Fuzzy, probabilistic and stochastic modelling of an elastically bedded beam

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Abstract

This article sets out to compare the effects of modelling uncertainty using fuzzy sets, random variables and stochastic analysis. With the aid of an example from civil engineering - the bending equation for an elastically bedded beam - we discuss what each model is or is not capable of capturing. All models may be adequately used for variability studies, but may fail to detect the effect of localized parameter fluctuations on the response of the system. In the stochastics setting, we show an instance of the linearization effect of large noise which says that under large stochastic excitations, the contributions of nonlinear terms may be annihilated.

Keywords. Fuzzy sets, random variables, stochastic differential equations, parameter uncertainty, civil engineering models.

1 Introduction

This article addresses the question of how to model parameter uncertainties in engineering problems. Among the many possibilities we wish to compare three approaches: fuzzy, probabilistic and stochastic. The question which approach is the most adequate can only be decided by looking at the consequences of using one or the other model - in particular, whether the desired or exptected effects are adequately captured. This we intend to demonstrate in a typical application in civil engineering - the computation of the displacement of an elastically bedded beam.

Suppose the response of an engineering structure is described e. g. by a differential equation. Apart from the design dimensions, certain parameters describing exterior and material properties enter as input data. These parameters exhibit variations and are only imprecisely known. The question is how the variability of the data affects the response and consequently the safety of the structure. To be more specific, assume that a certain soil parameter c has been measured or estimated from a geological assessment at various locations on a construction site, so that a list of values $c_1, \ldots, c_n$ is available. This could be interpreted as n realizations of a constant parameter c, or as realizations of a spatially variable field $c(x)$, where x denotes a spatial coordinate. Commonly, in civil engineering, the size of the sample is small and the data are augmented by prior expert knowledge. In the first, spatially constant case, the parameter c might thus be modelled either as a fuzzy number or as a random variable in the sense of subjective probability (a frequentist interpretation appears inappropriate with data as described above). In the second case, $c(x)$ might be modelled as a stationary random field, where the choice of correlation function depends on prior knowledge with possible adaptation using the data. We will address the following issues:

- What is the solution concept appropriate to the fuzzy, probabilistic or stochastic approach?
- Is the variability and the sensitivity of the response of the system adequately predicted?
- Can the model describe localization effects, that is, the possibly deviating response due to a localized, rapid change in the parameter c, even though within the boundaries given by its fuzzy or random range?
- How are nonlinear terms in the differential equation affected by a stochastic input?

Our principal findings which we wish to report here are: the fuzzy, probabilistic and stochastic approach all can - in their respective terms - describe the variability of the outcome and admit a study of its sensitivity. However, the response due to localization may considerably exceed the boundaries predicted by the fuzzy or probabilistic model. What concerns the stochastic model, we wish to point out the nonlinear trivialization effect of large noise: if the stochastic excitation is derived from a multiple of white noise which becomes larger and larger, the response of the nonlinear system may tend to the response of the linear part of the system in the mean.

The plan of the paper is as follows: In Section 2 we present the engineering example underlying our investigation and collect its properties when the parameters are deterministic. Section 3 is devoted to the fuzzy and probabilistic approach. Section 4 contains a typical stochastic model. We prove existence and uniqueness of the solution and show that the mentioned nonlinear triviality effect occurs in the example. In the last section we summarize the conclusions for engineering design suggested by our observations.

In the current engineering literature, especially in geotechnics, the problems addressed here are of high concern. The mechanisms of failure have been investigated, for example, in [3, 9, 22] using a fuzzy, probabilistic, and stochastic approach, whereas questions of modelling uncertainty in engineering problems are addressed e. g. in [8, 11, 12, 15, 21]; for the localization phenomenon we refer to [6, 7, 13]. The nonlinear trivialization effect was discoverd by [1, 2] and has been studied in a number of papers by the authors [17, 18, 19, 23]. For an exposition of various models of uncertainty in connection with fuzzy sets we refer to [14].

2 The elastically bedded beam

The displacement $w(x), x \in \mathbb{R}$ of an infinite beam on a linear elastic bedding is described by the bending equation

$$EI\, w^{IV}(x) + bc\, w(x) = q(x), \ -\infty < x < \infty\,,$$

see e. g. [5, Section 61]. Here EI is the flexural rigidity of the beam, b its effective width, c the bearing coefficient of the foundation and $q(x)$ the loading. One may imagine that the beam describes a buried pipeline, the loading $q(x)$ resulting from the covering soil. The parameters EI and b of the beam may be considered as precisely known, whereas the soil properties c and q vary in an imprecisely known fashion. A more refined model would allow for a nonlinear elastoplastic response of the founding soil, say

$$EI\, w^{IV}(x) + bc\, w(x) + F(w(x)) = q(x)\,.$$

In this situation, a typical nonlinear function would be antisymmetric and monotonically increasing with limits $\pm L$ at $\pm\infty$. The terms $bc\, w(x) + F(w(x))$ together would signify a hardening property of the foundation. We will study the singular boundary value problem for the standardized equation

$$u^{IV}(x) + 4k^4 u(x) = p(x), \ -\infty < x < \infty \qquad (1)$$

requiring that the solution should remain bounded at $\pm\infty$. In case k is a constant and $p(x)$ is an integrable function, both deterministic, its unique classical solution is given by

$$u(x) = \int_{-\infty}^{\infty} G(x,y)\, p(y)\, dy$$

in terms of its Green function

$$G(x,y) = \frac{1}{8k^3} e^{-k|x-y|} \big(\sin k|x-y| + \cos k|x-y| \big)\,.$$

In case the standardized load $p(x) \equiv p$ is constant, the displacement is constant as well and simply given by

$$u(x) \equiv \frac{p}{4k^4}\,.$$

For the computational examples to follow we let the parameters vary around central moduli of $k = 10^{-2}$, $p = 10^{-8}$. Approximately, this corresponds to the case of a buried cast-iron pipeline with an effective diameter of 6 [cm], covered by about 100 [cm] of top soil and bedded in loosely packed sand. The resulting overall displacement would amount to $u(x) \equiv 0.25$ [cm] in the deterministic case.

3 Fuzzy and probabilistic modelling

The collection of fuzzy subsets of a set X will be denoted by $\mathbb{F}(X)$. A fuzzy set $A \in \mathbb{F}(X)$ is determined by its membership function $m_A(x), 0 \le m_A(x) \le 1$, describing the degree of membership of the value x in A, respectively the degree of possibility that the object described by A assumes the value x. The sets $[A]^\alpha = \{x : m_A(x) \ge \alpha\}$ are the α-level sets of A. Given a function $f : X \to Y$, the Zadeh extension principles allows to extend it to a function $f : \mathbb{F}(X) \to \mathbb{F}(Y)$ by

$$m_{f(A)}(y) = \sup\left\{m_A(x), x \in f^{-1}(y)\right\}.$$

An element $A \in \mathbb{F}(\mathbb{R}^d)$ is called a fuzzy vector, if each of its α-level sets is convex and compact $(0 < \alpha < 1)$, and $[A]^1$ contains exactly one point. In the case of $d = 1$ we refer to A as a fuzzy number. If $f : \mathbb{R}^d \to \mathbb{R}$ is continuous and A a fuzzy vector, the function value $f(A)$ is a fuzzy number. In addition, its level sets are obtained as the images of the level sets of the data A under the function f, that is, $[f(A)]^\alpha = f([A]^\alpha)$. For more details on this and on fuzzy sets in general we refer to [4, 10].

We return to the model (1) of the elastically bedded beam, assuming that the parameter k is described by a fuzzy number K, whereas the load

$p(x) = \sum_{i=1}^{n} a_i p_i(x)$ is given by a combination of fixed shape functions $p_i(x)$ with parameters a_i, which in turn will be interpreted as fuzzy numbers $A_i, i = 1, \ldots, n$. We model the data as a non-interactive fuzzy vector in $\mathbb{R}^{(1+n)}$. This notion corresponds to independence in probability theory and signifies that the joint membership function is given by the formula $m_{(K,A_1,\ldots,A_n)}(k, a_1, \ldots, a_n) = \min\{m_K(k), m_{A_1}(a_1), \ldots, m_{A_n}(a_n)\}$. We shall compute the fuzzy point values of the fuzzy response $u(x)$ by applying the Zadeh extension principle to the solution operator

$$(k, a_1, \ldots, a_n) \longrightarrow u(x) = L_x(k, a_1, \ldots, a_n)\,. \tag{2}$$

By the discussion above, an α-level set of $L_x(K, A_1, \ldots, A_n)$ is computed as the collection of the values of the solutions attained when the parameters vary in the respective level sets $[K]^\alpha, [A_1]^\alpha, \ldots, [A_n]^\alpha$.

In fact, the Zadeh extension principle allows to derive a fuzzy solution concept for which this is the unique fuzzy solution. To achieve this, equation (1) is rewritten as a prolongued system for the variables $(u, h, b_1, \ldots, b_n)$,

$$u^{IV}(x) + 4h^4 u(x) - \sum_{i=1}^{n} b_i p_i(x) = 0\,, \tag{3}$$

$$h' = 0, b_1' = 0, \ldots, b_n' = 0\,, \tag{4}$$

$$h(0) = k, b_1(0) = a_1, \ldots, b_n(0) = a_n\,. \tag{5}$$

The parameters $h = k, b_i = a_i, i = 1, \ldots, n$ are actually constants. The device incorporated in (4), (5) allows to interpret them as variables, so formally equation (3) contains no "parameters". This is important, because substitution of a function with fuzzy parameters in a term depending on the same parameters need not preserve equalities (see e. g. the discussions in [10, 16]). The equation operator $E : (\mathcal{C}_b^4(\mathbb{R}))^{2+n} \to (\mathcal{C}_b^0(\mathbb{R}))^{2+n}$ is defined by the left hand sides of (3), (4) on the space of bounded and four times differentiable functions, the restriction operator $R : (\mathcal{C}_b^4(\mathbb{R}))^{2+n} \to \mathbb{R}^{1+n}$ by the left hand sides of (5). The solution operator is

$$L : \mathbb{R}^{1+n} \to (\mathcal{C}_b^4(\mathbb{R}))^{2+n}\,,$$
$$(k, a_1, \ldots, a_n) \to (u, h, b_1, \ldots, b_n),$$

where, by construction, $h = k$ and $b_i = a_i$. An element $X = (U, H, B_1, \ldots, B_n) \in \mathbb{F}\big((\mathcal{C}_b^4(\mathbb{R}))^{2+n}\big)$ is called a fuzzy solution of system (3), (4), (5), if

$$E(X) = 0 \text{ in } \mathbb{F}\big((\mathcal{C}_b^0(\mathbb{R}))^{2+n}\big)\,,$$
$$R(X) = (K, A_1, \ldots, A_n) \text{ in } \mathbb{F}(\mathbb{R}^{1+n})\,.$$

Here 0 denotes the crisp zero function; note that the derivatives of the fuzzy function X occurring in $E(X)$ are defined by means of the extension principle, as are the other operations performed on it.

Proposition 1 *Given $(K, A_1, \ldots, A_n) \in \mathbb{F}(\mathbb{R}^{1+n})$, problem (3), (4), (5) has a unique fuzzy solution $X = (U, H, B_1, \ldots, B_n) \in \mathbb{F}\big((\mathcal{C}_b^4(\mathbb{R}))^{2+n}\big)$, namely $X = L(K, A_1, \ldots, A_n)$.*

The proof follows by adapting the arguments given in [16] for the pure initial value problem. In addition, one can show that the restriction of the fuzzy solution function U to a spatial point x is simply given by fuzzifying (2), thus by the fuzzy number $L_x(K, A_1, \ldots, A_n)$.

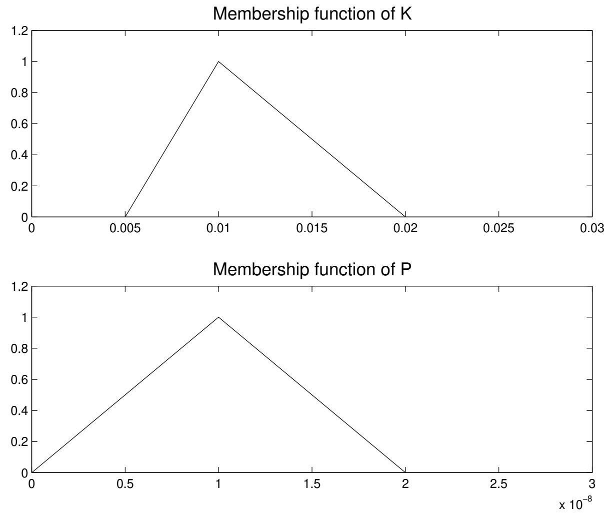

Figure 1: Fuzzy parameters K and P.

In the example to follow, we assume that p is constant. For the sake of exposition, we take both K and P as triangular fuzzy numbers, centered around the values indicated in Section 1, namely

$$K = \langle \tfrac{1}{2}, 1, 2 \rangle \cdot 10^{-2}, \tag{6}$$

$$P = \langle 0, 1, 2 \rangle \cdot 10^{-8}, \tag{7}$$

see Figure 1. Then the (constant) fuzzy solution is simply given by the fuzzy number $L_x(K, P) = P/4K^4$. It is depicted in Figure 2.

We observe that the fuzzy model correctly describes the fluctuations of the response in dependence of the data variability. In addition, the α-level structure provides a good picture of the sensitivity of the result.

On the other hand, assume now that the parameter p varies in the range given by its fuzzy description, in the example its support $[0, 2 \cdot 10^{-8}]$, but its realization is not actually constant. For example, it might jump from 0 to $2 \cdot 10^{-8}$ at the point $x = 0$. This means that

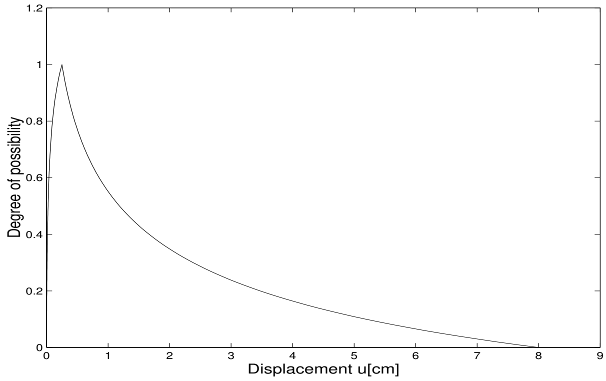

Figure 2: Fuzzy displacement u.

we have to solve equation (1) with a load

$$p(x) = \begin{cases} 0, & x < 0, \\ p = 2 \cdot 10^{-8}, & x > 0. \end{cases}$$

The corresponding displacement is

$$w(x) = \begin{cases} \frac{p}{8k^4}\, e^{kx} \cos kx, & x < 0, \\ \frac{p}{8k^4}\big(2 - e^{-kx} \cos kx\big), & x > 0. \end{cases}$$

Taking the admissible value $k = \frac{1}{2} \cdot 10^{-2}$, it is seen that the the graph of w exceeds the band described by the fuzzy displacement with constant parameters. This is an example of the effect, observed in other circumstances as well, that a localized parameter fluctuation may produce a response not predicted by the fuzzy model. In Figure 3, the 0-level set of the fuzzy displacement is indicated by the horizontals $u = 0$ and $u = 8$, the curve is the graph of w, and the other horizontals indicate level sets of the fuzzy displacement u for $\alpha = 0.2, 0.4, 0.6, 0.8$. Degree of possibility equal to one occurs at $u = 0.25$.

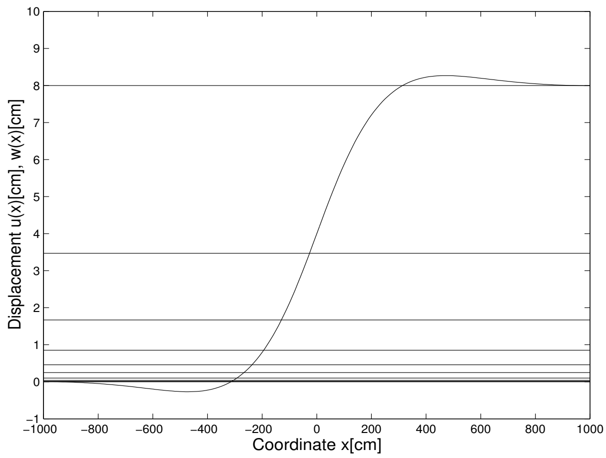

Figure 3: Fuzzy solution versus effect of localization.

We now turn to a probabilistic modelling of system (1). We assume again that the load is of the form $p(x) = \sum_{i=1}^{n} a_i p_i(x)$ as before. In the probabilistic approach, we describe the parameters $k, a_1, \ldots, a_n$ as random variables on a probability space (Ω, Σ, μ). In this case we obtain the solution as a random process

$$(x, \omega) \in \mathbb{R} \times \Omega \longrightarrow u(x, \omega)\,.$$

It is well known and easy to prove that, for almost all $\omega \in \Omega$, a pathwise solution exists. These pathwise solutions are almost surely unique and differentiable. The point values $u(x)$ are random variables. In continuation of the computational example above, we take p constant with a triangular distribution with defining boundaries as in (7), and similarly for the parameter k with boundaries as in (6). We approximate the distribution of the solution $u(x) \equiv p/4k^4$ numerically by simulation (using 10^5 realizations), the result being depicted in Figure 4.

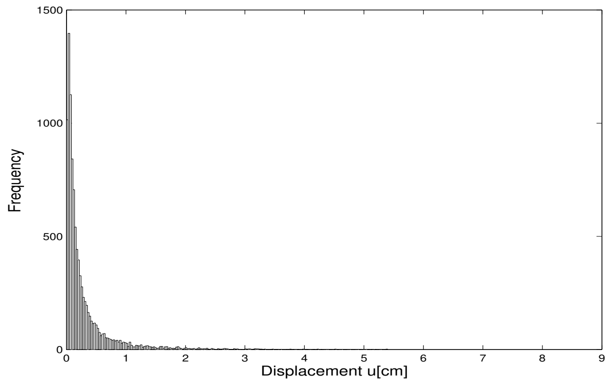

Figure 4: Frequency distribution of random displacement.

For reasons of comparison, we have chosen the same shape for the probability distribution of k and p as for the possibility distribution in the fuzzy case. As expected, the probabilistic result centers around the probable outcome, whereas the fuzzy result exhibits the possible variation of the outcome. The picture needs no further explication; it is clear that the probabilistic result fails to cover the response due to a localized parameter fluctuation all the more.

4 Stochastic modelling

By a stochastic model in the context of the elastically bedded beam we mean that the parameters are described by spatial random fields or, in one dimension, random processes. We begin with the linear case. To fix ideas, we assume that the parameter k in (1) is constant, and the load is modelled by a multiple of Gaussian white noise, $p(x) = a\dot{W}(x)$. To adapt the model to real data, one might actually take a more general diffusion process and add a deterministic mean value. However, this does not change the effects we want to describe and so we shall use this simpler model, namely,

$$v^{IV}(x) + 4k^4 v(x) = a\dot{W}(x)\,. \qquad (8)$$

As noted, white noise $\dot{W}$ is a stationary Gaussian process. Its mean and covariance are given by

$$\mathrm{E}\big(\dot{W}(x)\big) = 0,$$
$$\mathrm{COV}\big(\dot{W}(x), \dot{W}(y)\big) = \delta(x-y)\,,$$

where $\delta(.)$ denotes the Dirac measure. Thus $\dot{W}$ has to be interpreted as a generalized stochastic process, in fact, it is the generalized derivative of Wiener process W. In turn, the stochastic differential equation is interpreted as an Itô-differential equation (see e. g. [20] as a reference). Its (almost surely unique) solution is given by the Itô-integral

$$v(x) = a\int_{-\infty}^{\infty} G(x,y)\,dW(y)\,,$$

where $G(x,y)$ is the Green function. The Itô-integral can be defined as the mean square limit

$$\mathrm{L}^2 - \lim_{\Delta y\to 0} a \sum_{j=-\infty}^{\infty} G(x, j\Delta y)\,\Delta W_j\,,$$

where $\Delta W_j = W((j+1)\Delta y) - W(j\Delta y)$ denote the increments of Wiener Process. These are known to be independent and normally distributed according to $\mathcal{N}(0,\Delta y)$. This in turn indicates how the solution can be approximated by simulation. An approximate realization of white noise is shown in Figure 5, a number of sample paths of the Itô-solution are depicted in Figure 6. The constant a has been fixed in this computational example so that the standard deviation of the solution $v(x)$ coincides with the value 0.25 [cm] used for the deterministic displacement in Section 2. For this as well as later use we record the following result; the star denotes convolution.

Lemma 1 *The covariance of the Itô-solution v to problem (8) is*

$$\mathsf{COV}\big(v(x), v(z)\big) = a^2 S * S(|x-z|)$$

where $S(y) = G(0,y)$ denotes the fundamental solution of equation (1).

Proof: By the isometry property of Itô-integrals and using that $G(x,y) = S(x-y) = S(|x-y|)$, we have that

$$\mathsf{E}\big(v(x)v(z)\big) = a^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(x-y)S(z-y)\,dy$$
$$= a^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(x-z+y)S(y)\,dy = a^2 S * S(|x-z|)\,,$$

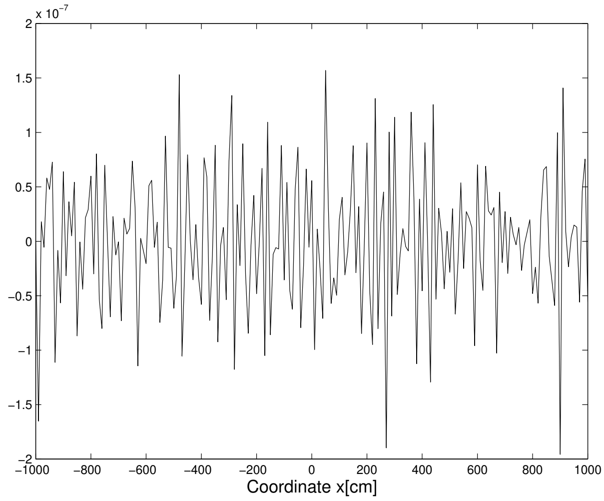

Figure 5: An approximate sample path of white noise.

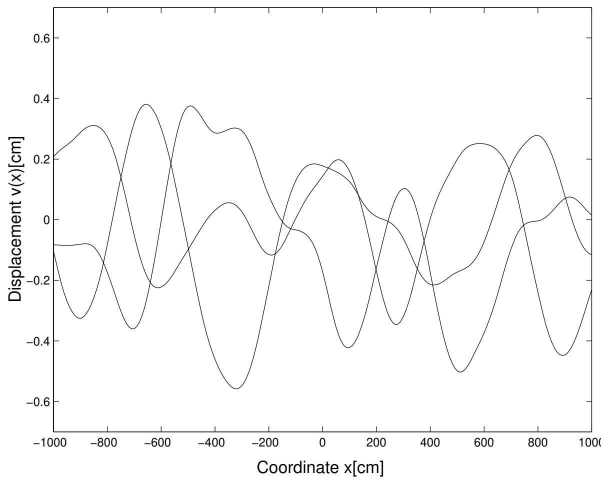

Figure 6: Three sample paths of the solution v.

which proves the result, since $\mathsf{E}(v(x)) \equiv 0$.

In particular,

$$\mathsf{V}\big(v(x)\big) = a^2 S * S(0) > \mathsf{COV}\big(v(x), v(z)\big) \qquad (9)$$

whenever $x \neq z$. In fact, a little Fourier analysis shows that

$$S * S(r) = \frac{1}{\pi}\int_0^{\infty} \cos r\xi\,\frac{d\xi}{(\xi^4+4k^4)^2}\,,$$

whence

$$S * S(0) - S * S(r) = \frac{1}{\pi}\int_0^{\infty} \big(1-\cos r\xi\big)\,\frac{d\xi}{(\xi^4+4k^4)^2}$$

is strictly greater than 0 whenever $r \neq 0$.

Our next task is the investigation of the nonlinear, stochastic bending equation

$$u^{IV}(x) + 4k^4 u(x) + F(u(x)) = a\dot{W}(x)\,, \qquad (10)$$

where we assume that F is Lipschitz continuous with a global Lipschitz constant $\mathrm{Lip}(F)$. We interpret (10) as an integral equation, namely

$$\begin{aligned} u(x) &= -\int_{-\infty}^{\infty} G(x,y)\,F(u(y))\,dy \\ &+ a\int_{-\infty}^{\infty} G(x,y)\,dW(y)\,. \end{aligned} \tag{11}$$

We shall construct a solution in the space $\mathcal{C}_b = \mathcal{C}_b^0(\mathbb{R} : \mathrm{L}^2(\Omega))$ of stochastic processes with bounded second moments. This is a Banach space, equipped with the norm

$$||u|| = \sup_{x\in\mathbb{R}} \left(\mathrm{E}|u(x)|^2\right)^{1/2}.$$

Proposition 2 *Under the condition Lip(F) $< 4k^4$, the integral equation (11) has a unique solution u in the space $\mathcal{C}_b$.*

Proof: First, the operator $\mathcal{L}$ defined by the right hand side of (11) maps the space $\mathcal{C}_b$ into itself. To see this, one has to employ the Lipschitz property of F, the mentioned isometry property of Itô-integrals, and the fact that the Green function $y \to G(x,y)$ is locally uniformly square integrable due to the inequality $e^{-|x-y|} \le e^{|x|-|y|}$. The arguments are similar to the ones proving the contraction property which we now present in more detail:

$$\begin{aligned} &\sup_{x\in\mathbb{R}} \Big(\mathsf{E}\big|\mathcal{L}u(x) - \mathcal{L}\tilde{u}(x)\big|^2\Big)^{1/2} \\ &= \sup_{x\in\mathbb{R}} \Big(\mathsf{E}\big(\int_{-\infty}^{\infty} G(x,y)(F(u(y)) - F(\tilde{u}(y)))\,dy\big)^2\Big)^{1/2} \\ &\le \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |G(x,y)|\,\mathrm{Lip}(F)\big(\mathsf{E}|u(y) - \tilde{u}(y)|^2\big)^{1/2} dy \\ &= \frac{1}{4k^4}\mathrm{Lip}(F) \sup_{y\in\mathbb{R}} \big(\mathsf{E}|u(y) - \tilde{u}(y)|^2\big)^{1/2} \end{aligned}$$

since the integral of the Green function at fixed x equals $1/4k^4$. Thus $\mathcal{L}$ is a contraction on $\mathcal{C}_b$ if the Lipschitz constant is less than $4k^4$; the integral equation (11) has a unique fixed point $u \in \mathcal{C}_b$, as required.

We remark that this fixed point $u(x)$ provides a pathwise solution of the differential equation (10) in the sense of generalized functions (Schwartz distributions). For the notions from distribution theory which we need here and below we refer to [24].

We now turn to the nonlinear trivialization effect of large noise. This refers to a comparison of the solutions of the nonlinear equation

$$u^{IV}(x) + 4k^4 u(x) + F(u(x)) = a\dot{W}(x)\,,$$

and the linear equation

$$v^{IV}(x) + 4k^4 v(x) = a\dot{W}(x)\,.$$

as a tends to infinity. We shall prove that - for a large class of nonlinearities F - the difference $u - v$ tends to zero in the mean. This class of nonlinearities has been introduced in [23]. For its definition we need the Fourier transform of bounded functions, hence the Schwartz theory of distributions is again required. Recall that $\mathcal{S}'(\mathbb{R})$ denotes the space of tempered distributions, the collection of continuous functionals on the space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions. The action of a distribution T on a function ψ is denoted by $\langle T, \psi\rangle$.

Definition 1 *A distribution $T \in \mathcal{S}'(\mathbb{R})$ is said to be massless at zero, if $\lim_{\varepsilon\to 0}\langle T, \psi(\cdot/\varepsilon)\rangle = 0$ for the function $\psi(y) = \exp(-y^2/2)$.*

We shall be concerned with bounded functions F whose Fourier transform $\mathcal{F}F$ is massless at zero. This is the case if and only if

$$\lim_{\varepsilon\to 0} \varepsilon \int_{-\infty}^{\infty} F(y)\, e^{-\varepsilon^2 y^2/2}\, dy = 0\,.$$

Here are some examples of functions F whose Fourier transform $\mathcal{F}F$ is massless at zero (see [23] for details and further examples):

(a) Every bounded, continuous function F with $\lim_{y\to -\infty} F(y) = -L, \lim_{y\to\infty} F(y) = +L$;

(b) every continuous, periodic function F whose first derivative is locally square integrable;

(c) every element $F \in \mathrm{L}^p(\mathbb{R})$ when $1 \le p \le 2$.

The significance of this notion lies in the following result, the proof of which can be found in [23] as well.

Proposition 3 *Let F be a bounded function whose Fourier transform $\mathcal{F}F$ is massless at zero. Let $(v_a, w_a), a > 0$, be non-degenerate Gaussian vectors and assume that the variance $V(v_a)$ tends to infinity as $a \to \infty$. Then*

$$\mathsf{E}\big(F(v_a)F(w_a)\big) \to 0 \text{ as a} \to \infty\,.$$

The nonlinear trivialization effect of large noise is expressed in the following result.

Theorem 1 *Assume that the function F is bounded with Fourier transform $\mathcal{F}F$ massless at zero. Further, assume that Lip(F) $< 2k^4$. Let u be the solution constructed above to the nonlinear stochastic differential equation (10) and v the solution to the linear equation (8). Then*

$$\mathsf{E}\Big(\int_K |u(x) - v(x)|\, dx\Big) \to 0 \text{ as } a \to \infty$$

for every compact interval $K \subset \mathbb{R}$.

Proof: The difference of u and v satisfies the equation

$$(u-v)^{IV} + 4k^4(u-v) + F(u) - F(v) = -F(v)\,.$$

Applying the Green function and taking the expectation we get

$$\mathsf{E}\Big(u(x)-v(x) + \int_{-\infty}^{\infty} G(x,y)\big(F(u(y))-F(v(y))\big)dy\Big)^2$$
$$= \mathsf{E}\Big(\int_{-\infty}^{\infty} G(x,y)F(v(y))\,dy\Big)^2$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y)G(x,z)\,\mathsf{E}\Big(F(v(y))F(v(z))\Big)dydz\,.$$

Denote the last term by $h^2(x)$. Due to Proposition 3, the boundedness of F and the integrability of the Green function, $h^2(x)$ tends to zero as $a \to \infty$ and remains bounded. By the triangle and Hölder's inequality it follows that

$$\mathsf{E}\,|u(x)-v(x)|$$
$$\leq \mathsf{E}\,\Big|\int_{-\infty}^{\infty} G(x,y)\big(F(u(y))-F(v(y))\big)dy\,\Big| + h(x)\,.$$

Using the fact that $G(x,y) \leq e^{-|kx-ky|}/4k^3$ and the elementary inequality $-\frac{|kx|}{1+b} - |kx-ky| \leq -\frac{|ky|}{1+b} - \frac{b}{1+b}|kx-ky|$ for $b>0$, we obtain

$$\mathsf{E}\int_{-\infty}^{\infty} e^{-|kx|/(1+b)}|u(x)-v(x)|\,dx$$
$$\leq \mathrm{Lip}(F)\,\mathsf{E}\Big(\frac{1}{4k^3}\int_{-\infty}^{\infty} e^{-b|kx-ky|/(1+b)}\,dx$$
$$\cdot\int_{-\infty}^{\infty} e^{-|ky|/(1+b)}|u(y)-v(y)|\,dydx\Big)$$
$$+\int_{-\infty}^{\infty} e^{-|kx|/(1+b)}h(x)\,dx$$
$$\leq \frac{1+b}{2bk^4}\,\mathrm{Lip}(F)\,\mathsf{E}\int_{-\infty}^{\infty} e^{-|ky|/(1+b)}|u(y)-v(y)|\,dy$$
$$+\int_{-\infty}^{\infty} e^{-|kx|/(1+b)}h(x)\,dx\,.$$

Thus if the coefficient in front of the first integral is less than one, which is the case when $\mathrm{Lip}(F) < 2k^4$ for b chosen large enough, then

$$\mathsf{E}\int_{-\infty}^{\infty} e^{-|kx|/(1+b)}|u(x)-v(x)|\,dx \to 0 \text{ as } a \to \infty$$

due to the properties of $h(x)$ derived above. This, in particular, implies the assertion of the theorem.

The theorem asserts that under quite general conditions on the nonlinear term, the stochastic solution to the nonlinear equation behaves like the stochastic solution to the corresponding linear equation in the mean. Thus large noise may annihilate nonlinearities in the model. This is the trivialization effect. We note that the condition on the Lipschitz constant of F can be improved by considering other norms. In addition, if the noise is concentrated on a compact interval, the trivialization effect occurs for all F as in the theorem with Lip$(F) < 4k^4$.

5 Summary and Conclusions

We have investigated the assertions of a fuzzy, probabilistic and a stochastic model in a typical example from civil engineering. The approaches formulate uncertainty in different ways; the modelling assumptions differ, as do the interpretations of the results. One may state that all models capture and process the uncertainty and provide insight into the sensitivity of the system. They highlight different aspects and neglect others. For example, the fuzzy and the probabilistic approach evaluate the width of fluctuations quite differently; the stochastic approach may suppress the effect of nonlinearities (this, by the way, is an example of a certain smoothing observed otherwise as well). But most importantly, neither of the models - if set up in the way as done here - fully detects the effects of a localized parameter fluctuation on the response of the system.

As a consequence for engineering practice and design, we conclude that a safety analysis must include both a study of the variability, as can be achieved by the models under discussion, and an investigation into specific failure mechanisms that may arise from local disturbances. We focused our discussion on the fuzzy, probabilistic and stochastic approach for reasons of exposition. A similar analysis can be carried through in the framework of other theories of imprecise probabilities as well.

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