The status of F-indicator-fields within the theory of interval-probability

K. Weichselberger
University of Munich, Forschungsgruppe Intervallwahrscheinlichkeit
Ludwigstr. 33, 80539 München
weichsel@stat.uni-muenchen.de

Abstract

In the theory of interval-probability the F-indicator-field of the random event A describes the information that A occurs and that nothing more is known. It will be shown how combined algebraic operations can be employed to trace back the set of F-fields to the set of F-indicator-fields. This is also a demonstration of mathematical analysis applied to the objects of the theory of interval-probability.

Keywords. Interval-probability, indicator-fields, Choquet-capacities.

1 Introduction

The concept of interval-probability has been developed as a means of unifying the treatment of uncertainty and ambiguity. The theory of interval-probability is aimed at as a self-dependent generalization of the costumary "classical" theory of probability. Up to now its results are presented in a German monograph [13] and in some English articles [10], [11], [12]. Of course it shows many similarities with related approaches. This applies in the first line to the school of T. Fine [9], [6] and to P. Walley's treatment of imprecise probabilities [8], but also to P. Huber [4] and to the employment of Choquet-capacities by Huber and Strassen [5] as well as by Dempster and Shafer [2], [3], [7]. To avoid misinterpretations the theory applies own terminology even to concepts very near to definitions given by other authors.

The theory of interval-probability generalizes the approach by Kolmogorov and is not restricted to any of the competing views on the nature of probability statements. Among others an important feature of the theory is the possibility to apply algebraic operations to probability fields. It will be shown how such operations may be used to characterize certain types of fields, for instance those generated by totally monotone capacities.

2 The theory of interval-probability

The theory is mainly characterized by three axioms in addition to the three Kolmogorov-axioms. The latter are understood to define the concept of K-functions on a given sample-space Ω with respect to a σ-field $\mathcal{A}$ of random events.

2.1 Definition.
An interval-valued set function $P(.)$ on $\mathcal{A}$ is called an R-probability if it obeys the following two axioms:
IV.

$$P(A) = [L(A); U(A)], \quad \forall A \in \mathcal{A}, \tag{1}$$

with

$$0 \leq L(A) \leq U(A) \leq 1, \quad \forall A \in \mathcal{A}. \tag{2}$$

V.
The set $\mathcal{M}$ of K-functions $p(.)$ on $\mathcal{A}$ with

$$L(A) \leq p(A) \leq U(A), \quad \forall A \in \mathcal{A}, \tag{3}$$

is not empty. □

If axioms IV and V are satisfied, $(\Omega; \mathcal{A}; L(.), U(.)) =: \mathcal{R}$ is named an R-(probability)field, and the set of K-functions p(.):

$$\mathcal{M} = \{p(.) \mid L(A) \leq p(A) \leq U(A), \forall A \in \mathcal{A}\}.$$

is named the structure of $\mathcal{R}$.
The higher quality level of probability assignments is characterized by

2.2 Definition.
R-probability in accordance with the following axiom is named F-probability:
VI.

$$\left.\begin{array}{rcl} \inf_{p(.)\in\mathcal{M}} p(A) & = & L(A) \\ \sup_{p(.)\in\mathcal{M}} p(A) & = & U(A) \end{array}\right\} \forall A \in \mathcal{A} \tag{4}$$

If VI holds, $(\Omega; \mathcal{A}; L(.)) = \mathcal{F}$ is named an F-(probability) field, where the upper interval-limits are determined by

$$U(A) = 1 - L(\neg A), \quad \forall A \in \mathcal{A}. \tag{5}$$

□

If the structure of an F-field consists of only one element, the field is called point-probability. Materially it equals classical probability. Otherwise any subset $\mathcal{V}$ of the structure $\mathcal{M}$, for which

$$\left.\begin{array}{rcl} \inf_{p(.)\in\mathcal{V}} p(A) & = & L(A) \\ \sup_{p(.)\in\mathcal{V}} p(A) & = & U(A) \end{array}\right\} \forall A \in \mathcal{A}, \tag{6}$$

holds, is called a prestructure of the F-field. R-fields not in concordance with Axiom VI are named redundant R-fields. If a redundant R-field obeys (5) and shows $L(\Omega) = 1$, it is designated as an adjusted one.

The theory of interval-probability understands F-probability to constitute the top quality of probability assignments using interval-limits – obviously not in concordance with concepts limiting the title of probability assignment to subgroups of F-probability, like Choquet-capacities of order two [5] or to totally monotone Choquet-capacities [7].

A feature of F-probability given by two-monotone capacities is, that for every monotone set of events $A_1, A_2, \ldots, A_s$ there exists at least one K-function $p(.)$ fulfilling $p(A_i) = L(A_i)$, $i = 1, \ldots, s$. An F-field with this property is called C-field.

If – according to the Dempster-Shafer theory – $L(A)$ is described by totally monotone Choquet-capacities on a finite Ω with $\mathcal{A} = \mathcal{P}(\Omega)$, the Moebius inversion

$$m(A) = \sum_{B\subseteq A} (-1)^{|A|-|B|} L(B), \quad \forall A \in \mathcal{A}, \tag{7}$$

produces $m(A) \geq 0$ and together with $\sum_{A\subseteq\Omega} m(A) = 1$ allows an interpretation of $m(A)$ as "basic probability numbers" with

$$L(A) = \sum_{B\subseteq A} m(B), \quad \forall A \in \mathcal{A}. \tag{8}$$

If $L(A)$ is not totally monotone, for at least one $A \subseteq \Omega$ the Moebius inversion gives $m(A) < 0$ making impossible this type of interpretation.

F-fields, for which the lower interval-limits $L(.)$ are given by a totally monotone Choquet-capacity are shortly nominated as CT-fields.

2.1 Operations with F-fields

In classical probability theory every assignment in Ω_k – the sample space of k elements – corresponds to a point in the hyperplane $\sum_{i=1}^{k} p_i = 1$ in R^k. In the theory of interval-probability any assignment by F-probability one-to-one determines a structure in R^k: a polyhedron created by the interval-limits in the hyperplane $\sum_{i=1}^{k} p_i = 1$. Since on the one hand the structure is a set, but on the other hand it is generated by the vector of lower interval-limits, both set-operations and vector-operations can be executed on F-fields and on structures.

3.1 Definition.

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \Longleftrightarrow L_2(A) \leq L_1(A), \quad \forall A \in \mathcal{A}:$$

$\mathcal{F}_1$ is at least as strong as $\mathcal{F}_2$. □

3.2 Definition.
J not empty:

$$\tilde{\mathcal{F}} = \bigcup_{i\in J} \mathcal{F}_i \Longleftrightarrow \tilde{L}(A) = \inf_{i\in J} L_i(A), \quad \forall A \in \mathcal{A}, \tag{9}$$

□

$\tilde{\mathcal{F}} = (\Omega; \mathcal{A}; \tilde{L}(.))$ is the union of $\mathcal{F}_i$, $i \in J$.
With respect to the structures $\mathcal{M}(\mathcal{F})$, one arrives at

3.3 Lemma.
$\bigcup_{i\in I} \mathcal{M}(\mathcal{F}_i) \subseteq \mathcal{M}(\tilde{\mathcal{F}}) = \mathcal{M}(\bigcup_{i\in I} \mathcal{F}_i)$. □

3.4 Definition.
For any R-probability field $\mathcal{R}$ with structure $\mathcal{M}(\mathcal{R})$ the derivable F-field $\mathcal{F} = \mathcal{R}^*$ is the F-field, for which $\mathcal{M}(\mathcal{F}) = \mathcal{M}(\mathcal{R})$ holds. □

3.5 Lemma.
a.) $\mathcal{R}^* \subseteq \mathcal{R}$,
b.) $\mathcal{F} \subseteq \mathcal{R} \Rightarrow \mathcal{F} \subseteq \mathcal{R}^*$. □

There are two types of intersection:

3.6 Definition.
J not empty; $\mathcal{F}_i = (\Omega; \mathcal{A}; L_i(.))$, $U_i(A) = 1 - L_i(\neg A)$:

$$\overline{L}(A) = \sup_{i\in J} L_i(A), \overline{U}(A) = \inf_{i\in J} U_i(A). \tag{10}$$

If $\overline{L}(.)$, $\overline{U}(.)$ satisfy Axiom V: $\overline{\mathcal{R}} = (\Omega; \mathcal{A}; \overline{L}(.), \overline{U}(.))$ constitutes the R-intersection of $\mathcal{F}_i$, $i \in J$: $\overline{\mathcal{R}} =$

$\bigcap_{i \in J} \mathcal{F}_i$.
Otherwise: $\mathcal{F}_i$ have an empty intersection. □

3.7 Definition.
If $\overline{\mathcal{R}} = \bigcap_{i \in J} \mathcal{F}_i$, the derivable F-field of $\overline{\mathcal{R}}$ is called the F-intersection of $\mathcal{F}_i$, $i \in J$:

$$\overline{\mathcal{R}}^* = \underset{i \in J}{\circledast} \mathcal{F}_i$$

□

3.8 Definition.
$\mathcal{F}_i = (\Omega;\, \mathcal{A};\, L_i(.))$, $i = 1, \ldots, n$;
$\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $\sum_{i=1}^{n} \lambda_i = 1$, $\lambda_i \geq 0$,
$i = 1, \ldots, n$.

$$\mathcal{F}_0 = (\Omega;\, \mathcal{A};\, L_0(.)) = \sum_{i=1}^{n} \lambda_i \tilde{\mathcal{F}}_i \Longleftrightarrow$$

$$L_0(A) = \sum_{i=1}^{n} \lambda_i L_i(A), \quad \forall A \in \mathcal{A}. \tag{11}$$

$\mathcal{F}_0$ is called convex-combination of the F-fields $\mathcal{F}_i$ with weights $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$. □

A combination of the concepts described above leads to two further concepts, which are useful in the theory of interval-probability.

3.9 Definition.
Let for all $j \in J$ the F-field $\mathcal{F}_0^{(j)}$ be the convex-combination of
$\mathcal{F}_i = (\Omega;\, \mathcal{A};\, L_i(.))$, $i = 1, \ldots, n$, with weights $\vec{\lambda}^{(j)}$. Then
$\mathcal{F}_{EX} := (\Omega;\, \mathcal{A};\, L_{EX}(.)) = \bigcup_{j \in J} \mathcal{F}_0^{(j)}$
is named the extern-combination of $\mathcal{F}_i$ with the weight-core $\Lambda = \{\vec{\lambda}^{(j)},\, j \in J\}$.
Obviously

$$L_{EX}(A) = \inf_{j \in J} \sum_{i=1}^{n} \lambda_i^{(j)} L_i(A), \quad \forall A \in \mathcal{A}, \tag{12}$$

holds. □

3.10 Definition.
If, under the circumstances of 3.9,

$$L_{IN}(A) = \sup_{j \in J} \sum_{i=1}^{n} \lambda_i^{(j)} L_i(A), \quad \forall A \in \mathcal{A}, \tag{13}$$

defines an F-field $\mathcal{F}_{IN} = (\Omega;\, \mathcal{A};\, L_{IN}(.))$,
the equations $\mathcal{F}_{IN} = \bigcap_{j \in J} \mathcal{F}_0^{(j)}$ hold and $\mathcal{F}_{IN}$ is called the intern-combination of $\mathcal{F}_i$, $i = 1, \ldots, n$, with the weight-core $\Lambda = \{\vec{\lambda}^{(j)},\, j \in J\}$.
The concepts of extern-combination and of intern-combination are comprised under the name of interval-combination.
Definitions up to 3.8 and their consequences are thoroughly described and generalized in [13], the last two in the forthcoming second volume.

3 F-indicator-fields

A special class of F-fields are the F-indicator-fields of each event $A \neq \emptyset$ in $\mathcal{A}$. The F-indicator-field $\mathcal{I}(A)$ for event A formulates the information that event A occurs and that nothing more is known. Its lower interval-limits[1] therefore are determined by

$$\begin{array}{lll} L(B) = 0, & \forall B: & A \not\subseteq B \\ L(B) = 1, & \forall B: & A \subseteq B \end{array} \tag{14}$$

Consequently the conjugate upper interval-limits read as

$$\begin{array}{lll} U(B) = 0 & \forall B: & A \cap B = \emptyset \\ U(B) = 1 & \forall B: & A \cap B \neq \emptyset \end{array}$$

In a sample space $\Omega_k = \bigcup_{i=1, \ldots, k} E_i$ with $\mathcal{A} = \mathcal{P}(\Omega_k)$ there are $2^k - 1$ events $A \neq \emptyset$ and therefore the same number of F-indicator-fields $\mathcal{I}(A)$, $A \in \mathcal{A}$, $A \neq \emptyset$. A special role is that of $\mathcal{I}(E_1), \ldots, \mathcal{I}(E_k)$: The information, that a certain singleton has occurred, leaves no question open, the statement is deterministic. In all other cases the assignment of an F-indicator-field contains some ambiguity: If, for instance, it is certain, that $A_{12} = E_1 \cup E_2$ occurs, but nothing more is known, $P(E_1) = [0;\, 1]$ and $P(E_2) = [0;\, 1]$ must be accepted.
In accordance with the interpretation of F-indicator-fields the following relations hold.

4.1 Lemma.
For $(\Omega;\, \mathcal{A})$ let $A_i \in \mathcal{A} \setminus \{\emptyset\}, \quad \forall i \in J \neq \emptyset:$
a) $\mathcal{I}(A_1) \subseteq \mathcal{I}(A_2) \Leftrightarrow A_1 \subseteq A_2$;
b) $\mathcal{I}(\bigcup_{i \in J} A_i) = \bigcup_{i \in J} \mathcal{I}(A_i)$
c) $\mathcal{I}(\bigcap_{i \in J} A_i) = \circledast_{i \in J} \mathcal{I}(A_i)$, if $\bigcap_{i \in J} A_i \neq \emptyset$. □

Each of the relations is to be verified immediately using (13).
It is obvious that generally the application of convex-combination or of interval-combination to F-indicator-fields leads out of this class of fields. It will be shown how these procedures may be applied in characterizing F-fields on finite sample spaces.
In order to generate probability-fields equivalent to classical probability the application of convex-combinations to F-indicator-fields is a promising device.

4.2 Lemma.
Any convex-combination of the k $\mathcal{F}$-indicator-fields $\mathcal{I}(E_1), \ldots, \mathcal{I}(E_k)$ produces a point-probability-field. □

The proof of this lemma is obvious:

$$\mathcal{F} = \sum_{i=1}^{k} \lambda_i \mathcal{I}(E_i)$$

[1] Note that in Shafer's theory this is a special case of a simple support function ([7])

generates

$$L(B) = U(B) = \sum_{i=1}^{k} \lambda_i \cdot 1_B(E_i)$$

where

$$1_B(E) = \begin{cases} 1: & E \subseteq B \\ 0: & E \not\subseteq B \end{cases}$$

Therefore $p(E_i) = \lambda_i$, $i = 1, \ldots, k$, describes the single element of the structure. □

Evidently every point-probability can be represented in that way by choosing

$$\lambda_i = \begin{cases} p(A_i): & \mid A_i \mid = 1 \\ 0: & \text{otherwise.} \end{cases}$$

While there is no ambiguity in such assignment, ambiguity is produced if $\lambda_i > 0$ for any A_i containing more than one singleton E. □

4.3 Lemma.
Any convex-combination of F-indicator-fields produces a CT-field. □
The proof of this lemma is gained by showing:
1.) Any F-indicator-field is a CT-field.
2.) If the set-functions $L_i(.)$ are totally monotone capacities, this is true also for any convex-combination of $L_i(.)$.
Both parts of the proof can easily be verified. □
In addition it can be shown that every CT-field is a convex-combination of F-indicator-fields. For every CT-field $\mathcal{F}$, weights $\lambda_{\mathcal{F}}(A)$ exist, producing

$$\mathcal{F} = \sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} \lambda_{\mathcal{F}}(A) \mathcal{I}(A).$$

Because of (11) and (14) this amounts to

$$L_{\mathcal{F}}(B) = \sum_{A \subseteq B} \lambda_{\mathcal{F}}(A). \tag{15}$$

(15) allows the identification of $\lambda_{\mathcal{F}}(A)$ via the Moebius inversion:

$$\lambda_{\mathcal{F}}(A) = \textstyle\sum_{B \subseteq A} (-1)^{|A|-|B|} L_{\mathcal{F}}(B)$$
$$\forall A \in \mathcal{A},\ A \neq \emptyset,$$

demonstrating the identity of $\lambda_{\mathcal{F}}(A)$ and the "basic probability numbers" $m(A)$ belonging to the assignment $\mathcal{F}$ according to the Dempster-Shafer-theory.
It remains to investigate, by what means an F-field can be combined from F-indicator-fields, if is not a CT-field.

4 Extern-combination and intern-combination

5.1 Lemma.
Any F-field on $(\Omega_k; \mathcal{P}(\Omega_k))$can be represented as extern-combination of F-indicator-fields. □

The proof of this lemma uses the fact, that the structure of an F-field is a bounded polyhedron and therefore possesses vertices as extreme points and that the set of these vertices is a prestructure of the F-field. Each F-field suiting to a vertex is point-probability and according to the commentary following Lemma 4.2 can be represented by a convex-combination of F-indicator-fields with $\lambda_i = 0$, $\forall A_i :\mid A \mid > 1$. The representation of the union of these point-probabilities is achieved by the union of convex-combinations, therefore by an extern-combination. Moreover this extern-combination due to (6) and (9) is a representation of the F-field itself – but of course not the only one.[2] □

Employing two cases where F-fields of simple design are not CT-fields, the representation by extern-combinations of F-indicator-fields is demonstrated.

5.2 Example.
For $k = 3$ let
$P(E_1) = [0.2; 0.4]$
$P(E_2) = [0.1; 0.2]$
$P(E_3) = [0.4; 0.6]$
According to (5) this entails:
$P(E_1 \cup E_2) = [0.4; 0.6]$
$P(E_1 \cup E_3) = [0.8; 0.9]$
$P(E_2 \cup E_3) = [0.6; 0.8]$.
The structure of this F-field in barycentric representation is shown as Figure 1. It possesses four vertices:
$A: p(E_1) = 0.4, p(E_2) = 0.2, p(E_3) = 0.4$
$B: p(E_1) = 0.4, p(E_2) = 0.1, p(E_3) = 0.5$
$C: p(E_3) = 0.3, p(E_2) = 0.1, p(E_3) = 0.6$
$D: p(E_1) = 0.2, p(E_2) = 0.2, p(E_3) = 0.6$
An attempt to calculate "basic probability numbers" by means of (7) fails because of the result $m(\Omega) = -0.1$.
The convex-combinations of $\mathcal{I}(E_1)$, $\mathcal{I}(E_2)$ and $\mathcal{I}(E_3)$ representing the F-point-probabilities of the four vertices are:
$A: \mathcal{F}_A(.) = 0.4 \cdot \mathcal{I}(E_1) + 0.2 \cdot \mathcal{I}(E_2) + 0.4 \cdot \mathcal{I}(E_3)$
$B: \mathcal{F}_B(.) = 0.4 \cdot \mathcal{I}(E_1) + 0.1 \cdot \mathcal{I}(E_2) + 0.5 \cdot \mathcal{I}(E_3)$
$C: \mathcal{F}_C(.) = 0.3 \cdot \mathcal{I}(E_1) + 0.1 \cdot \mathcal{I}(E_2) + 0.6 \cdot \mathcal{I}(E_3)$
$D: \mathcal{F}_D(.) = 0.2 \cdot \mathcal{I}(E_1) + 0.2 \cdot \mathcal{I}(E_2) + 0.6 \cdot \mathcal{I}(E_3)$
Translated into lower interval-limits:

[2] The distinction between this result and well known lower envelope theorems is due to the exclusive use of F-indicator-fields.

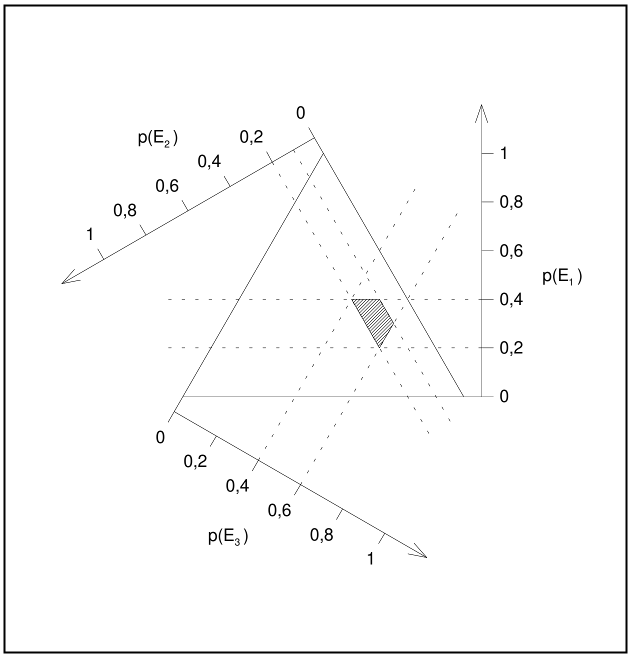

Figure 1: The structure of the F-field represented in Examples 5.2, 5.5, and 5.8.

$L_A(E_1) = 0.4$, $L_A(E_2) = 0.2$, $L_A(E_3) = 0.4$, $L_A(E_1 \cup E_2) = 0.6$, $L_A(E_1 \cup E_3) = 0.8$, $L_A(E_2 \cup E_3) = 0.6$
$L_B(E_1) = 0.4$, $L_B(E_2) = 0.1$, $L_B(E_3) = 0.5$, $L_B(E_1 \cup E_2) = 0.5$, $L_B(E_1 \cup E_3) = 0.9$, $L_B(E_2 \cup E_3) = 0.6$
$L_C(E_1) = 0.3$, $L_C(E_2) = 0.1$, $L_C(E_3) = 0.6$, $L_C(E_1 \cup E_2) = 0.4$, $L_C(E_1 \cup E_3) = 0.9$, $L_C(E_2 \cup E_3) = 0.7$
$L_D(E_1) = 0.2$, $L_D(E_2) = 0.2$, $L_D(E_3) = 0.6$, $L_D(E_1 \cup E_2) = 0.4$, $L_D(E_1 \cup E_3) = 0.8$, $L_A(E_2 \cup E_3) = 0.8$
According to Definition 3.9 the extern-combination is given by the minima of the four set-functions $L_A(.)$, $L_B(.)$, $L_C(.)$ and $L_D(.)$:
$L_{EX}(E_1) = 0.2$, $L_{EX}(E_2) = 0.1$, $L_{EX}(E_3) = 0.4$, $L_{EX}(E_1 \cup E_2) = 0.4$,$L_{EX}(E_1 \cup E_3) = 0.8$, $L_{EX}(E_2 \cup E_3) = 0.6$.
Obviously the lower interval-limit of the given F-field has been represented. □

5.3 Example.
In the case of $k = 4$ a uniform F-probability is given by:

$$P(A) = \begin{cases} P^{(1)}(A) &= \left[L^{(1)}(A); U^{(1)}(A)\right] \\ &= [0.1; 0.4] \ \forall A : \mid A \mid = 1 \\ P^{(2)}(A) &= \left[L^{(2)}(A); U^{(2)}(A)\right] \\ &= [0.4; 0.6] \ \forall A : \mid A \mid = 2 \\ P^{(3)}(A) &= \left[L^{(3)}(A); U^{(3)}(A)\right] \\ &= [0.6; 0.9] \ \forall A : \mid A \mid = 3 \end{cases}$$

The structure of this F-field is shown in Figure 2.

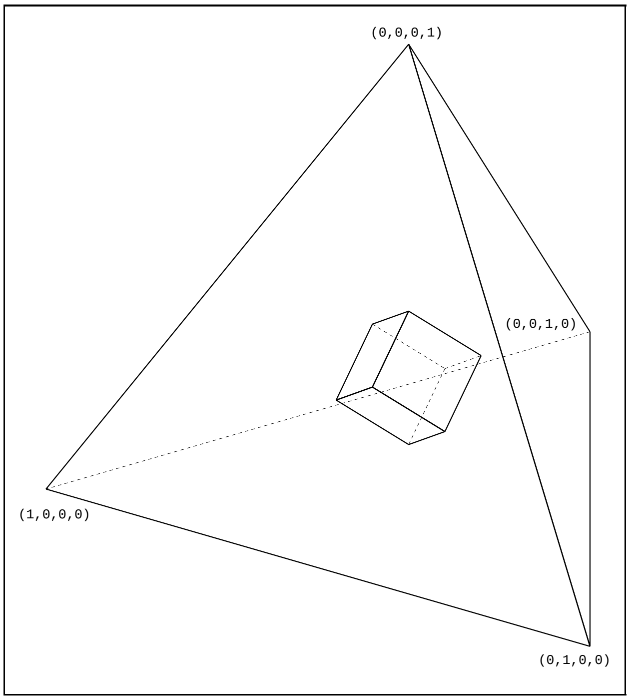

Figure 2: The structure of the F-field represented in Examples 5.3, 5.6, and 5.9.

Its eight vertices are
$A : p_A(E_1) = 0.1$, $p_A(E_2) = 0.3$, $p_A(E_3) = 0.3$, $p_A(E_4) = 0.3$
$B : p_B(E_1) = 0.3$, $p_B(E_2) = 0.1$, $p_B(E_3) = 0.3$, $p_B(E_4) = 0.3$
$C : p_C(E_1) = 0.3$, $p_C(E_2) = 0.3$, $p_C(E_3) = 0.1$, $p_C(E_4) = 0.3$
$D : p_D(E_1) = 0.3$, $p_D(E_2) = 0.3$, $p_D(E_3) = 0.3$, $p_D(E_4) = 0.1$
$E : p_E(E_1) = 0.4$, $p_E(E_2) = 0.2$, $p_E(E_3) = 0.2$, $p_E(E_4) = 0.2$
$F : p_F(E_1) = 0.2$, $p_F(E_2) = 0.4$, $p_F(E_3) = 0.2$, $p_F(E_4) = 0.2$
$G : p_G(E_1) = 0.2$, $p_G(E_2) = 0.2$, $p_G(E_3) = 0.4$, $p_G(E_4) = 0.2$
$H : p_H(E_1) = 0.2$, $p_H(E_2) = 0.2$, $p_H(E_3) = 0.2$, $p_H(E_4) = 0.4$
$L(.)$ is not totally monotone: According to (7): $m(E_1 \cup E_2 \cup E_3) = -0.3$. The procedure demonstrated in Example 5.2 leads to
$L_{EX}(E_1) = \min(0.1, 0.3, 0.4, 0.2) = 0.1 = L_{EX}(E_2) = L_{EX}(E_3) = L_{EX}(E_4)$
$L_{EX}(E_1 \cup E_2) =$
$\min(0.1 + 0.3, 0.3 + 0.3, 0.4 + 0.2, 0.2 + 0.2) = 0.4 = L_{EX}(A), \forall A : \mid A \mid = 2$
$L_{EX}(E_1 \cup E_2 \cup E_3) = \min(0.1+0.3+0.3, 0.3+0.3+0.3, 0.4 + 0.2 + 0.2, 0.2 + 0.2 + 0.2) = 0.6 = L_{EX}(A), \forall A : \mid A \mid = 3$.

Again the given $P(.)$ is represented by P_{EX}. □

For the sake of simplicity one may prefer representations by convex-combinations to those by extern-combinations and among the extern-combinations those which are nearest to a convex-combination. This can be achieved by means of a measure of distance between the convex-combinations employed in one extern-combination.
One way to choose a formula for such a measure is described by

5.4 Definition.
For any interval-combination with weight-core $\Lambda = \{\vec{\lambda}^{(j)},\, j \in J\}$, the measure of distinction in case of weight-vectors $\vec{\lambda}^{(j)} = (\lambda_1^{(j)}, \ldots, \lambda_n^{(j)})$ is given by

$$\triangle(\Lambda) = \sum_{i=1}^{n} (\sup_{j\in J} \lambda_i^{(j)} - \inf_{j\in J} \lambda_i^{(j)}), \qquad (16)$$

providing the result $\triangle(\Lambda) = 0$ only to characterize convex-combinations. □
The representation of the F-field used in Example 5.2 by the extern-combination resulting from the vertices produces the value $\triangle(\lambda) = 0.5$ of the measure of distinction. Another representation of the same F-field is described by

5.5 Example.
The F-field of Example 5.2 is represented by the extern-combination of the F-indicator-fields for all $A \in \mathcal{P}(\Omega_4),\, A \neq \emptyset$, resulting from $\Lambda = \{\vec{\lambda}^{(1)}, \vec{\lambda}^{(2)}, \vec{\lambda}^{(3)}\}$ with
$\lambda_\Omega^{(i)} = 0,\, i = 1,\, 2,\, 3$ and
$\lambda_1^{(1)} = 0.2,\, \lambda_2^{(1)} = 0.15,\, \lambda_3^{(1)} = 0.45,$
$\lambda_{12}^{(1)} = 0.05,\, \lambda_{13}^{(1)} = 0.15,\, \lambda_{23}^{(1)} = 0.00,$
$\lambda_1^{(2)} = 0.25,\, \lambda_2^{(2)} = 0.10,\, \lambda_3^{(2)} = 0.45,\, \lambda_{12}^{(2)} = 0.05,$
$\lambda_{13}^{(2)} = 0.10,\, \lambda_{23}^{(2)} = 0.05,$
$\lambda_1^{(3)} = 0.25,\, \lambda_2^{(3)} = 0.15,\, \lambda_3^{(3)} = 0.40,\, \lambda_{12}^{(3)} = 0.00,$
$\lambda_{13}^{(3)} = 0.15,\, \lambda_{23}^{(3)} = 0.05.$
The lower interval-limits produced by each of the three convex-combinations are
$L^{(1)}(E_1) = 0.20,\, L^{(1)}(E_2) = 0.15,\, L^{(1)}(E_3) = 0.45,$
$L^{(1)}(E_1 \cup E_2) = 0.40,\, L^{(1)}(E_1 \cup E_3) = 0.80,\, L^{(1)}(E_2 \cup E_3) = 0.60$
$L^{(2)}(E_1) = 0.25,\, L^{(2)}(E_2) = 0.10,\, L^{(2)}(E_3) = 0.45,$
$L^{(2)}(E_1 \cup E_2) = 0.40,\, L^{(2)}(E_1 \cup E_3) = 0.80,\, L^{(2)}(E_2 \cup E_3) = 0.60$
$L^{(3)}(E_1) = 0.25,\, L^{(3)}(E_2) = 0.15,\, L^{(3)}(E_3) = 0.40,$
$L^{(3)}(E_1 \cup E_2) = 0.40,\, L^{(3)}(E_1 \cup E_3) = 0.80,\, L^{(3)}(E_2 \cup E_3) = 0.60$
and the extern-combination given by the minimal values:
$L(E_1) = 0.20,\, L(E_2) = 0.10,\, L(E_3) = 0.40,\, L(E_1 \cup E_2) = 0.40,\, L(E_1 \cup E_3) = 0.80,\, L(E_2 \cup E_3) = 0.60.$
The measure of distinction amounts to $\triangle(\Lambda) = 0.30$. □

5.6 Example.
The extern-combination representing the uniform F-field of Example 5.3 yields $\triangle(\Lambda) = 1.2$. A slight improvement to $\triangle(\Lambda) = 1.0$ is achieved by the employment of a weight-core symmetric in E_i and consisting of four convex-combinations, the first of which uses the weights $\lambda_1^{(1)} = 0.1,\, \lambda_2^{(1)} = \lambda_3^{(1)} = \lambda_4^{(1)} = 0.2,\, \lambda_{12}^{(1)} = \lambda_{13}^{(1)} = \lambda_{14}^{(1)} = 0.1,\, \lambda_{23}^{(1)} = \lambda_{24}^{(1)} = \lambda_{34}^{(1)} = 0.0,\, \lambda_{123}^{(1)} = \lambda_{124}^{(1)} = \lambda_{134}^{(1)} = \lambda_{234}^{(1)} = 0.0,\, \lambda_\Omega^{(1)} = 0.0$. This convex-combination hits the values of $L(E_1),\, L(E_1 \cup E_2),\, L(E_1 \cup E_3),$
$L(E_1 \cup E_4),\, L(E_2 \cup E_3 \cup E_4)$ and surmounts the other lower interval-limits. □

Both examples demonstrate the practical way of finding an appropriate weight-core Λ: For each event $B \in \mathcal{P}(\Omega)$ at least one $\vec{\lambda}^{(B)} \in \Lambda$ must be provided, so that

$$L(B) = \sum_{A\subseteq B} \lambda^{(B)}(A)$$

and

$$L(C) \leq \sum_{A\subseteq C} \lambda^{(B)}(A), \quad \forall C \in \mathcal{P}(\Omega) \setminus \{\emptyset\}\,. \qquad (17)$$

Lemma 5.1 allows some commentaries:

- Since no redundant R-field can be represented by an extern-combination of F-indicator-fields, another necessary and sufficient criterion for F-fields is established.
- For the set of all CT-fields the process of closing with respect to unions results in the set of all F-fields. This relation between the two concepts has been described by Th.Augustin [1; pp. 107-108] in 1996 already.
- For the set of F-indicator-fields belonging to singletons, combined application of union and convex-combination leads into more general systems – dependent upon the order of application:
  - Unions of convex-combinations – resulting in extern-combinations – produce the system of all F-fields.
  - Convex-combination of unions generate the set of CT-fields.

With respect to intern-combinations the situation is not quite the same.

5.7 Lemma.
Any F-field on $(\Omega_k; \mathcal{P}(\Omega_k))$ can be represented as intern-combination of F-indicator-fields. $\square$

The proof of this lemma is based on the possibility to generate every F-field $\mathcal{F} = (\Omega_k; \mathcal{P}(\Omega_k); L(.))$ as an intersection of convex-combinations, each of them combining two F-indicator-fields:

$$\mathcal{F} = \bigcap_{A \in \mathcal{P}(\Omega_k) \setminus \{\emptyset, \Omega\}} (L(A) \cdot \mathcal{I}(A) + (1 - L(A)) \cdot \mathcal{I}(\Omega))$$

(See [13], Corollary A. 4. 23). $\square$

The construction of intern-combinations representing a given F-field obeys the same principles as that of extern-combinations with one difference: Instead of (17)

$$L(C) \geq \sum_{A \subseteq C} \lambda^{(B)}(A), \quad \forall C \in \mathcal{P}(\Omega) \setminus \{\emptyset\} \tag{18}$$

is needed. Again a small value of $\triangle(\Lambda)$ may to be aimed at.

5.8 Example.
The F-field of Example 5.2 is represented by the intern-combination of the F-indicator-fields for all $A \in \mathcal{P}(\Omega_4)$, $A \neq \emptyset$, produced by $\Lambda = \{\vec{\lambda}^{(1)}, \vec{\lambda}^{(2)}\}$ with $\lambda_\Omega^{(i)} = 0.0$, $i = 1, 2$,
$\lambda_1^{(1)} = 0.2$, $\lambda_2^{(1)} = 0.1$, $\lambda_3^{(1)} = 0.4$, $\lambda_{12}^{(1)} = 0.1$, $\lambda_{13}^{(1)} = 0.1$, $\lambda_{23}^{(1)} = 0.1$,
$\lambda_1^{(2)} = 0.2$, $\lambda_2^{(2)} = 0.1$, $\lambda_3^{(2)} = 0.4$, $\lambda_{12}^{(2)} = 0.05$, $\lambda_{13}^{(2)} = 0.2$, $\lambda_{23}^{(2)} = 0.05$,
The two convex-combinations yield the following lower interval-limits:
$L^{(1)}(E_1) = 0.2$, $L^{(1)}(E_2) = 0.1$, $L^{(1)}(E_3) = 0.4$,
$L^{(1)}(E_1 \cup E_2) = 0.4$, $L^{(1)}(E_1 \cup E_3) = 0.7$, $L^{(1)}(E_2 \cup E_3) = 0.6$
$L^{(2)}(E_1) = 0.2$, $L^{(2)}(E_2) = 0.1$, $L^{(2)}(E_3) = 0.4$,
$L^{(2)}(E_1 \cup E_2) = 0.35$, $L^{(2)}(E_1 \cup E_3) = 0.8$, $L^{(2)}(E_2 \cup E_3) = 0.55$.
Therefore the intern-combination generated by the maximal values equals the given $L(.)$. The measure of distinction for this weight-core amounts to 0.1. $\square$

5.9 Example.
An intern-combination representing the uniform F-field of Example 5.3 is given by $\Lambda = \{\vec{\lambda}^{(1)}, \vec{\lambda}^{(2)}, \vec{\lambda}^{(3)}\}$ with
$\lambda^{(1)}(A) = \lambda^{(2)}(A) = \lambda^{(3)}(A) = 0.1, \forall A : |A| = 1$,
$\lambda^{(1)}(E_1 \cup E_2) = \lambda^{(1)}(E_3 \cup E_4) = 0.2$,
$\lambda^{(1)}(E_1 \cup E_3) = \lambda^{(1)}(E_1 \cup E_4) = \lambda^{(1)}(E_2 \cup E_3) = \lambda^{(1)}(E_2 \cup E_4) = 0.05$
$\lambda^{(2)}(E_1 \cup E_3) = \lambda^{(2)}(E_2 \cup E_4) = 0.2$,
$\lambda^{(2)}(E_1 \cup E_2) = \lambda^{(2)}(E_1 \cup E_4) = \lambda^{(2)}(E_2 \cup E_3) = \lambda^{(2)}(E_3 \cup E_4) = 0.05$
$\lambda^{(3)}(E_1 \cup E_4) = \lambda^{(3)}(E_2 \cup E_3) = 0.2$,
$\lambda^{(3)}(E_1 \cup E_2) = \lambda^{(3)}(E_1 \cup E_3) = \lambda^{(3)}(E_2 \cup E_4) = \lambda^{(3)}(E_3 \cup E_4) = 0.05$
$\lambda^{(1)}(A) = \lambda^{(2)}(A) = \lambda^{(3)}(A) = 0.0 \ \forall A : |A| = 3, 4$.
Each of the three convex-combinations hits
$L(E_1) = L(E_2) = L(E_3) = L(E_4) = 0.1$
and $L(E_1 \cup E_2 \cup E_3) = L(E_1 \cup E_2 \cup E_4) = L(E_1 \cup E_3 \cup E_4) = L(E_2 \cup E_3 \cup E_4) = 0.6$.
For A, $|A| = 2$, Λ produces:
$L^{(1)}(E_1 \cup E_2) = L^{(1)}(E_3 \cup E_4) = 0.4$,
$L^{(1)}(E_1 \cup E_3) = L^{(1)}(E_2 \cup E_3) = L^{(1)}(E_1 \cup E_4)$
$= L^{(1)}(E_2 \cup E_4) = 0.25$
$L^{(2)}(E_1 \cup E_3) = L^{(2)}(E_2 \cup E_4) = 0.4$,
$L^{(2)}(E_1 \cup E_2) = L^{(2)}(E_2 \cup E_3)$
$= L^{(2)}(E_1 \cup E_4) = L^{(2)}(E_3 \cup E_4) = 0.25$,
$L^{(3)}(E_1 \cup E_4) = L^{(3)}(E_2 \cup E_3) = 0.4$,
$L^{(3)}(E_1 \cup E_2) = L^{(3)}(E_1 \cup E_3)$
$= L^{(3)}(E_2 \cup E_4) = L^{(3)}(E_3 \cup E_4) = 0.25$
Therefore the six maximal values yield $L(A) = 0.4, \quad \forall A : |A| = 2$. The measure of distinction is calculated as $\triangle(\Lambda) = 6 \cdot 0.15 = 0.90$.
In comparison with extern-combinations generally it is easier to find an intern-combination of F-indicator-fields representing the given F-field.
In another respect, too, the role of intern-combinations is different.

- Any F-field can be represented by an intern-combination of F-indicator-fields, but this is true also for redundant R-fields if they are adjusted. Representation by an intern-combination of F-indicator-fields therefore may be used as a necessary, but not as a sufficient criterion for F-fields.

Since the process of R-intersection of CT-fields leads to the set of adjusted R-fields too, it may be stated: Concerning the operations union and intersection there is no difference between CT-fields and other F-fields.
Beyond characterizing the role of F-indicator-fields the present article demonstrates the kind of formalism becoming available, if uncertainty thoroughly is expressed by means of interval-probability.

Acknowledgements

I thank two referees, Anton Wallner and Thomas Augustin for their valuable remarks.

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