for short) is defined as

$$N(m) = \sum_{A \in \mathcal{P}(X) - \{\emptyset\}} m(A) \log_2 |A|,$$

where m denotes a basic probability assignment corresponding to the given belief function. The measure of nonspecificity is a weighted average of Hartley measure of all focal elements weighted by their basic probability numbers. It has many desirable properties, most notably additivity and subadditivity. For a more detailed discussion of properties of nonspecificity and some alternative proposals see [69, 40, 61, 58, 48]. Recently, Abellán and Moral [1] proposed to use the same expression as a measure of nonspecificity of a convex set of probability measures via the Möbius inverse of its associated lower probability (capacity). This generalization appears to preserve all the desirable properties from the case of belief functions. Klir and Yuan [47] proposed a new measure of nonspecificity defined for convex subsets of the n-dimensional Euclidean space. Given a convex universal set $X \subseteq \mathbf{R}^n$ for some finite $n \geq 1$, the proposed measure, ${}^cH$, is defined for a convex subset A of X by the formula

$${}^cH(A) = \min_{t \in T} \ln \left[\prod_{i=1}^{n} [1 + \mu^1(A_{i_t})] + \mu^n(A) - \prod_{i=1}^{n} \mu^1(A_{i_t}) \right],$$

where μ^j denotes the j-dimensional Lebesgue measure ($j = 1, n$), T denotes the set of all transformations from one orthogonal coordinate system to another, and A_{i_t} denotes the i-th projection of A within the coordinate system t. The function ${}^cH$ has been proven to satisfy all basic properties expected from an intuitive measure of nonspecificity (monotonicity, additivity, continuity, and invariance with respect to isomeric transformations of the coordinate system) [47, 71, 59].

The first attempt to measure the conflict was done by Höhle [20, 21]. He proposed to use the function C defined by the formula

$$C(m) = - \sum_{A \in \mathcal{P}(X)} m(A) \log_2 Bel(A)$$

for any basic probability assignment m on X and its corresponding belief function Bel. The function C is usually called the *measure of confusion*.

Yager [69] proposed to use the function E, usually called the *measure of dissonance*, defined by

$$E(m) = - \sum_{A \in \mathcal{P}(X)} m(A) \log_2 Pl(A)$$

for any basic probability assignment m on X and its corresponding plausibility function Pl.

Klir and Ramer [43] introduced a new measure defined by

$$D(m) = - \sum_{A \in \mathcal{P}(X)} m(A) \log_2 \sum_{B \in \mathcal{P}(X)} m(B) \frac{|A \cap B|}{|B|}$$

for all basic probability assignments m on X. They gave it the name *discord*. (For additional discussion of discord see also [8, 65, 60].)

Klir and Parvitz [41] suggested yet another measure, which is called *strife*, and defined by the formula

$$S(m) = - \sum_{A \in \mathcal{P}(X)} m(A) \log_2 \sum_{B \in \mathcal{P}(X)} m(B) \frac{|A \cap B|}{|A|}$$

for all basic probability assignments m on X. (See also [45, 67].)

All four of the above measures have the same interpretation: they define conflict as a weighted average of a logarithmic transformation of a 'local' measure of conflict for individual focal elements, i.e., measure of conflict of a particular focal element with all the other focal elements. The proposals differ in the definition of such 'local' measure. Unfortunately, none of the proposed measures is subadditive. The measure of dissonance also does not have the proper range.

Slightly different, but similar, approach was taken by George and Pal [10]. They defined conflict as a weighted average over all pairs of focal elements of a measure of conflict of two focal elements. They postulated a set of desirable properties for a measure of conflict of two focal elements. Then they proved that there is only one measure satisfying them. The resulting measure is defined by

$$GP(m) = \sum_{A \in \mathcal{P}(X)} m(A) \sum_{B \in \mathcal{P}(X)} m(B)[1 - \frac{|A \cap B|}{|A \cup B|}]$$

for all basic probability assignments m on X. This measure is not a generalization of the Shannon entropy but rather quadratic entropy, hence it is neither additive nor subadditive.

Lamata and Moral [49] were the first to propose the sum of the measure of nonspecificity and entropy-like measure as a measure of total uncertainty. They suggested to use the measure of dissonance as the measure of conflict. Later the original authors of the respective measures of conflict also suggested to use them in a measure of total uncertainty. In the same paper [49], Lamata and Moral also proposed an alternative measure of total uncertainty. The alternative measure is defined as

$$LM(Bel) = \int_0^\infty Bel(\{x \in X \mid -\log(\frac{Pl(\{x\})}{\sum_{y \in X} Pl(\{y\})}) \geq \alpha\}) d\alpha$$

for any belief function Bel on X, where Pl denotes the plausibility function corresponding to Bel. This measure can be understood as the "expected value" of

$$-\log(\frac{Pl(\{x\})}{\sum_{y \in X} Pl(\{y\})})$$

with respect to Bel. The main appeal of this proposal was the fact that it is readily generalizable to a larger class of coherent lower probabilities than belief functions. Unfortunately the measure LM is neither additive nor subadditive.

Pal *et al.* [55, 56] started by postulating requirements on a 'local' measure of total uncertainty for a single focal element. They proved that their requirements uniquely determine this 'local' measure to be the following expression

$$\log_2 \frac{|A|}{m(A)}$$

for a given focal element A and basic probability assignment m. As a measure of total uncertainty, they proposed to use the weighted average of this 'local' measure:

$$PE(m) = \sum_{A \in \mathcal{P}(X)} m(A) \log_2 \frac{|A|}{m(A)}$$

for any basic probability assignment m on X.

Smets [64] pursued a quite different approach than the rest of researchers in the area. He was trying to identify a measure of "information content" of a given belief function. The main requirement for such a measure was additivity with respect to the Dempster rule of combination (i.e., $\mathcal{U}(Bel_1 \oplus Bel_2) = \mathcal{U}(Bel_1) + \mathcal{U}(Bel_2)$). This requirement together with few other basic requirements can be satisfied only when the belief functions of interest are so called non-dogmatic belief functions. A belief function Bel is called *non-dogmatic* if it holds for its corresponding basic probability assignment m that $m(X) > 0$. For an arbitrary non-dogmatic belief function Bel, a measure of information content has to be in the form:

$$IN(Bel) = -\sum_{A \in \mathcal{P}(X)} c(A) \log Q(A),$$

where Q denotes the commonality function corresponding to Bel and c is an arbitrary non-negative monotonic set function. Smets suggested to use $c(A) = 1$ for all A. For a discussion of Smets's approach see [12].

From the above it is clear that the search for a satisfactory measure was not straightforward. Several groups of researchers worked independently at approximately the same time on related research ideas that ultimately resulted in finding a measure of total uncertainty for belief functions that satisfies all five basic requirements stated in the previous section [50, 3, 51, 54, 13]. Harmanec and Klir [13] proved that the measure satisfies all five requirements from the previous section, most notably subadditivity. The measure is defined as follows:

Definition 1. *The measure of the total amount of uncertainty contained in Bel, denoted as $AU(Bel)$, is defined by*

$$AU(Bel) = \max\{-\sum_{x \in X} p_x \log_2 p_x\},$$

where the maximum is taken over all $\{p_x\}_{x \in X}$ such that $p_x \in [0,1]$ for all $x \in X$, $\sum_{x \in X} p_x = 1$, and for all $A \subseteq X$, $Bel(A) \leq \sum_{x \in A} p_x$.

Observe that in the above definition we maximize a continuous function on a closed convex set; therefore, the maximum always exists.

An interesting question is whether the measure AU is the only measure satisfying the requirements. It is still an open question, but Harmanec [11] proved that the measure AU is the smallest measure (if any other exists) that satisfies a set of eight basic requirements. The requirements used in [11] are more elementary but as a set stronger than the ones discussed in the previous section. Unfortunately, as shown by Vejnarová [66] the measure AU is not *strictly* subadditive and submodular. (A measure $\mathcal{U}$ is strictly subadditive if Eq. 1 holds if and only if the condition of the requirement 5 from the previous section is satisfied; it is submodular if

$$\mathcal{U}(Bel) + \mathcal{U}(Bel \downarrow Y_1) \leq \mathcal{U}(Bel \downarrow Y_3 \cap Y_1) + \mathcal{U}(Bel \downarrow Y_2 \cap Y_1)$$

for all belief functions Bel and all triplets Y_1, Y_2, Y_3 of orthogonal partitions of X). The properties of strict subadditivity and (strict) modularity are useful in classical information theory as tools for characterization of (conditional) independence. If one accepts the requirements from [11] it follows using the counter example from [66] and the fact that the measure AU is minimal that there cannot be any measure satisfying these requirements that is strictly subadditive. Clearly, the measure AU can be directly generalized to any class of coherent lower probabilities. It is an open question, however, whether the generalization would preserve all the desirable properties also.

The measure AU is defined implicitly as a solution to an optimization problem. Using standard optimization techniques would be computationally infeasible in this case. Fortunately, a relatively efficient algorithm for its computation was developed by Meyerowitz *et al.* [53], and, independently, by Harmanec *et al.* [16]. For some preliminary results, see also [52]. Jaffray developed a generalization of the algorithm for 2-monotone capacities [22]. For some related results see also [26].

3. Applications of measures of uncertainty

As noted in the previous section, even the measure AU, which is the best justified measure so far, does not seem to be a suitable tool for work related to independence concepts. However, well justified measure of uncertainty can be used in many other contexts. Three basic principles of uncertainty were developed to guide the use of uncertainty measures in different situations [28, 34]. These principles are:

a principle of minimum uncertainty, a principle of maximum uncertainty, and a principle of uncertainty invariance.

The principle of minimum uncertainty is an arbitration principle. It guides the selection of meaningful alternatives from possible solutions of problems in which some of the initial information is inevitably lost but in different solutions it is lost in varying degrees. The principle states that we should accept only those solutions with the least loss of information. This principle is applicable, for example, in simplification and conflict resolution problems. For some development of this principle see [23].

The principle of maximum uncertainty is applicable in situations in which we need to go beyond conclusions entailed by verified premises. The principle states that any conclusion we make should maximize the relevant uncertainty within constraints given by the verified premises. In other words, the principle guides us to utilize all the available information but at the same time fully recognize our ignorance. This principle is useful, for example, when we need to reconstruct an overall system from the knowledge of (some) subsystems. The principle is widely used within classical probability framework [4, 5, 57], but has yet to be developed in a more general setting.

The last principle, the principle of uncertainty invariance, is of relatively recent origin [29, 30]. Its purpose is to guide meaningful transformations between various theories of uncertainty. The principle postulates that the amount of uncertainty should be preserved in each transformation of uncertainty from one mathematical framework to another. The principle was first studied in the context of probability-possibility transformations [29, 30, 32, 9, 42, 68]. For an alternative approach see also [24, 25]. Unfortunately, at the time, no well justified measure of uncertainty was available. Consequently, these transformations as well as transformations to/from belief functions were again investigated using the measure AU after its discovery [37, 14, 15].

Though important, the principles of uncertainty are not the only situations a measure of uncertainty can be used. Some other examples where the measures of uncertainty were used include measuring of closeness of possibility distribution [19] and investigating dynamics of combination of evidence [12].

4. Conclusion

This introduction aims to present the current status quo in the field of generalized information theory. It is clear from the presentation that most of the work has concentrated on justification and investigation of various uncertainty measures. This is only a start and much remains to be done, especially in developing methodologies based on the uncertainty principles.

I hope I covered the main results and developments, but any suggestions or pointers to work that should also be included is very welcome and appreciated.

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References

[1] J. Abellán and S. Moral. A non-specificity measure for convex sets of probability distributions. In G. de Cooman, F. G. Cozman, S. Moral, and P. Walley, editors, *Proceedings of the First International Symposium on Imprecise Probabilities and Their Applications*, pages 1–7, Ghent, Belgium, 1999. Imprecise Probabilities Project.

[2] G. J. Chaitin. *Information, Randomness and Incompleteness: Papers on Algorithmic Information Theory*. World Scientific, Singapore, 1987.

[3] C. W. R. Chau, P. Lingras, and S. K. M. Wong. Upper and lower entropies of belief functions using compatible probability functions. In J. Komorowski and Z. W. Raś, editors, *Methodologies for Intelligent Systems, Proceedings of 7th International Symposium ISMIS'93*, number 689 in LNAI, pages 306–315. Springer-Verlag, 1993.

[4] R. Christensen. Entropy minimax multivariate statistical modeling I: Theory. *International Journal of General Systems*, 11:231–277, 1985.

[5] R. Christensen. Entropy minimax multivariate statistical modeling II: Applications. *International Journal of General Systems*, 12(3):227–305, 1985.

[6] D. Dubois and H. Prade. A note on measures of specificity for fuzzy sets. *International Journal of General Systems*, 10(4):279–283, 1985.

[7] D. Dubois and H. Prade. *Possibility Theory*. Plenum Press, New York, 1988.

[8] J. F. Geer and G. J. Klir. Discord in possibility theory. *International Journal of General Systems*, 19(2):119–132, 1991.

[9] J. F. Geer and G. J. Klir. A mathematical analysis of information-preserving transformations between probabilistic and possibilistic formulations of uncertainty. *International Journal of General Systems*, 20(2):143–176, 1992.

[10] T. George and N. R. Pal. Quantification of conflict in Dempster-Shafer framework: a new approach. *International Journal of General Systems*, 24(4):407–423, 1996.

[11] D. Harmanec. Toward a characterization of uncertainty measure for the Dempster-Shafer theory. In P. Besnard and S. Hanks, editors, *Proceedings of the Eleventh Conference on Uncertainty in Artificial Intelligence*, pages 255–261, San Mateo, California, 1995. Morgan Kaufmann Publishers.

[12] D. Harmanec. A note on uncertainty, Dempster rule of combination, and conflict. *International Journal of General Systems*, 26(1-2):63–72, 1997.

[13] D. Harmanec and G. J. Klir. Measuring total uncertainty in Dempster-Shafer theory: A novel approach. *International Journal of General Systems*, 22(4):405–419, 1994.

[14] D. Harmanec and G. J. Klir. Principle of uncertainty invariance revisited. In P. A. S. Ralston and T. L. Ward, editors, *Proceedings of the 1996 International Fuzzy Systems and Intelligent Control Conference (IFSICC)*, pages 331–339, Louisville, KY, 1996. Institution for Fuzzy Systems and Intelliegnt Control.

[15] D. Harmanec and G. J. Klir. On information-preserving transformations. *International Journal of General Systems*, 26(3):265–290, 1997.

[16] D. Harmanec, G. Resconi, G. J. Klir, and Y. Pan. On the computation of the uncertainty measure for the Dempster-Shafer theory. *International Journal of General Systems*, 25(2):153–163, 1996.

[17] R. V. Hartley. Transmission of information. *Bell System Technical Journal*, 7:535–563, 1928.

[18] M. Higashi and G. J. Klir. Measures of uncertainty and information based on possibility distributions. *International Journal of General Systems*, 9(1):43–58, 1983.

[19] M. Higashi and G. J. Klir. On the notion of distance representing information closeness: possibility and probability distributions. *International Journal of General Systems*, 9(2):103–115, 1983.

[20] U. Höhle. Fuzzy plausibility measures. In E. P. Klement, editor, *Proceedings of the 3rd International Seminar on Fuzzy Set Theory*, pages 249–260, Johannes Kepler University, Linz, 1981.

[21] U. Höhle. Entropy with respect to plausibility measures. In *Proceedings of the 12th IEEE International Symposium on Multiple-Valued Logic*, pages 167–169, Paris, 1982.

[22] J. Y. Jaffray. On the maximum-entropy probability which is consistent with convex capacity. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 3(1):27–33, 1995.

[23] C. Joslyn and G. J. Klir. Minimal information loss possibilistic approximations of random sets. In *Proceedings of IEEE International Conference on Fuzzy Systems*, pages 1081–1088, San Diego, 1992.

[24] G. Jumarie. Expert systems and fuzzy systems: A new approach via possibility-probability conversion. *Kybernetes*, 22(8):21–36, 1993.

[25] G. Jumarie. Possibility-probability transformation: A new result via information theory of deterministic functions. *Kybernetes*, 23(5):56–59, 1994.

[26] J. N. Kapur, G. Baciu, and H. K. Kesavan. The MinMax information measure. *International Journal of Systems Science*, 26(1):1–12, 1995.

[27] G. J. Klir. Where do we stand on measures of uncertainty, ambiguity, fuzziness, and the like? *Fuzzy Sets and Systems*, 24(2):141–160, 1987.

[28] G. J. Klir. The role of uncertainty principles in inductive systems modelling. *Kybernetes*, 17(2):24–34, 1988.

[29] G. J. Klir. Probability-possibility conversion. In *Proceedings of Third IFSA Congress*, pages 408–411, Seattle, 1989.

[30] G. J. Klir. A principle of uncertainty and information invariance. *International Journal of General Systems*, 17(2-3):249–275, 1990.

[31] G. J. Klir. Generalized information theory. *Fuzzy Sets and Systems*, 40(1):127–142, 1991.

[32] G. J. Klir. Some applications of the principle of uncertainty invariance. In *Proceedings of International Fuzzy Engineering Symposium*, pages 15–26, Yokohama, Japan, 1991.

[33] G. J. Klir. Developments in uncertainty-based information. In M. C. Yovits, editor, *Advances in Computers*, volume 36, pages 255–332. Academic Press, San Diego, 1993.

[34] G. J. Klir. Principles of uncertainty: What are they? Why do we need them? *Fuzzy Sets and Systems*, 74(1):15–31, 1995.

[35] G. J. Klir. Uncertainty and information measures for imprecise probabilities: An overview. In G. de Cooman, F. G. Cozman, S. Moral, and P. Walley, editors, *Proceedings of the First International Symposium on Imprecise Probabilities and Their Applications*, pages 234–240, Ghent, Belgium, 1999. Imprecise Probabilities Project.

[36] G. J. Klir and T. A. Folger. *Fuzzy Sets, Uncertainty, and Information*. Prentice-Hall, Englewood Cliffs, NJ, 1988.

[37] G. J. Klir and D. Harmanec. On some bridges to possibility theory. In G. de Cooman, D. Ruan, and E. E. Kerre, editors, *Foundations and Applications of Possibility Theory (Proceedings of FAPT'95)*, volume 8 of *Advances in Fuzzy Systems — Applications and Theory*, pages 3–19, New Jersey, 1995. World Scientific.

[38] G. J. Klir and D. Harmanec. Generalized information theory: Recent developments. *Kybernetes*, 25(7/8):50–67, 1996.

[39] G. J. Klir and D. Harmanec. Types and measures of uncertainty. In J. Kacprzyk, H. Nurmi, and M. Fedrizzi, editors, *Consensus Under Fuzziness*, International Series in Intelligent Technologies, pages 29–51. Kluwer Academic Publishers, Boston, MA, 1997.

[40] G. J. Klir and M. Mariano. On the uniqueness of possibilistic measure of uncertainty and information. *Fuzzy Sets and Systems*, 24(2):197–219, 1987.

[41] G. J. Klir and B. Parviz. A note on measure of discord. In D. Dubois et al., editors, *Proceedings of the Eighth Conference on Uncertainty in Artificial Intelligence*, pages 138–141, San Mateo, California, 1992. Morgan Kaufman.

[42] G. J. Klir and B. Parviz. Probability-possibility transformations: A comparison. *International Journal of General Systems*, 21(3):291–310, 1992.

[43] G. J. Klir and A. Ramer. Uncertainty in the Dempster-Shafer theory: A critical reexamination. *International Journal of General Systems*, 18(2):155–166, 1990.

[44] G. J. Klir and M. J. Wierman. *Uncertainty–Based Information: Elements of Generalized Information Theory*, volume 15 of *Studies in Fuzziness and Soft Computing*. Physica Verlag, 1998.

[45] G. J. Klir and B. Yuan. On measures of conflict among set-valued statements. In *Proceedings of 1993 World Congress on Neural Networks*, Portland, Oregon, 1993.

[46] G. J. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Prentice Hall PTR, Upper Saddle River, NJ, 1995.

[47] G. J. Klir and B. Yuan. On nonspecificity of fuzzy sets with continuous membership functions. In *Proceedings of 1995 IEEE International Conference on Systems, Man, and Cybernetics*, pages 25–29, Vancouver, 1995.

[48] R. Körner and W. Näther. On the specificity of evidences. *Fuzzy Sets and Systems*, 71:183–196, 1995.

[49] M. T. Lamata and S. Moral. Measures of entropy in the theory of evidence. *International Journal of General Systems*, 14(4):297–305, 1988.

[50] P. J. Lingras. *Qualitative and Quantitative Reasoning under Uncertainty in Intelligent Information Systems*. PhD thesis, Department of Computer Science, University of Regina, 1991.

[51] Y. Maeda and H. Ichihashi. An uncertainty measure with monotonicity under the random set inclusion. *International Journal of General Systems*, 21(4):379–392, 1993.

[52] Y. Maeda, H. T. Nguyen, and H. Ichihashi. Maximum entropy algorithms for uncertainty measures. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 1(1):69–93, 1993.

[53] A. Meyerowitz, F. Richman, and E. A. Walker. Calculating maximum-entropy probability densities for belief functions. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 2(4):377–389, 1994.

[54] H. T. Nguyen and E. A. Walker. On decision making using belief functions. In R. R. Yager, M. Fedrizzi, and J. Kacprzyk, editors, *Advances in the Dempster-Shafer Theory of Evidence*, pages 311–330. John Wiley, New York, 1994.

[55] N. R. Pal, J. C. Bezdek, and R. Hemasinha. Uncertainty measures for evidential reasoning I: A review. *International Journal of Approximate Reasoning*, 7(3,4):165–183, 1992.

[56] N. R. Pal, J. C. Bezdek, and R. Hemasinha. Uncertainty measures for evidential reasoning II: A new measure of total uncertainty. *International Journal of Approximate Reasoning*, 8(1):1–16, 1993.

[57] J. B. Paris. *The uncertain reasoner's companion – a mathematical perspective.* Cambridge University Press, 1994.

[58] A. Ramer. Uniqueness of information measure in the theory of evidence. *Fuzzy Sets and Systems*, 24(2):183–196, 1987.

[59] A. Ramer. Solution to problem 96-9: A projection inequality. *SIAM Review*, 39(3):516–517, 1997.

[60] A. Ramer and G. J. Klir. Measures of discord in the Dempster-Shafer theory. *Information Sciences*, 67(1-2):35–50, 1993.

[61] A. Ramer and L. Lander. Classification of possibilistic uncertainty and information functions. *Fuzzy Sets and Systems*, 24(2):221–230, 1987.

[62] G. Shafer. *A Mathematical Theory of Evidence.* Princeton University Press, Princeton, New Jersey, 1976.

[63] C. E. Shannon. The mathematical theory of communication. *The Bell Technical Journal*, 27(July, October):379–423,623–656, 1948.

[64] P. Smets. Information content of an evidence. *International Journal of Man-Machine Studies*, 19(1):33–43, 1983.

[65] J. Vejnarová. A few remarks on measures of uncertainty in Dempster-Shafer theory. In *Proceedings of Workshop on Uncertainty in Expert Systems*, Prague, 1991. Czechoslovak Academy of Sciences.

[66] J. Vejnarová. Measures of uncertainty and independence concept in different calculi. Technical Report LISp-97-11, Laboratory of Intelligent Systems, University of Economics, Prague, Czech Republic, 1997.

[67] J. Vejnarová and G. J. Klir. Measure of strife in Dempster-Shafer theory. *International Journal of General Systems*, 22(1):25–42, 1993.

[68] S. Wonneberger. Generalization of an invertible mapping between probability and possibility. *Fuzzy Sets and Systems*, 64(2):229–240, 1994.

[69] R. R. Yager. Entropy and specificity in a mathematical theory of evidence. *International Journal of General Systems*, 9(4):249–260, 1983.

[70] Y. Y. Yao, S. K. M. Wong, and L. S. Wang. A non-numeric approach to uncertain reasoning. *International Journal of General Systems*, 23(4):343–359, 1995.

[71] B. Yuan, G. J. Klir, and J. Fridrich. A projection inequality, Problem 96-9. *SIAM Review*, 38(2):315, 1996.