Functions and measures with linear ordinal scales, (a-)symmetric Sugeno integral and ordinal Ky Fan distance

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Abstract

We develop a purely ordinal model for aggregation operators for lattice valued functions, comprising as special cases quantiles, Ky Fan metrics and Sugeno integrals. For modelling findings of psychological experiments like the reflection effect in decision behaviour under risk or uncertainty, we introduce reflection lattices. These are complete linear lattices endowed with an order reversing bijection like the reflection at 0 on the real interval $[-1, 1]$. Furthermore the lattice operations $\vee$, $\wedge$ are modified in order to mimic addition and multiplication of numbers of both signs.

Keywords. Quantile, Ky Fan metric, Sugeno integral, aggregation, reflection lattice.

1 Motivation and survey

Aggregation has a long tradition, mean values are a very common concept, perceivable by everybody in the discrete case, the continuous case referring to integration w.r.t. a measure μ. In the simplest discrete case, the measure is the counting measure. Here numbers play an important role. But how do humans perceive numbers? The numbers, say the set $\mathbb{R}$ of reals, support two basic structures, the algebraic structure defined by $+$ and $\cdot$ and the ordinal structure given by $\leq$. There are many situations where only order is relevant, cardinals being used merely by tradition and convenience.

We are interested in the question if aggregation can be done in purely ordinal terms and what results can be attained. Of course many partial results are available. Since often they are formulated in terms of numbers, we ask what can be sustained if one ignores the algebraic structure or what weaker additional structure has to be imposed on the linear ordinal scale in order to formulate some well known important issues. It turns out that enough structure is given by an order reversing bijection of the scale leaving one point fixed. Thus the scale decomposes into two symmetric parts. This can be interpreted as the first step to numbers. Since, repeating the procedure with each resulting part of the scale infinitely often, one ends up with the binary representation of the numbers in the unit interval $[0, 1] \subset \mathbb{R}$.

There are several ordinal concepts for aggregating values with respect to a measure. The oldest and best known takes a certain quantile, say the median, of a sample as the aggregated value. Next Ky Fan's [2] metric on the space $L_0(\mu)$ of μ-measurable functions is essentially ordinal. More recently and independently Sugeno [5] developed his integral which employs the same idea as Ky Fan. One aim of the present paper is to develop a common model for these three examples and this is done with a complete linear lattice M as scale, comprising the classical case $M = [0, 1] \subset \mathbb{R}$. For the Sugeno integral the scales used for functions and the measure are identical, but for the quantiles we need separate scales for the functions and the measure which are related by a commensurability application, as we call it.

The structure of a linear lattice seems not to be sufficient to model elementary human behaviour. There is some empirical psychological evidence (cf. reflection effect, inverse S-shaped decision weights) that in certain decision situations humans have a point $\mathbb{O}$ of reference on their scale which allows to distinguish good and bad or gains and losses, i.e. values above respectively below the reference point. Then the attainable gains and the attainable losses are aggregated separately and finally these two aggregated values are compared to reach the final decision. In the cardinal case this behaviour can be modelled with the symmetric Choquet integral. The Choquet integral not being a purely ordinal issue, we introduce the symmetric Sugeno integral in order to model the essentials of this behaviour in purely ordinal terms.

Formally we model the scale with neutral reference point as a complete linear reflection lattice R corresponding to $[-1, 1] \subset \mathbb{R}$. On R we use the binary relations from [3] to get operations corresponding to addition and multiplication on $\mathbb{R}$.

For defining the analogue of the asymmetric Choquet integral in our context an order preserving bijection of the scale M for the measure μ with ordinal values has to be supposed. This additional structure on the scale allows to define the conjugate of μ which is not needed for the symmetric Sugeno integral. A pair of mutual conjugate ordinal measures might also be used to model imprecision in an ordinal context.

2 Reflection lattices and ordinal measures

First we recall some definitions playing central roles throughout the paper. Let $L \neq \emptyset$ be an ordered set, i.e. L is endowed with a reflexive, antisymmetric and transitive binary relation denoted $\leq$. We further suppose that the order is *linear* or *total*, i.e. $x \leq y$ or $y \leq x$ for arbitrary $x, y \in L$. For avoiding technicalities we additionally suppose that L is a *complete linear lattice*, i.e. the least upper bound $\bigvee_{x \in S} x$ and the greatest lower bound $\bigwedge_{x \in S} x$ exist for all subsets $S \subset L$. Especially the *top* $\mathbb{I} := \bigvee_{x \in L} x$ and the *bottom* $\mathbb{O} := \bigwedge_{x \in L} x$ of L exist.

In order to get a scale for the decision processes described in the introduction we take a copy L_+ of a complete linear lattice with bottom $\mathbb{O}$ and top $\mathbb{I}$ and a second copy L_- with reversed order, $-a$ being the image of $a \in L_+$. Then $\mathbb{O} \in L_+$ and $-\mathbb{O} \in L_-$ are identified in the disjoint union $L_+ \cup L_-$ to get the *linear reflection lattice* R. Bottom and top are now $-\mathbb{I}$ and $\mathbb{I}$, respectively. Examples of reflection lattices are $[-1, 1] \subset \mathbb{R}$ or $\mathbb{R} \cup \{-\infty, \infty\}$. The *negative* $-a$, the *absolute value* $|a|$ and the *signum* sign a of an element $a \in R$ are defined like for the reals.

For use in Section 4 we introduce, by means of the usual lattice operations $\vee$ and $\wedge$, two new operations on R [3],

$$a \bigtriangledown b \ := \begin{cases} -(|a| \vee |b|) & \text{if } b \neq -a \text{ and} \\ & |a| \vee |b| = -a \text{ or } = -b \\ \mathbb{O} & \text{if } b = -a \\ |a| \vee |b| & \text{else} \end{cases} ,$$

$$a \bigtriangleup b \ := \begin{cases} -(|a| \wedge |b|) & \text{if sign } a \neq \text{sign } b \\ |a| \wedge |b| & \text{else} \end{cases} ,$$

which will play the roles of addition and multiplication, respectively, of numbers in our context. For example, $|x \bigtriangledown (-y)|$ will play the role of a distance (see Section 4). But one has to be cautious with the new binary operations since $\bigtriangledown$ is not associative and for both distributivity does not hold. On L_+ the operators $\bigtriangledown$, $\bigtriangleup$ coincide with the lattice operators $\vee$, $\wedge$, respectively.

Let M be a complete linear lattice. A M-valued set function $\mu : 2^\Omega \to M$ on a set Ω is called an *ordinal measure*, if $\mu(\emptyset) = \mathbb{O}$, $\mu(\Omega) = \mathbb{I}$ and it is *monotone*, $A \subset B$ implies $\mu(A) \leq \mu(B)$.

An ordinal measure μ is called *minitive* (necessity measure in decision theoretic terminology) if for all families of sets $\mathcal{A} \subset 2^\Omega$

$$\mu(\bigcap_{A \in \mathcal{A}} A) = \bigwedge_{A \in \mathcal{A}} \mu(A) ,$$

that is the intersection has the largest measure allowed on $\mathcal{A}$ by the monotonicity condition for μ. A *maxitive* measure (possibility measure) is just the dual with $\cup$ and $\vee$.

If M is a linear reflection lattice with the reflection now denoted $a \mapsto \bar{a}$, the *conjugate* of μ can be defined,

$$\bar{\mu}(A) := \overline{\mu(\Omega \setminus A)} \quad A \subset \Omega .$$

It is easily checked that μ is minitive iff $\bar{\mu}$ is maxitive.

Like in the cardinal case one may employ the pair $(\mu, \bar{\mu})$ to model imprecision. Thus the self conjugate ordinal measures, $\mu(\Omega \setminus A) = \overline{\mu(A)}$ for all A, might be considered as a counterpart of probability measures in the present context.

3 Fan-Sugeno operators

Let L be a complete linear lattice and $f : \Omega \to L$ a function (an act in decision theoretic terminology). Like in probability theory the decreasing *distribution function* $G_{\mu,f} : L \to M$ of f is defined as

$$G_{\mu,f}(x) := \mu(\{\omega \in \Omega \mid f(\omega) \geq x\}) .$$

We relate the scale L of the functions f to the scale M of the ordinal measure μ by a *commensurability function*, an order preserving application $\ell : L \to M$ with $\ell(\mathbb{O}) = \mathbb{O}$, $\ell(\mathbb{I}) = \mathbb{I}$. The *lower* and *upper Fan-Sugeno operators* $L^\Omega \to L$ are now defined by

$$\underline{S}_{\mu,\ell}(f) := \ell_{\text{invs}}(G_{\mu,f} * \ell) , \quad \overline{S}_{\mu,\ell}(f) := \ell^{\text{invs}}(G_{\mu,f} \mathbin{\overline{*}} \ell) .$$

Here we use the *(inner) products*

$$\varphi * \psi \ := \bigvee_{x \in L} \varphi(x) \wedge \psi(x) \ \in M , \quad \varphi, \psi \in M^L ,$$

$$\varphi \mathbin{\overline{*}} \psi \ := \bigwedge_{x \in L} \varphi(x) \vee \psi(x)$$

and the lower and upper pseudo inverse functions of ℓ,

$$\ell_{\text{invs}}(y) := \bigwedge_{\ell(x) \geq y} x , \quad \ell^{\text{invs}}(y) := \bigvee_{\ell(x) \leq y} x , \quad y \in M .$$

We get

$$\underline{S}_{\mu,\ell} \leq \overline{S}_{\mu,\ell}$$

and equality $\underline{S}_{\mu,\ell} = \overline{S}_{\mu,\ell} =: S_{\mu,\ell}$ holds in the classical case $M = [0, 1] \subset \mathbb{R}$.

In our definition it is essential that the inner product is performed with a decreasing and an increasing function, $G_{\mu,f}$ and ℓ, respectively. The inverse of ℓ is needed only to bring back this result to the right scale L, used for the functions.

Example 3.1 *Let $L = M = [0,1]$ and $\ell(x) = x$ the identity mapping on M, then $S_{\mu,\ell}(f)$ is the Sugeno integral of f w.r.t. the fuzzy measure (or capacity) μ ([5], cf. [1]).*

Example 3.2 *If the commensurability function is essentially constant with value $p \in M$, $\ell(x) := p$ for $x \neq \mathbb{O}, \mathbb{I}$, then $\underline{S}_{\mu,\ell}(f)$ is the lower p-quantile of f and $\overline{S}_{\mu,\ell}(f)$ the upper one.*

The last example can elucidate the role of the commensurability function ℓ. The scale for the values of the function to be aggregated is reduced by ℓ to a scale of three values, $\mathbb{O}$, p and $\mathbb{I}$.

Example 3.2 also makes clear why both pseudo inverse functions of ℓ, the lower and upper one, are needed in our definition of the Fan-Sugeno operator. Already in the cardinal case a quantile is not, in general, a single number but an interval. The same would happen if the usual Sugeno integral would be defined with a finite descretisation of the scale $[0,1] \subset \mathbb{R}$.

Here are the elementary properties of the Fan-Sugeno operators (omitting the distinction between upper and lower operators).

(i) $S_{\mu,\ell}(\mathbb{I}_A) = \mu(A)$ for $A \subset \Omega$.

(ii) $f \leq g$ implies $S_{\mu,\ell}(f) \leq S_{\mu,\ell}(g)$.

(iii) $S_{\mu,\ell}(f \vee g) \geq S_{\mu,\ell}(f) \vee S_{\mu,\ell}(g)$ and equality holds for all f,g iff μ is a maxitive measure.

(iv) $S_{\mu,\ell}(f \wedge g) \leq S_{\mu,\ell}(f) \wedge S_{\mu,\ell}(g)$ and equality holds for all f,g iff μ is a minitive measure.

(v) $S_{\mu,\ell}(f \vee g) = S_{\mu,\ell}(f) \vee S_{\mu,\ell}(g)$ if f, g are comonotonic.

(vi) If M is a reflection lattice, then $S_{\bar{\mu},\ell}(\bar{f}) = \overline{S_{\mu,\ell}(f)}$.

Here, comontonicity is defined like in the cardinal case. Clearly the upper level sets $\{f \geq x\} := \{\omega \in \Omega \mid f(\omega) \geq x\}$ of f for the levels $x \in L$ form a chain $\mathcal{K}_f \subset 2^\Omega$. A set $F \subset L^\Omega$ of functions is called *comonotonic* if $\bigcup_{f \in F} \mathcal{K}_f$ forms again a chain.

4 Ky Fan distance and (a-)symmetric extension of the operators

In order to model the situation described in paragraph 4 of Section 1, the range of the function f has to be a linear reflection lattice R, $\mathbb{O}$ being the reference point. Like in the cardinal case a function $f : \Omega \to R = L_+ \cup L_-$ decomposes in positive and negative parts, $f = f^+ \bigtriangledown (-f^-)$. With a commensurability function $\ell : L_+ \to M$ the Fan-Sugeno operator is defined for functions $f \geq \mathbb{O}$.

In the spirit of Ky Fan [2] we define the (μ, ℓ)-*distance*

$$\mathrm{dist}_{\mu,\ell}(f,g) := S_{\mu,\ell}(|f \bigtriangledown (-g)|)\,, \quad f, g \in R^\Omega\,.$$

The usual properties of a distance hold with a restriction for the triangle inequality,

$$\begin{aligned} \mathrm{dist}_{\mu,\ell}(f,g) &\geq 0 \quad \text{and} \quad \mathrm{dist}_{\mu,\ell}(f,g) = 0 \ \text{ iff } \ f = g\,, \\ \mathrm{dist}_{\mu,\ell}(f,g) &= \mathrm{dist}_{\mu,\ell}(g,f)\,, \\ \mathrm{dist}_{\mu,\ell}(f,h) &\leq \mathrm{dist}_{\mu,\ell}(f,g) \vee \mathrm{dist}_{\mu,\ell}(g,h) \\ & \text{ if } \mu \text{ is maxitive}\,. \end{aligned}$$

For arbitrary R-valued functions there are, like for the Choquet integral, two possibilities to extend the operator. First we define the *symmetric extension* of the Fan-Sugeno operator (omitting again the distinction between upper and lower operators)

$$\check{S}_{\mu,\ell}(f) := S_{\mu,\ell}(f^+) \bigtriangledown (-S_{\mu,\ell}(f^-))\,.$$

The following symmetry property holds,

$$\check{S}_{\mu,\ell}(-f) = -\check{S}_{\mu,\ell}(f)\,.$$

The *asymmetric extension* needs the conjugate $\bar{\mu}$ of the ordinal measure μ, hence M has to be a reflection lattice, too,

$$\hat{S}_{\mu,\ell}(f) := S_{\mu,\ell}(f^+) \bigtriangledown (-S_{\bar{\mu},\ell}(f^-))\,.$$

Now we get the asymmetry property

$$\hat{S}_{\mu,\ell}(-f) = -\hat{S}_{\bar{\mu},\ell}(f)\,.$$

If μ is self conjugate then both extensions coincide,

$$\hat{S}_{\mu,\ell} = \check{S}_{\mu,\ell} \quad \text{if } \mu = \bar{\mu}\,.$$

5 Conclusion

The present paper is a first approach leaving many questions open. Which subclass of ordinal measures corresponds to the probability measures (cf. end of Section 2)? Related with this question, how one should define imprecision in the ordinal context? From decision behaviour can there be found empirical evidence for different shapes of the commensurability function ℓ in a purely ordinal setting? We pointed out that the symmetric extension of the Fan-Sugeno operator has a decision behavioural interpretation. Are there also applications for the asymmetric extension of the Fan-Sugeno operators?

Like in the cardinal case one could introduce ordinal *signed* measures. The scale for the ordinal measure could then be the same reflection lattice R as we had in Section

4 for the functions. Which should be the corresponding aggregation operator? Could this be a useful tool for modeling human behaviour? In this context our ordinal difference $x \bigtriangledown (-y)$ would be of importance.

Are there other applications of the model presented here? Is the ordinal Ky Fan distance a useful issue?

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