Coherent Lower Previsions as Exact Functionals and their (Sigma-)Core

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Abstract

Coherent lower previsions and exact cooperative games are mathematically essentially the same. We investigate in this paper the smallest class containing these functionals resp. games. This class will be denoted to consist of exact functionals which coincide with coherent lower previsions up to normalization. We investigate the exact functionals from a functional analytic point of view, i.e. we characterize this class by a norm, present a Hahn-Banach type theorem, a powerful construction method and adopt the concept of the core resp. σ-core from cooperative game theory.

Keywords. exact functionals, coherent lower previsions, exact cooperative games, core.

1 Introduction

Coherent lower previsions have turned out to be "sufficiently general" to model various kinds of uncertainty, partial information and ignorance. Mathematically, there are considerable similarities between this class and exact cooperative games but in contrast to cooperative game theory, lower previsions are normalized, i.e. $\Gamma(1) = 1$ holds for a lower prevision Γ. Therefore in this paper we enlarge the class of coherent lower previsions by dropping this normalization condition to cover exact games and denote this class to contain exact functionals taking over the language of cooperative game theory. Analogously, we generalize previsions avoiding sure loss to exactifiable functionals which cover balanced games.

The class of exact functionals contains amongst other things the classes of

- coherent lower previsions [13]
- exact cooperative games [11]
- coherent risk measures [1], [3]
- maxmin expected utility functionals [6].

The class of exactifiable functionals generalizes the classes of

- previsions avoiding sure loss [13]
- balanced cooperative games [11].

Since normalized exact functionals will be proved to be coherent lower previsions we do not generalize the concept of prevision. We show that not only structural assumptions on the domain (e.g. in cooperative game theory by supposing the domain to be an algebra) but also the normalization condition (e.g. in the theory of imprecise previsions) are mathematically not relevant for the results obtained.

This article is organized as follows. In Section 2 exact and exactifiable functionals are defined and each characterized by a norm. We also formulate an extension theorem of Hahn-Banach type. Relations to some theories mentioned above, especially to that of coherent lower previsions, are established in Section 3. Exact functionals will also be proved to be interpretable as a generalization of superadditive Choquet integrals where comonotonic additivity is relaxed to constant additivity. In Section 4 we provide a powerful construction method for exact functionals.
In the central Section 5 we introduce the core and σ-core for functionals, a concept mainly known from cooperative game theory. The core of a functional allows us to analyze exact and exactifiable functionals with methods from functional analysis as well as measure and integration theory.

2 Definitions and basic properties

Throughout the paper, Ω denotes a non-empty set, 2^Ω the power set of Ω, $\mathcal{A}$ an algebra in 2^Ω, $B(\mathcal{A})$ the Banach space spanned by the characteristic functions $\{1_A | A \in \mathcal{A}\}$ with the sup norm $\|\cdot\|_\infty$ and M a non-empty subset of $B(2^\Omega)$. A real functional Γ on a linear space $S \subset B(2^\Omega)$ is called superlinear if it is superadditive and positively homogeneous. It is called constant additive or translation invariant, if $\Gamma(f+c) = \Gamma(f) + \Gamma(c)$ for all $c, f \in S$, c constant.

Definition 2.1 *Let $\Gamma : M \to \mathbb{R}$ be the restriction of a monotone, constant additive, superlinear functional $\Gamma' : B(2^\Omega) \to \mathbb{R}$. Then Γ is called **exact** and Γ' an exact extension of Γ.*
*Let $\Gamma : M \to \mathbb{R}$ be a functional that can be dominated by an exact functional $\Gamma' : B(2^\Omega) \to \mathbb{R}$ on M. Then Γ is called **exactifiable** and Γ' an exactification of Γ.*

Obviously, exact functionals in particular are exactifiable functionals. It is easy to prove that monotone linear functionals as well as the infimum are exact. Due to the one-to-one correspondence between sets and their characteristic functions we identify set functions with functionals on characteristic functions. By this means, we will call a set function exact if the corresponding functional on the set of characteristic functions is exact.

We now define two norm-type functions (norms for short) closely related to the class of exact and exactifiable functionals. The first one is a generalization of a norm introduced by Schmeidler for cooperative games in [11]. Each norm will be proved in Theorem 2.3 to characterize either class of functionals introduced in Definition 2.1. For an arbitrary real functional $\Gamma : M \to \mathbb{R}$ we define

$$|\Gamma| := \sup\left\{ \sum_{i=1}^n \lambda_i \Gamma(f_i) \;\middle|\; \sum_{i=1}^n \lambda_i f_i \le 1, \; n \in \mathbb{N}, \lambda_i \ge 0, f_i \in M \right\} \quad (1)$$

$$\|\Gamma\| := \inf\left\{ c \in \mathbb{R}_+ \;\middle|\; \forall\, n \in \mathbb{N}, \lambda_i \ge 0, \; \lambda_0 \in \mathbb{R}, f, f_i \in M : \; f \ge \sum_{i=1}^n \lambda_i f_i + \lambda_0 \; \Rightarrow \; \Gamma(f) \ge \sum_{i=1}^n \lambda_i \Gamma(f_i) + \lambda_0 c \right\} \quad (2)$$

Both norms are extended real valued, nonnegative, zero if $\Gamma = 0$, positively homogeneous and subadditive, i.e. the classes of functionals with finite $|\cdot|$- resp. $\|\cdot\|$-norm are convex cones. Additionally, $\|\Gamma\| = 0$ implies $\Gamma = 0$ and the $|\cdot|$-norm is monotone, i.e. $\Gamma_1 \le \Gamma_2$ implies $|\Gamma_1| \le |\Gamma_2|$. For a real functional Γ and an extension $\Gamma' : B(2^\Omega) \to \mathbb{R}$ of Γ we obviously have $|\Gamma| \le |\Gamma'|$ resp. $\|\Gamma\| \le \|\Gamma'\|$ and $|\Gamma| \le \|\Gamma\|$. By an indirect proof one can show that $\|\Gamma\|$ is contained in the set on the right of Equation (2) and this implies that Equation (4) holds when replacing Γ_* by Γ.

We now show that for an exact functional in most interesting cases both norms are equivalent and coincide with the operator norm if M is a linear space. This result holds on condition that $1 \in M$ which is met e.g. in game theory (Ω corresponds to 1) or in (non-linear) functional analysis when functionals are defined on a linear space $B(\mathcal{A})$.

Proposition 2.2 *For an exact functional $\Gamma : M \to \mathbb{R}$ with $1 \in M$ we have*

$$\|\Gamma\| = |\Gamma| = \Gamma(1) = \sup_{\substack{f \in M \\ \|f\|_\infty \ne 0}} \frac{|\Gamma(f)|}{\|f\|_\infty}. \quad (3)$$

Proof. By definition of the norms, $\|\Gamma\| \ge |\Gamma| \ge \Gamma(1)$. Let $\Gamma' : B(2^\Omega) \to \mathbb{R}$ be an exact extension of Γ. Then for all $\lambda_i \ge 0, \lambda_0 \in \mathbb{R}, f, f_i \in M$, $f \ge \sum_{i=1}^n \lambda_i f_i + \lambda_0$

$$\Gamma(f) \ge \Gamma'(\sum_{i=1}^n \lambda_i f_i + \lambda_0) \ge \sum_{i=1}^n \lambda_i \Gamma(f_i) + \lambda_0 \Gamma(1),$$

i.e. $\|\Gamma\| \le \Gamma(1) < \infty$, hence $\|\Gamma\| = |\Gamma| = \Gamma(1)$. Obviously, $\Gamma(1) \le \sup_{\|f\|_\infty \ne 0} \frac{|\Gamma(f)|}{\|f\|_\infty}$. For every $f \in M$ with $\|f\|_\infty \ne 0$, by exactness of Γ

$$\frac{|\Gamma(f)|}{\|f\|_\infty} = \left| \Gamma' \left(\frac{f}{\|f\|_\infty} \right) \right| \le \Gamma(1).$$

This proves the last equation in (3). □

The following theorem characterizes, by means of our two norms, the class of exact and exactifiable functionals and provides a method for extending exact functionals (cf. [9]).

Theorem 2.3 *Let Γ be a real functional on a non-empty set M.*

(a) Equivalent are
- *Γ is exact.*
- *$\|\Gamma\| < \infty$.*
- *The functional $\Gamma_* : B(2^\Omega) \to \mathbb{R}$ defined by*

$$\Gamma_*(f) := \sup\left\{ \sum_{i=1}^n \lambda_i \Gamma(f_i) + \lambda_0 \|\Gamma\| \;\middle|\; \sum_{i=1}^n \lambda_i f_i + \lambda_0 \le f, \; n \in \mathbb{N}, \; \lambda_0 \in \mathbb{R}, \; \lambda_i \ge 0 \text{ and } f_i \in M \right\} \quad (4)$$

is an exact extension of Γ with $\|\Gamma_*\| = \|\Gamma\|$.

(b) Equivalent are

- Γ *is exactifiable.*
- $|\Gamma| < \infty$.
- *The functional* $\Gamma_\bullet : B(2^\Omega) \to \mathbb{R}$ *defined by*

$$\Gamma_\bullet(f) := \sup \left\{ \sum_{i=1}^{n} \lambda_i \Gamma(f_i) + \lambda_0 |\Gamma| \;\middle|\; \sum_{i=1}^{n} \lambda_i f_i + \lambda_0 \le f,\, n \in \mathbb{N},\; \lambda_0 \in \mathbb{R},\, \lambda_i \ge 0 \text{ and } f_i \in M \right\} \quad (5)$$

is an exactification of Γ *with* $|\Gamma_\bullet| = |\Gamma|$.

The first equivalence relation in Theorem 2.3 (a) shows that the definition of exactness does not rely on structural assumption on the domain (which seems to be the case when defining exactness via functionals on a linear space) but only on the relations between the values of the functionals (when calculating the $\|\cdot\|$-norm).

Theorem 2.3 (a) is of Hahn-Bach type (and can be proved in an analogous way) because it says that every real functional with finite $\|\cdot\|$-norm can be extended maintaining the norm.

The following corollary summarizes important results implied by Proposition 2.2 and Theorem 2.3 and shows that the definitions of the norms $|\cdot|$ and $\|\cdot\|$ as well as those of the functionals $\Gamma_\bullet$ and Γ_* are quite natural.

Corollary 2.4 *Let* Γ *be a real functional on* M. *Then*

(a) $\Gamma_* = \inf \left\{ \Gamma' : B(2^\Omega) \to \mathbb{R} \;\middle|\; \Gamma' \text{ exact},\; \Gamma'|M \ge \Gamma,\, \|\Gamma'\| = \|\Gamma\| \right\}$.

(b) $\|\Gamma\| = \inf \left\{ \Gamma'(1) \;\middle|\; \Gamma' : B(2^\Omega) \to \mathbb{R} \text{ exact},\; \Gamma'|M = \Gamma \right\}$.

(c) $\Gamma_\bullet = \inf \left\{ \Gamma' : B(2^\Omega) \to \mathbb{R} \;\middle|\; \Gamma' \text{ exact},\; \Gamma'|M \ge \Gamma,\, |\Gamma'| = |\Gamma| \right\}$.

(d) $|\Gamma| = \inf \left\{ \Gamma'(1) \;\middle|\; \Gamma' : B(2^\Omega) \to \mathbb{R} \text{ exact},\; \Gamma'|M \ge \Gamma \right\}$.

At this point we have to stress that for an exact set function μ neither the exact extension μ_* nor the exactification $\mu_\bullet$ (each restricted to 2^Ω) coincides with the inner set function or the inner measure known in (non-additive) measure theory. But in all cases we have a sort of "inner extensions" of a given set function resp. functional.

The functionals Γ_* resp. $\Gamma_\bullet$ are of great importance in the following analysis of exact resp. exactifiable functionals Γ because they are closely related to Γ (cf. Theorem 2.3 and Corollary 2.4) and have a domain with more structure than Γ. This allows us to demonstrate some properties of the two classes of functionals examined in the present paper by investigating Γ_* resp. $\Gamma_\bullet$ particularly using functional analytical methods. Hence these functionals will be denoted as follows.

Definition 2.5 *For an exact functional* $\Gamma : M \to \mathbb{R}$, Γ_* *is called the* ***natural extension*** *of* Γ.
For an exactifiable functional $\Gamma : M \to \mathbb{R}$, $\Gamma_\bullet$ *is called the* ***natural exactification*** *of* Γ.

The following example shows that if there is not any positive constant contained in the domain M it can happen that $|\Gamma| \neq \|\Gamma\|$ even for an exact functional Γ defined on M. Therefore the exact extension not necessarily coincides with the exactification.

Example 2.6 *Let* $\Omega = \{1, 2\}$, $M = \{-1_{\{1\}}\}$ *and* $\Gamma : M \to \mathbb{R}$ *be defined by* $\Gamma(-1_{\{1\}}) := -0.5$. *Then* $|\Gamma| = 0$ *and* $\|\Gamma\| = 0.5$, *i.e.* Γ *is exact by Theorem 2.3. The natural extension then is* $\Gamma_*(f) = 0.5 \cdot \inf f$ *for all* $f \in B(2^\Omega)$.

The natural extension can be used for proving boundedness of the values and Lipschitz-continuity of exact functionals (cf. [9]), i.e. for all $f, g \in M$ we have

$$\|\Gamma\| \inf f \le \Gamma(f) \le \|\Gamma\| \sup f, \quad (6)$$

$$|\Gamma(f) - \Gamma(g)| \le \|\Gamma\| \cdot \|f - g\|_\infty. \quad (7)$$

3 Relations to other theories

In this section we prove the classes of coherent lower previsions, exact cooperative games and Choquet integrals w.r.t. supermodular set functions to be contained in the class of exact functionals. Analogous results for exactifiable functionals are also stated.

First the relation of our classes of functionals to the **theory of imprecise probabilities** is investigated. Walley examined in [13] mainly two classes of functionals on an arbitrary non-empty subset of $B(2^\Omega)$ to model rational behaviour in decision situations. These are the lower previsions avoiding sure loss and the coherent lower previsions.

A real functional Γ on a non-empty subset M of $B(2^\Omega)$ is called a *lower prevision avoiding sure loss* (cf. [13, Definition 2.4.1 and Lemma 2.4.4]) if for all $n \in \mathbb{N}$, $\lambda_i \ge 0, f_i \in M$

$$\sup \sum_{i=1}^{n} \lambda_i f_i \ge \sum_{i=1}^{n} \lambda_i \Gamma(f_i). \quad (8)$$

A real functional Γ on a non-empty subset M of $B(2^\Omega)$ is called a *coherent lower prevision* (cf. [13, Definition 2.5.1 and Lemma 2.5.4]) if for all $n \in \mathbb{N}$, $\lambda_0, \lambda_i \geq 0$, $f_0, f_i \in M$

$$\sup\left(\sum_{i=1}^n \lambda_i f_i - \lambda_0 f_0\right) \geq \sum_{i=1}^n \lambda_i \Gamma(f_i) - \lambda_0 \Gamma(f_0). \quad (9)$$

Proposition 3.1 *Let Γ be a real functional on a non-empty subset M of $B(2^\Omega)$.*

(a) Γ is a lower prevision avoiding sure loss iff it is exactifiable and there exists an exactification Γ' of Γ with $\|\Gamma'\| = 1$.

(b) Γ is a coherent lower prevision iff it is exact and there exists an exact extension of Γ with $\|\Gamma'\| = 1$.

Proof. First, observe that the natural extension $\underline{E}$ for a real functional $\Gamma : M \to \mathbb{R}$ defined by Walley in [13] coincides with the natural extension introduced in Definition 2.1 iff $\|\underline{E}\| = 1$ (cf. [13, Definition 3.1.1 and Lemma 3.1.3 (b)]). Since it is well-known that Γ is a prevision avoiding sure loss iff $\underline{E}$ is real valued (cf. [13, p. 123]) we obtain (a) using Theorem 2.3 (b).
Another well-known result is that Γ is a coherent lower prevision iff $\underline{E}$ extends Γ (cf. [13, Proposition 3.1.2 and Lemma 3.1.3]). Analogously to (a), the assertion holds using Theorem 2.3 (a). □

Corollary 3.2 *Let Γ be a real functional on a non-empty subset M of $B(2^\Omega)$. Γ is exact iff there exists a $\lambda \geq 0$ and a coherent lower prevision $\Gamma' : M \to \mathbb{R}$ such that $\Gamma = \lambda\Gamma'$.*

To emphasize the difference between coherent lower previsions and exact functionals we consider again the situation given in Example 2.6. In this case Γ obviously is a coherent lower prevision. Since $\|\Gamma\| = \|\Gamma_*\| = 0.5$ but $\|\underline{E}\| = 1$ we obtain that $\underline{E}$ is not a natural extension when considering a coherent lower prevision as an element of the class of exact functionals. $\underline{E}$ coincides with Γ_* if there exist a positive constant in the domain, i.e. in most interesting cases.

We now demonstrate the relation between exact resp. exactifiable functionals and **cooperative game theory**. A cooperative game v is a bounded, nonnegative, real valued set function on an algebra $\mathcal{A}$ in Ω, mapping the empty set to 0. Two classes of cooperative games are of special interest here, the balanced games and the exact games.

A cooperative game v is called a *balanced game* if for all $n \in \mathbb{N}$, $\lambda_i \geq 0, A_i \in \mathcal{A}$

$$\sum_{i=1}^n \lambda_i 1_{A_i} \leq 1_\Omega \;\Rightarrow\; \sum_{i=1}^n \lambda_i v(A_i) \leq v(\Omega). \quad (10)$$

Obviously, a game v is balanced iff $v(\Omega) = |v|$. Equation (10) implies $|v| < \infty$ since $v(\Omega) < \infty$. Therefore balanced games are a subclass of the exactifiable functionals.

A cooperative game v is called an *exact game* if for all $n \in \mathbb{N}$, λ_0, $\lambda_i \geq 0$, $A, A_i \in \mathcal{A}$

$$\sum_{i=1}^n \lambda_i 1_{A_i} - \lambda_0 \leq 1_A$$
$$\Rightarrow \sum_{i=1}^n \lambda_i v(A_i) - \lambda_0 |v| \leq v(A). \quad (11)$$

Obviously, exact cooperative games are balanced. Thus $|v| < \infty$ and by definition of $|\cdot|$ the above implication remains true when admitting negative λ_0. Applying Proposition 2.2 a game v is exact iff it is exact in the sense of Definition 2.1. Therefore exact games are a subclass of the exact functionals.

Comparing the relations between exact functionals, coherent lower previsions and exact cooperative games we obtain that the intersection of the classes of exact games and coherent lower previsions consists of all exact games satisfying $v(\Omega) = 1$. Hence both classes are mathematically essentially equal. The differences consist only in the restriction on the domain (cf. exact games) and the restriction to normalized functionals (cf. coherent lower previsions).

We now can give another characterization of exact functionals in terms of coherent lower previsions and exact cooperative games.

Proposition 3.3 *The class of exact functionals is the smallest convex cone containing coherent lower previsions and exact games.*

Proof. Corollary 3.2 together with positive homogeneity and subadditivity of $\|\cdot\|$ imply that the convex cone induced by coherent lower previsions consists of all exact functionals. Since we proved exact cooperative games to be exact in the sense of Definition 2.1 we obtain the assertion. □

Finally, the relation between exact functionals and **non-additive measure and integration theory** is outlined. The next proposition being essentially a reformulation of one by Schmeidler (cf. [12, Theorem and Proposition 3]) shows that the main difference between exact functionals and Choquet integrals is the additivity property and that superadditive Choquet integrals build up a subclass of exact functionals.

Proposition 3.4 *An exact functional Γ on $B(\mathcal{A})$ is representable as a Choquet integral iff it is comonotonic additive. A Choquet integral w.r.t. a set function μ is exact iff μ is supermodular.*

As an easy consequence of Proposition 3.4 finite supermodular set functions turn out to be exact. Furthermore the functionals $\inf : B(2^\Omega) \to \overline{\mathbb{R}}$ as well as $\liminf : \mathbb{R}^\mathbb{N} \to \overline{\mathbb{R}}$ are exact as Choquet integrals w.r.t. supermodular set functions.

Supermodular set functions are important exact set functions because their natural extension can be proved to coincide with their corresponding Choquet integral (cf. [9]).

Another class of functionals closely related to exact functionals are the coherent risk measures. In [8] it has been proved that coherent risk measures are the negatives of coherent lower previsions, i.e. $\Gamma : B(\mathcal{A}) \to \mathbb{R}$ is exact with $\|\Gamma\| = 1$ iff $-\Gamma$ is a coherent risk measure.

4 Construction of exact functionals

We focus now on the "space" of exact functionals and present in Proposition 4.1 a powerful construction method. Almost all operations on a set of exact functionals are only valid if the latter have the same $\|\cdot\|$-norm. Hence exact functionals with this property will be called equinormed.

Proposition 4.1 *Let $\{\Gamma_i\}_{i\in I}$ be a non-empty indexed set of equinormed exact functionals on $M \subset B(2^\Omega)$ and $\Gamma : B(2^I) \to \mathbb{R}$ be exact. Then the functional*

$$M \to \mathbb{R}, \quad f \mapsto \Gamma(i \mapsto \Gamma_i(f)) \tag{12}$$

is exact. Additionally, if Γ is linear then the condition of equinormation can be dropped.

Proof. The function defined in (12) is well-defined because the function $i \mapsto \Gamma_i(f)$ is bounded for every $f \in M$. Exactness is easily verified for the functional $B(2^\Omega) \to \mathbb{R}, f \mapsto \Gamma(i \mapsto (\Gamma_i)_*(f))$, and therefore for the function defined in (12). □

Since exact cooperative games are a subclass of exact functionals which are only characterized by restrictions on the domain, the constructed functional in Proposition 4.1 is an exact game if all Γ_i are.

For exact Choquet integrals Γ_i and Γ the resulting functional in (12) is not a Choquet integral in general since, as we will see in the following section, every exact functional has a representation of Choquet integrals of the form of (12) (cf. Corollary 5.4 (b)).

The subsequent corollary summarizes some theorems and proposition proved individually until now e.g. for coherent lower previsions (cf. [13, 2.6.3 - 2.6.7]).

Corollary 4.2 *Let $\{\Gamma_i\}_{i\in I}$ be a non-empty indexed set of equinormed exact functionals on $M \subset B(2^\Omega)$. Then*

(a) Convex combinations of some Γ_i are exact.

(b) The lower envelope $\inf_{i\in I} \Gamma_i$ is exact.

If $I = \mathbb{N}$ then

(c) The limit inferior $\liminf_{i\to\infty} \Gamma_i$ is exact.

(d) If the sequence (Γ_i) is pointwise convergent then the limit $\lim_{i\to\infty} \Gamma_i$ is exact.

(e) If the sequence (Γ_i) is increasing then the supremum $\sup_{i\to\infty} \Gamma_i$ is exact.

5 The core of functionals

In this section we adopt the core concept from cooperative game theory to our theory of functionals on arbitrary subsets of $B(2^\Omega)$. Similar concepts are known in all theories mentioned in the introduction. The core allows us to analyze exact functionals with methods from functional analysis as well as measure and integration theory.

Throughout the remaining part of the paper we identify $B^*(\mathcal{A})$ with the space of bounded additive set functions on $\mathcal{A}$, $ba(\mathcal{A})$, due to the existence of a natural isometric isomorphism between these spaces (cf. [5, Theorem IV.5.1]), i.e. linear functionals are sometimes interpreted as additive set functions and vice versa. Another important space used in this section is the space of bounded countably additive set functions on $\mathcal{A}$, $ca(\mathcal{A})$. For a set function $\mu : \mathcal{A} \to \mathbb{R}_+$ denote $I_\mu f$ the Choquet integral $\int f d\mu$ w.r.t. μ for all $f \in B(\mathcal{A})$.

Definition 5.1 *Let $\Gamma : M \to \mathbb{R}$ be a functional and $\mathcal{A}$ an algebra satisfying $M \subset B(\mathcal{A})$. Then*

$$\mathcal{C}_\mathcal{A}(\Gamma) := \Big\{\Lambda \in B^*(\mathcal{A}) \,\Big|\, \Lambda|M \geq \Gamma, \; \Lambda \textit{ monotone}, |\Lambda| = |\Gamma|\Big\} \tag{13}$$

*is called the **$\mathcal{A}$-core** of Γ. If no confusion about the algebra used is possible we will call the $\mathcal{A}$-core just core and denote it by $\mathcal{C}(\Gamma)$.*

Due to our identification of $B^*(\mathcal{A})$ with $ba(\mathcal{A})$ we have

$$\mathcal{C}_\mathcal{A}(\Gamma) = \Big\{\lambda \in ba(\mathcal{A}) \,\Big|\, I_\lambda|M \geq \Gamma, \; \lambda \geq 0, \lambda(\Omega) = |\Gamma|\Big\} \tag{14}$$

since additivity and non-negativity of λ imply exactness of λ, hence $\lambda(\Omega) = |\lambda|$ and $|I_\lambda| = |\lambda|$.

For a cooperative game v on an algebra $\mathcal{A}$ our definition of the core coincides with the one in cooperative game theory. The core of a cooperative game $v : \mathcal{A} \to \mathbb{R}_+$ is defined by

$$core(v) := \{\lambda \in ba(\mathcal{A}) \mid \lambda \geq v, \lambda(\Omega) = v(\Omega)\}. \quad (15)$$

The definitions (14) and (15) of the core coincide. This is obvious if $v(\Omega) = |v|$, i.e. v is balanced. If v is not balanced then on the one hand $\lambda(\Omega) = |\lambda| \geq |v| > v(\Omega)$ for all additive λ dominating v, i.e. $core(v) = \emptyset$, and on the other hand $|v| = \|v\| = \infty$ because v is not exact, i.e. $\mathcal{C}_{\mathcal{A}}(v) = \emptyset$.

In Corollary 5.5 we will obtain that the core of a functional is non-empty iff it is exactifiable. This implies the well-known result that a cooperative game has a non-empty core iff it is balanced.

There is concept similar to the 2^Ω-core in the theory of imprecise previsions. For a real functional Γ on M Walley uses in [13] a set $\mathcal{M}$ representable as

$$\{\Lambda \in B(2^\Omega) \mid \Lambda|M \geq \Gamma, \Lambda \text{ monotone}, |\Lambda| = 1\}. \quad (16)$$

Analogously to the definition of the natural extension, the norm of $\mathcal{M}$ is "arbitrarily" fixed to 1 which shows that implicitly $1 \in \mathcal{M}$ and $\Gamma(1) = 1$ is assumed due to the probabilistic point of view.

We now show that on the one hand the core of a functional and its exactification and on the other hand the different cores of a functional are essentially identical. The reason why we therefore do not restrict the definition of the core to the 2^Ω-core will be answered after the definition of the σ-core.

Proposition 5.2 *Let Γ be a real functional on M and $\mathcal{A}_1, \mathcal{A}_2$ two algebras satisfying $M \subset B(\mathcal{A}_1) \subset B(\mathcal{A}_2)$. Then*

(a) $\mathcal{C}_{\mathcal{A}_1}(\Gamma) = \{\Lambda|B(\mathcal{A}_1) \mid \Lambda \in \mathcal{C}_{2^\Omega}(\Gamma_\bullet)\}$

(b) $\mathcal{C}_{\mathcal{A}_1}(\Gamma) = \{\Lambda|B(\mathcal{A}_1) \mid \Lambda \in \mathcal{C}_{\mathcal{A}_2}(\Gamma)\}$.

Proof. (a) To prove the "$\subset$"-part, we observe that every $\Lambda \in \mathcal{C}_{\mathcal{A}_1}(\Gamma)$ is exact with $|\Lambda| = 1$, hence

$$\Lambda(f) = \sum_{i=1}^n \lambda_i \Lambda(f_i) + \lambda_0|\Lambda| \geq \sum_{i=1}^n \lambda_i \Gamma(f_i) + \lambda_0|\Gamma|$$

for every $f \geq \sum_{i=1}^n \lambda_i f_i + \lambda_0$ with $n \in \mathbb{N}$, $\lambda_0 \in \mathbb{R}$, $\lambda_i \geq 0$ and $f, f_i \in B(\mathcal{A}_1)$. Thus $\Lambda \geq \Gamma_\bullet|B(\mathcal{A}_1)$. Using the Hahn-Banach Theorem, Λ can be extended to an element of $\mathcal{C}_{\mathcal{A}_1}(\Gamma_\bullet)$. The reverse implication is almost trivial.
(b) The assertion follows from (a) by replacing $\mathcal{C}_{\mathcal{A}_2}$ in (b) by $\{\Lambda|B(\mathcal{A}_2) \mid \Lambda \in \mathcal{C}_{2^\Omega}(\Gamma_\bullet)\}$. □

The subsequent theorem is essential to adopt results from (σ-)additive measure and integration theory to the theory of exact functionals like for example convergence theorems (cf. Theorem 5.7). Special versions have been proved e.g. by Walley [13, Theorem 3.6.1] and by Huber [7, Proposition 10.2.1]. A more general version has already been proved by Bonsall in 1954 (cf. [2, Lemma 6 and Theorem 11]).

Theorem 5.3 *There is a one-to-one correspondence between exact functionals on $B(\mathcal{A})$ and non-empty convex weak*-compact sets $\mathcal{C} \subset B^*(\mathcal{A})$ of equinormed functionals, determined by the identities*

$$\Gamma(f) = \min_{\Lambda \in \mathcal{C}} \Lambda(f) \quad \textit{resp.} \quad \mathcal{C} = \mathcal{C}(\Gamma). \quad (17)$$

Proof. We only outline the central part of the proof, non-emptyness of the core and $\Gamma = \min_{\Lambda \in \mathcal{C}(\Gamma)} \Lambda$. Let $f_0 \in B(\mathcal{A})$ be arbitrary and the linear functional Λ' on the linear space spanned by the functions 1 and f_0, $sp(1, f_0)$, be defined by $\Lambda'(1) := \Gamma(1)$ and $\Lambda'(f_0) := \Gamma(f_0)$. Then $\Lambda' \geq \Gamma|sp(1, f_0)$ because of $\Lambda'(-f_0) = -\Lambda'(f_0) = -\Gamma(f_0) \geq \Gamma(-f_0)$. Using the Hahn-Banach Theorem we can extend Λ' to $B(\mathcal{A})$ such that this extension is contained in $\mathcal{C}(\Gamma)$. □

The following corollaries establish the relationship of exact (exactifiable) functionals with the natural extension (natural exactification) w.r.t. the core. Analogous results appear in the theory of imprecise previsions (cf. [13, Theorem 3.3.3 and 3.4.1]) and in game theory (cf. [11, Corollary 2.4 and 2.6]).

Corollary 5.4 *Let $\Gamma : M \to \mathbb{R}$ be a functional.*

(a) If Γ is exactifiable then $\Gamma_\bullet = \min_{\Lambda \in \mathcal{C}_{2^\Omega}(\Gamma)} \Lambda$.

(b) If Γ is exact then

$$\Gamma(f) = \min\{\Lambda(f) \mid \Lambda \in B(2^\Omega),\ \Lambda|M \geq \Gamma,\ \Lambda \textit{ monotone},\ |\Lambda| = \|\Gamma\|\}.$$

(c) If Γ is a coherent lower prevision then

$$\underline{E}(f) = \min\{\Lambda(f) \mid \Lambda \in B(2^\Omega),\ \Lambda|M \geq \Gamma,\ \Lambda \textit{ monotone},\ |\Lambda| = 1\}.$$

As another easy consequence of Theorem 5.3 we obtain an equivalent condition for non-emptyness of the core which is well-known in game theory.

Corollary 5.5 *The core of a functional $\Gamma : M \to \mathbb{R}$ is non-empty iff it is exactifiable.*

Corollary 5.5 remains true if the condition $|\Lambda| = |\Gamma|$ in Definition 5.1 is omitted: For every monotone

$\Lambda \in B^*(\mathcal{A})$ with $\Lambda|M \geq \Gamma$ we have $|\Lambda| \geq |\Gamma|$. Hence exactifiability is necessary for non-emptyness of the core. The previous corollary says that this condition is already sufficient.

We now investigate some continuity properties of exact functionals. For this, we define the σ-core for real functionals like in game theory.

Definition 5.6 *Let $\Gamma : M \to \mathbb{R}$ be a functional, $\mathcal{A}_M$ the σ-algebra generated by the upper level sets of M, i.e. $\mathcal{A}_M := \mathcal{A}(\{f \geq \alpha\},\ f \in M, \alpha \in \mathbb{R})$. Then*

$$\mathcal{C}^\sigma(\Gamma) := \Big\{\lambda \in ca(\mathcal{A}_M) \,\Big|\, I_\lambda|M \geq \Gamma, \lambda \geq 0, \lambda(\Omega) = |\Gamma|\Big\} \quad (18)$$

is called the ***σ-core*** *of Γ.*

Comparing the definitions of the core and the σ-core, a unique "small" algebra $\mathcal{A}_M$ is used in the latter because the elements of the σ-core cannot be extended to σ-additive measures on greater domains in general (cf. [10, Example 1]) in contrast to the elements of the core (cf. Proposition 5.2). Due to the close connection between exact functionals and their core the continuity properties of exact functionals correspond directly to those of the elements of the $\mathcal{A}_M$-core like in game theory (cf. [11, Theorem 3.2 and Proposition 3.15]). The proof of the subsequent Monotone Convergence Theorem is a simple generalization of that given by Parker for exact games in [10].

Theorem 5.7 *Let $\Gamma : M \to \mathbb{R}$ be an exact functional satisfying $\mathcal{C}(\Gamma) = \mathcal{C}^\sigma(\Gamma)$ and $(f_n)_{n\in\mathbb{N}}$ a monotone sequence in M such that f_n converges pointwise to a function $f \in M$. Then*

$$\lim_{n\to\infty} \Gamma(f_n) = \Gamma(f). \quad (19)$$

Fatou's Lemma and Lebesgue's Dominated Convergence Theorem can be deduced from the Monotone Convergence Theorem analogously to integration theory.

6 Summary and Conclusions

Two classes of functionals have been presented which slightly generalize the coherent lower previsions resp. previsions avoiding sure loss. This is done to build a common mathematical basis for the theory of imprecise previsions and cooperative game theory which provides with the core concept a well-developed analyzing method for the presented theory. Additionally, this generalization allows us to apply functional analytic methods like the Hahn-Banach type extension or the construction method presented in section 4.

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